Applications of Tree Automata Theory
Lecture I: Tree Automata

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Roadmap

1. Theory of Tree Automata
2. Parsing — Basics and Evaluation
3. Parsing — Advanced Topics
4. Machine Translation — Basics and Evaluation
5. Theory of Tree Transducers
6. Machine Translation — Advanced Topics

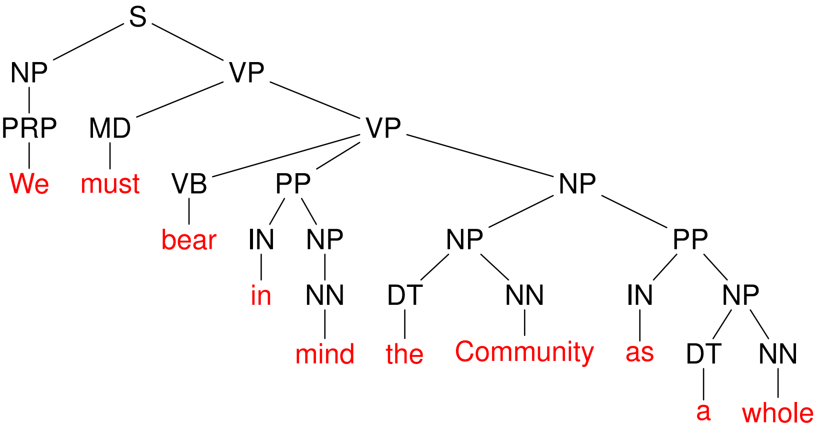
Always ask questions right away!

Trees

Motivation and Notation

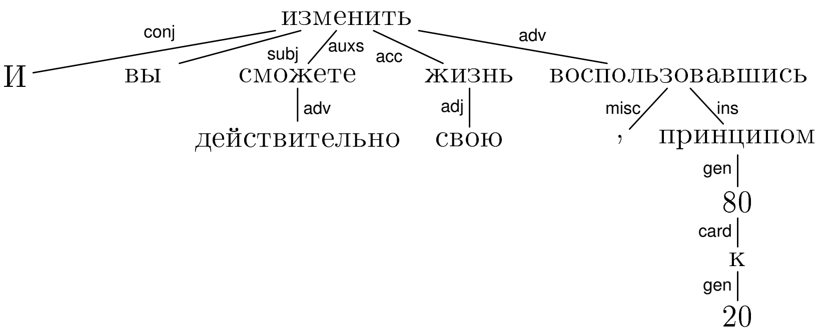
Trees — Parses

We must bear in mind the Community as a whole



Trees — Parses

И вы действительно сможете изменить свою жизнь, воспользовавшись принципом 80 к 20



Trees — XML

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<library>
 <book>
  <title>The Golden Ticket</title>
  <author>Lance Fortnow</author>
  <publisher>Princeton Univ. Press</publisher>
 </book><book>
  <title>Anna Karenina</title>
  <author>Leo Tolstoy</author>
  <publisher>The Russian Messenger</publisher>
 </book>
</library>
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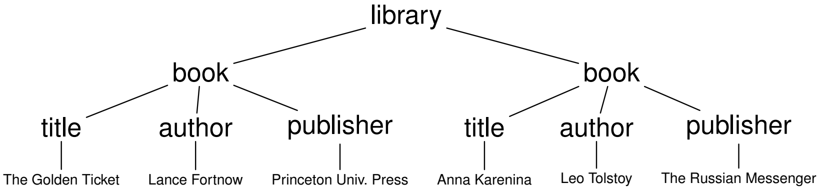


Trees

Sets Σ and Q

Definition

Set $T_\Sigma(Q)$ of Σ-trees indexed by Q is smallest T

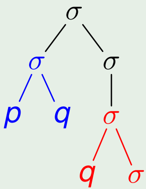
- $q \in T$ for all $q \in Q$
- $\sigma(t_1, \ldots, t_k) \in T$ for all $k \in \mathbb{N}$, $\sigma \in \Sigma$, and $t_1, \ldots, t_k \in T$

We assume $\Sigma \cap Q = \emptyset$ and write σ instead of $\sigma()$

Trees

Example

- $\Sigma = \{\sigma, \alpha\}$ and $Q = \{q, p\}$
- $\sigma(\sigma(p, q), \sigma(\sigma(q, \sigma))) \in T_\Sigma(Q)$



Trees

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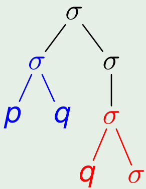

Notes

- obvious recursion & induction principle

Trees — Recursion

Definition (Gorn address)

Mapping $\text{pos}\colon T_\Sigma(Q) \to 2^{\mathbb{N}^*}$ assigning positions

$$\text{pos}(q) = \{\varepsilon\}$$

$$\text{pos}(\sigma(t_1, \ldots, t_k)) = \{\varepsilon\} \cup \{i.w \mid 1 \leq i \leq k,\ w \in \text{pos}(t_i)\}$$

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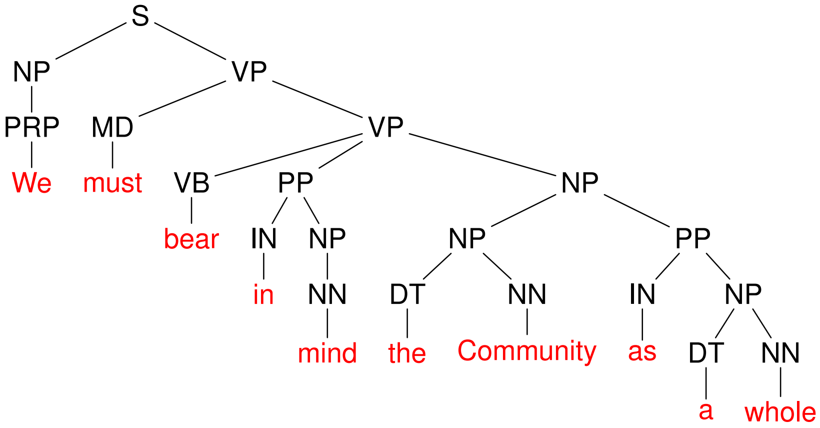
Definition

Leaves of $t \in T_\Sigma(Q)$:

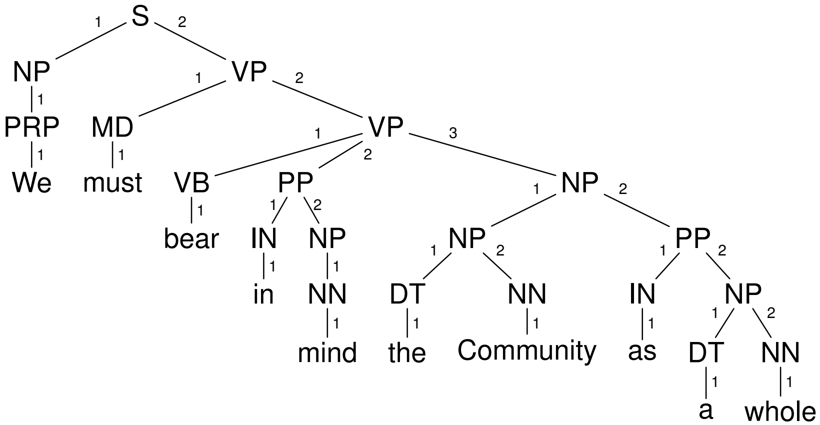
$$\text{leaves}(t) = \{w \in \text{pos}(t) \mid w.1 \notin \text{pos}(t)\}$$

Trees — Recursion

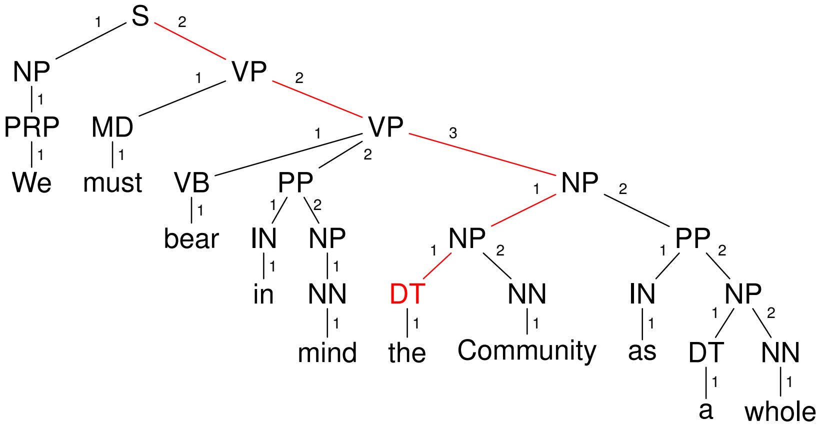


Leaves marked in red

Trees — Recursion



Trees — Recursion



Address of marked ‘DT’: 2.2.3.1.1

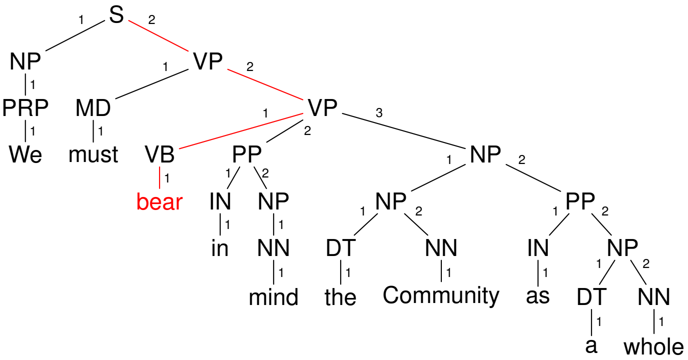
Trees

Trees $t, u \in T_\Sigma(Q)$ and position $w \in \text{pos}(t)$ in t

Notation

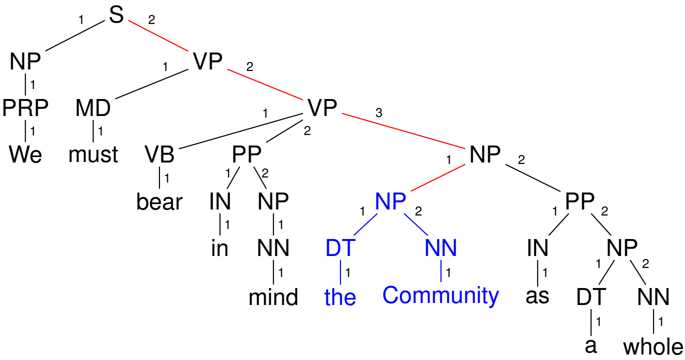
- $t(w)$ = label of t at position w
- $t|_w$ = subtree rooted in w
- $t[u]_w$ = tree obtained by replacing the subtree at w in t by u

Trees



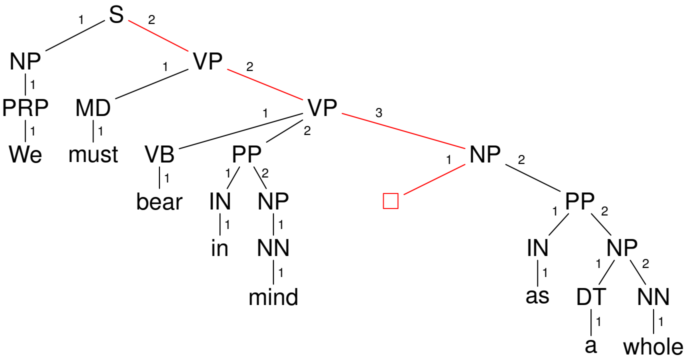
- $t(2.2.1.1) =$ bear

Trees



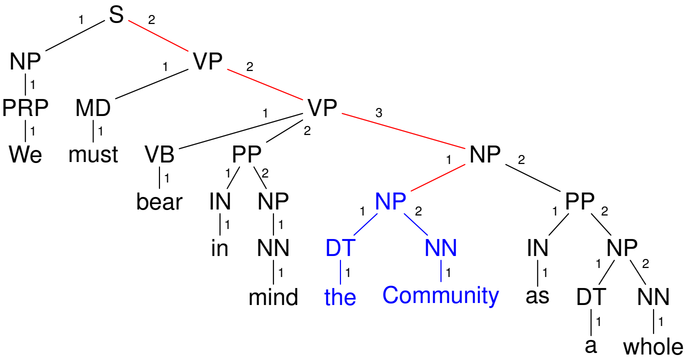
- $t(2.2.1.1) =$ bear
- $t|_{2.2.3.1} =$ NP(DT(the), NN(Community))

Trees



- $t(2.2.1.1) =$ bear
- $t|_{2.2.3.1} = \text{NP}(\text{DT}(\text{the}), \text{NN}(\text{Community}))$
- $t = t'[\text{NP}(\text{DT}(\text{the}), \text{NN}(\text{Community}))]_{2.2.3.1}$

Trees



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Tree Language

Representations

Tree Language

Tree language (or forest) = subset of $T_\Sigma(Q)$

Motivation

- parses of a sentence — parse forest
- translations of an input sentence — translation forest
- valid XML documents — XML schema
- …

Tree Language

How to represent a set of trees?

- enumerate them

Tree Language

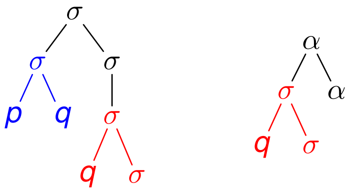
How to represent a set of trees?

- ~~enumerate them~~
- enumerate them cleverly (e.g., add sharing)

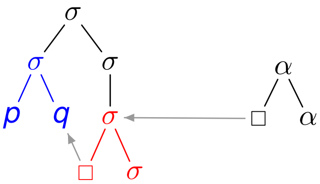
Tree Language

$L = \{\sigma(\sigma(p, q), \sigma(\sigma(q, \sigma)))\,,\ \alpha(\sigma(q, \sigma), \alpha)\}$

Enumeration of L:



Subtree-shared enumeration of L: (packed forest)



Tree Language

How to represent a set of trees?

- ~~enumerate them~~
- enumerate them cleverly (packed forest)

Tree Language

How to represent a set of trees?

- ~~enumerate them~~
- enumerate them cleverly (packed forest)
- parse forest of a CFG

Parse Forest of a CFG

Example

$$S \to NP\ VP \qquad VP \to MD\ VP$$

$$NP \to NP\ PP \qquad VP \to VB\ PP\ NP$$

$$MD \to must \qquad \ldots$$

Parse Forest of a CFG

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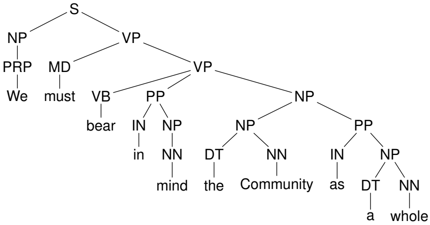

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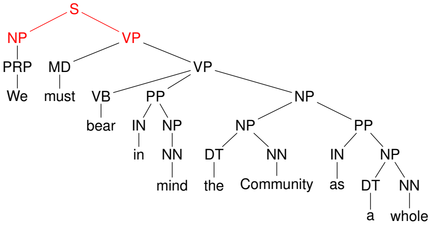

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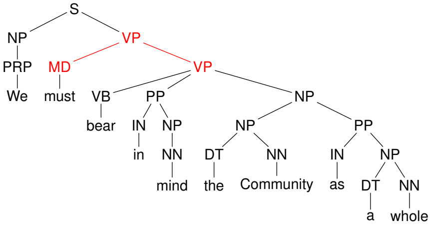

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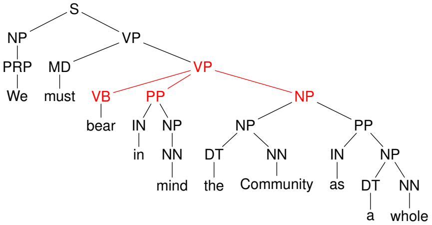

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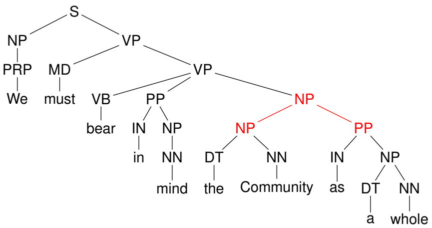

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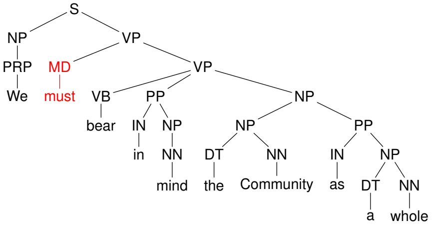

Local Tree Grammar

Definition

A local tree grammar (LTG) is a CFG $G = (N, Q, S, P)$

- finite set N — nonterminals
- finite set Q — terminals
- $S \subseteq N$ — start nonterminals
- finite set $P \subseteq N \times (N \cup Q)^*$ — productions

It will compute the derivation trees of the CFG

Local Tree Grammar

LTG $G = (N, Q, S, P)$

Definition (Generated tree language)

$L(G)$ contains exactly the trees $t \in T_N(Q)$

- $t(\varepsilon) \in S$ — root label in S
- $t(w) \to t(w.1) \cdots t(w.k) \in P$ for every internal node $w \in \text{pos}(t)$ with $\{i \in \mathbb{N} \mid w.i \in \text{pos}(t)\} = \{1, \ldots, k\} \neq \emptyset$

  "label $\to$ child labels" is a production of G
- $t(w) \in Q$ for all $w \in \text{leaves}(t)$ — leaves labeled by Q

Local Tree Grammar

Observation

LTGs generate exactly the parse forests of CFGs

Local Tree Grammar

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Theorem

Local tree languages (LTL) are generated by LTGs

closed under	**CFL (strings)**	**LTL (trees)**
union		
intersection		
(rank-bounded) complement		
relabeling		

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relabeling	✓	✗

Local Tree Grammar

Observation

LTGs generate exactly the parse forests of CFGs

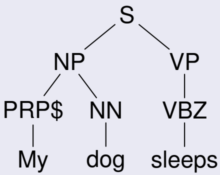
Properties

- ✓ simple
- ✓ no ambiguity (unique explanation for each generated tree)
- ✗ not closed under (most) BOOLEAN operations (union/intersection/complement: ✗/✓/✗)
- ✗ not closed under (non-injective) relabelings
- ✗ ...

Local Tree Grammar

LTG $G = (N, Q, S, P)$

No ambiguity



is in $L(G)$ if and only if

1. S is a start nonterminal
2. all the productions in it are in P
3. all leaves are labeled by elements of Q

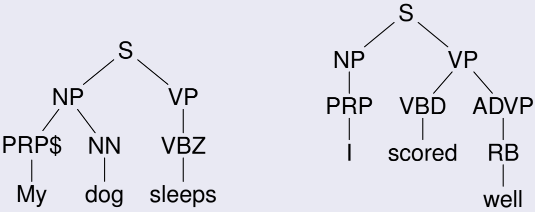
Local Tree Grammar

Observation

Local tree languages are not closed under union

Proof.

The following single-element tree languages are local:



But their union is not local as it must also generate trees for
My dog scored well and **I sleeps* □

Tree Language

How to represent a set of trees?

- ~~enumerate them~~
- enumerate them cleverly (packed forest)
- parse forest of a CFG (local tree languages)

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Tree Language

How to represent a set of trees?

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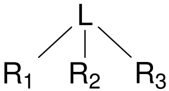
- tree substitution grammar

Local Tree Grammar

- CFG production $L \rightarrow R_1\ R_2\ R_3$ represented by tree



- “Glue” fragments together to obtain larger trees:

S

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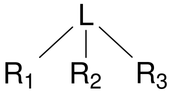

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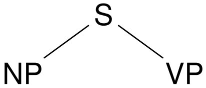

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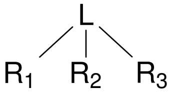

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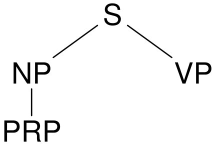

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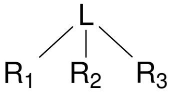

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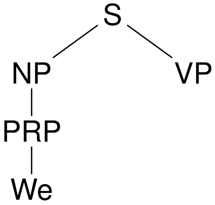

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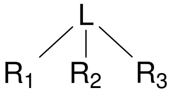

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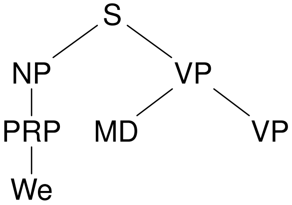

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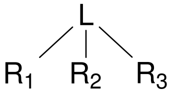

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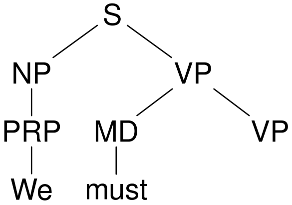

Local Tree Grammar

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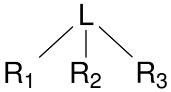

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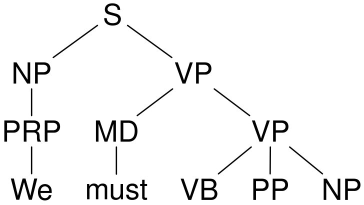

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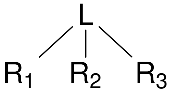

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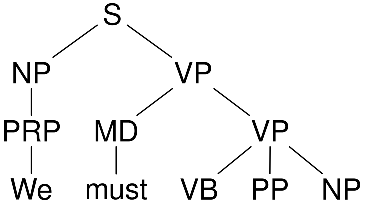

- But why only small tree fragments?

Tree Substitution Grammar

Definition (Joshi 1969)

A tree substitution grammar (TSG) is a tuple (N, Q, S, P)

- finite set N — nonterminals
- finite set Q — terminals
- $S \subseteq N$ — start nonterminals
- finite set $P \subseteq T_N(Q) \setminus Q$ — tree fragments

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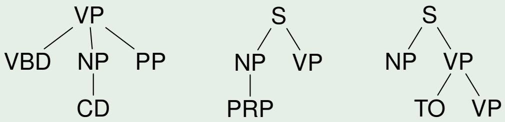
Typical fragments [POST 2011]



Tree Substitution Grammar

TSG $G = (N, Q, S, P)$ and sentential forms $\xi, \zeta \in T_N(Q)$

Definition

$\xi \Rightarrow_G \zeta$ if there exist

- a tree fragment $t \in P$ and
- a position $w \in \text{leaves}(\xi)$

such that $\xi = \xi[t(\varepsilon)]_w$ and $\zeta = \xi[t]_w$

Tree Substitution Grammar

TSG $G = (N, Q, S, P)$ and sentential forms $\xi, \zeta \in T_N(Q)$

Definition

$\xi \Rightarrow_G \zeta$ if there exist

- a tree fragment $t \in P$ and
- a position $w \in \text{leaves}(\xi)$

such that $\xi = \xi[t(\varepsilon)]_w$ and $\zeta = \xi[t]_w$

Intuition

In a derivation step:

1. Find a leaf labeled by $n \in N$
2. Find a fragment $t \in P$ such that $t(\varepsilon) = n$
3. Replace selected n by t

Tree Substitution Grammar

Example

Fragments:

S(NP, VP)	NP(PRP)
VP(MD, VP)	PRP(We)
VP(VB, PP, NP)	MD(must)

S

Tree Substitution Grammar

Example

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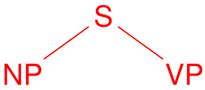

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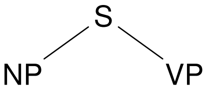

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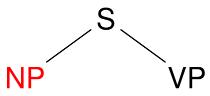

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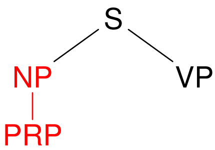

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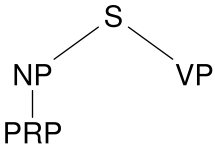

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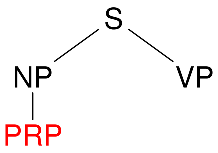

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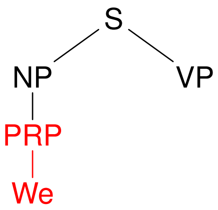

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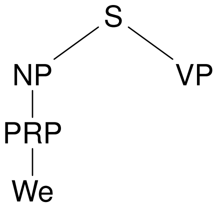

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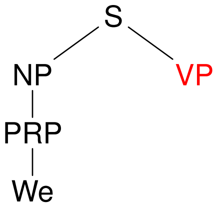

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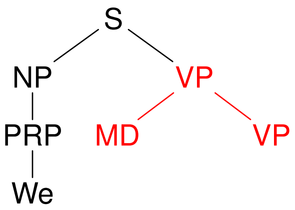

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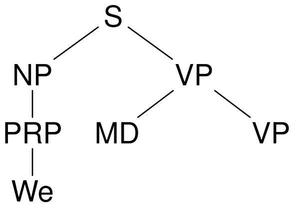

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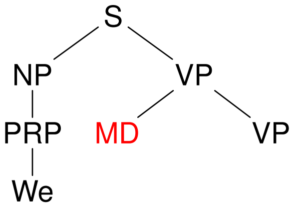

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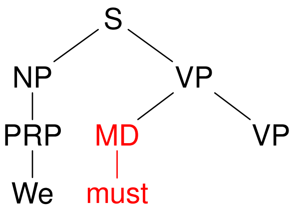

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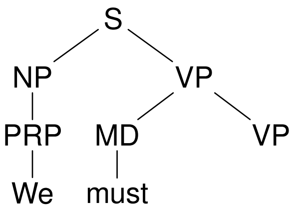

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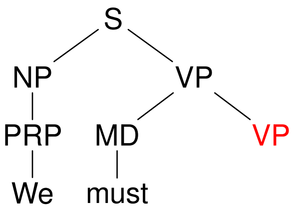

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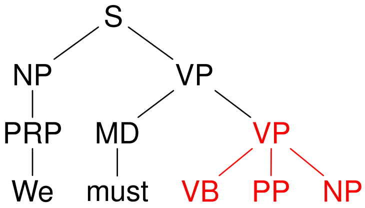

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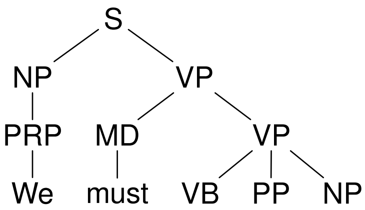

Tree Substitution Grammar

TSG $G = (N, Q, S, P)$

Definition

- for all $n \in N$:

  $$L(G, n) = \{t \in T_N(Q) \mid \forall w \in \text{leaves}(t) \colon t(w) \in Q,\ n \Rightarrow_G^* t\}$$

- $L(G) = \bigcup_{s \in S} L(G, s)$

Tree Substitution Grammar

TSG $G = (N, Q, S, P)$ and TSL $= \{L(G) \mid G \text{ TSG}\}$

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Theorem

1. FIN $\subsetneq$ TSL — all finite languages are TSL

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Theorem

1. $\text{FIN} \subsetneq \text{TSL}$ — all finite languages are TSL
2. $\text{LTL} \subsetneq \text{TSL}$ — all local tree languages are TSL

Tree Substitution Grammar

Theorem

Tree substitution languages (TSL) have the following properties:

closed under	**CFL**	**LTL**	**TSL**
union	✓	✗	
intersection	✗	✓	
(rank-bounded) complement	✗	✗	
relabeling	✓	✗	

Tree Substitution Grammar

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Tree Substitution Grammar

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Tree Substitution Grammar

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Tree Substitution Grammar

Properties

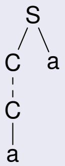
- ✓ simple
- ✓ more expressive than local tree grammars
- ✗ ambiguity (several explanations for a generated tree)
- ✗ not closed under BOOLEAN operations
  (union/intersection/complement: ✗/✗/✗)
- ✗ not closed under (non-injective) relabelings
- ✗ …

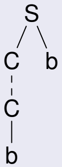
Theorem

Tree substitution languages are not closed under union

Proof.

Counterexample must be infinite $\rightsquigarrow$ artificial example





$L_1 = \{S(C^n(a), a) \mid n \in \mathbb{N}\}$ $L_2 = \{S(C^n(b), b) \mid n \in \mathbb{N}\}$

Their union is not a tree substitution language □

Tree Substitution Grammar

Theorem

Tree substitution languages are not closed under intersection

Proof.

Ideas? □

Tree Language

How to represent a set of trees?

- ~~enumerate them~~
- enumerate them cleverly (packed forest)
- ~~parse forest of a CFG~~ (local tree languages)
- tree substitution grammar
- regular tree grammar

Regular Tree Grammar

Definition (Brainerd, 1969)

A regular tree grammar (RTG) is a tuple $G = (N, \Sigma, S, P)$ with

- finite set N — nonterminals
- finite set Σ — terminals
- $S \subseteq N$ — start nonterminals
- finite set $P \subseteq N \times T_\Sigma(N)$ — productions

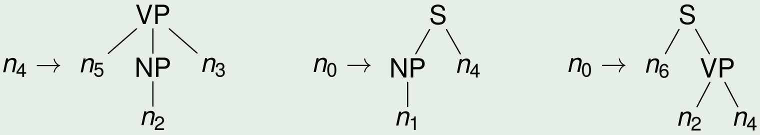
Remark

Instead of (n, t) we write $n \to t$

Regular Tree Grammar

Example

- $N = \{n_0, n_1, n_2, n_3, n_4, n_5, n_6\}$
- $\Sigma = \{\mathrm{VP}, \mathrm{NP}, \mathrm{S}, \dots\}$
- $S = \{n_0\}$
- and the following productions:



Regular Tree Grammar

RTG $G = (N, \Sigma, S, P)$ and sentential forms $\xi, \zeta \in T_\Sigma(N)$

Definition (Derivation Semantics)

$\xi \Rightarrow_G \zeta$ if there exist

- a production $n \to t \in P$ and
- a position $w \in \text{leaves}(\xi)$

such that $\xi = \xi[n]_w$ and $\zeta = \xi[t]_w$

Regular Tree Grammar

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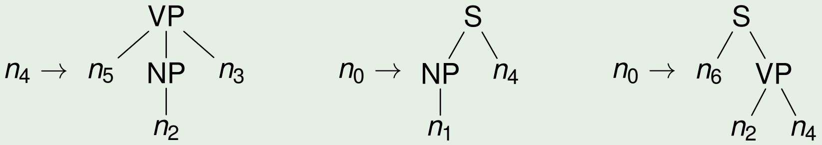
Definition (Recognized tree language)

$$L(G) = \{t \in T_\Sigma \mid \exists s \in S\colon s \Rightarrow_G^* t\}$$

Regular Tree Grammar

Productions

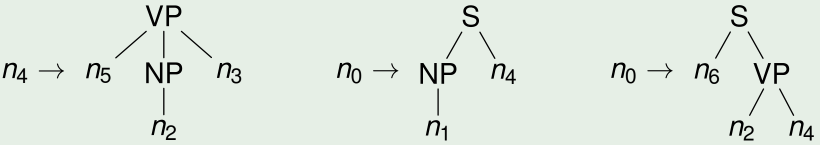


Derivation:

n_0

Regular Tree Grammar

Productions



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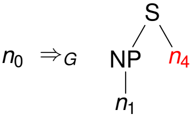

Regular Tree Grammar

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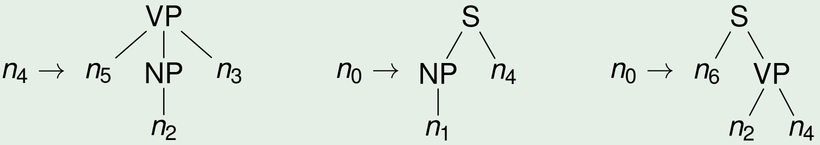

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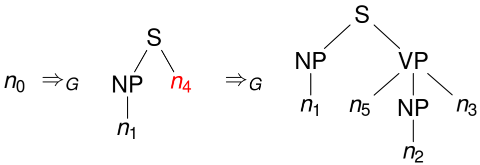

Regular Tree Grammar

regular tree languages $\text{RTL} = \{L(G) \mid G \text{ RTG}\}$

Theorem

tree substitution languages $\subsetneq$ *regular tree languages*

Proof.

We can express the union counterexample easily □

Regular Tree Grammar

Theorem

Regular tree languages (RTL) have the following properties:

closed under	CFL	LTL	TSL	RTL
union	✓	✗	✗	
intersection	✗	✓	✗	
(rank-bounded) complement	✗	✗	✗	
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Regular Tree Grammar

Properties

- ✓ simple
- ✓ more expressive than tree substitution grammars
- ✗ ambiguity (several explanations for a generated tree)
- ✓ closed under all BOOLEAN operations
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- ✓ …

Regular Tree Grammar

RTG $G = (N, \Sigma, S, P)$

Definition (Brainerd, 1969)

G is in normal form if $t = \sigma(n_1, \ldots, n_k)$ with $\sigma \in \Sigma$ and $n_1, \ldots, n_k \in N$ for all $n \to t \in P$

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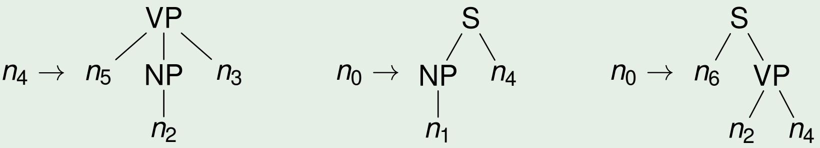
Example productions



Regular Tree Grammar

Theorem (Brainerd, 1969)

Every RTG is equivalent to an RTG in normal form

Proof.

Simply cut large rules introducing new states

$$n_0 \to S(n_6, VP(n_2, n_4)) \quad \mapsto \quad n_0 \to S(n_6, \overline{n}) \qquad \overline{n} \to VP(n_2, n_4)$$

$\square$

Standard Representation

Tree Automata

Tree Automaton

Definition (Thatcher, 1970; Rounds, 1970)

A tree automaton (TA) is an RTG in normal form

Tree Automaton

Definition (THATCHER, 1970; ROUNDS, 1970)

A tree automaton (TA) is an RTG in normal form

Remarks

- bottom-up: rules written as $\sigma(n_1, \ldots, n_k) \to n$
- top-down: rules written as $n \to \sigma(n_1, \ldots, n_k)$

Tree Automaton

Definition

- top-down deterministic if $\forall n \in N, k \in \mathbb{N}, \sigma \in \Sigma$
  $\exists$ at most one $n_1, \ldots, n_k \in N$: $n \to \sigma(n_1, \ldots, n_k) \in P$
  and $|S| = 1$
- bottom-up deterministic if $\forall k \in \mathbb{N}, \sigma \in \Sigma, n_1, \ldots, n_k \in N$
  $\exists$ at most one $n \in N$: $\sigma(n_1, \ldots, n_k) \to n \in P$

(red determines blue)

Tree Automaton

Theorem (Thatcher, Wright, 1968; Doner, 1970)

top-down det. RTL $\subsetneq$ bottom-up det. RTL $=$ RTL

Tree Automaton

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Proof.

Let $G = (N, \Sigma, S, P)$ be a tree automaton. We construct bottom-up det. TA $G' = (2^N, \Sigma, S', P')$ with

- $S' = \{N' \subseteq N \mid N' \cap S \neq \emptyset\}$ contains start nonterminal
- for each $\sigma \in \Sigma$, $N_1, \dots, N_k \subseteq N$, and $k \in \text{rk}_P(\sigma)$

$$\{n \mid n \to \sigma(n_1, \dots, n_k) \in P,\ n_i \in N_i\} \to \sigma(N_1, \dots, N_k) \in P'$$

- no other productions are in P'

□

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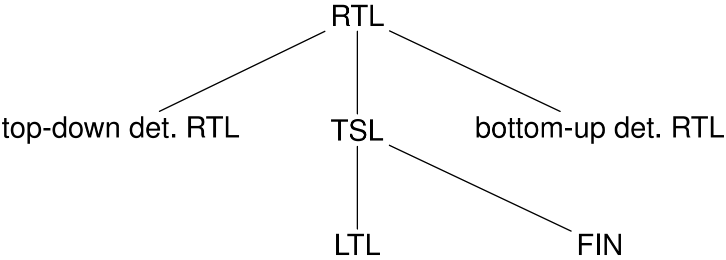
$$\{n \mid n \to \sigma(n_1, \ldots, n_k) \in P,\ n_i \in N_i\} \to \sigma(N_1, \ldots, N_k) \in P'$$

- no other productions are in P'

Strictness by the tree language $L = \{\sigma(\alpha, \beta), \sigma(\beta, \alpha)\}$ □

Tree Automaton



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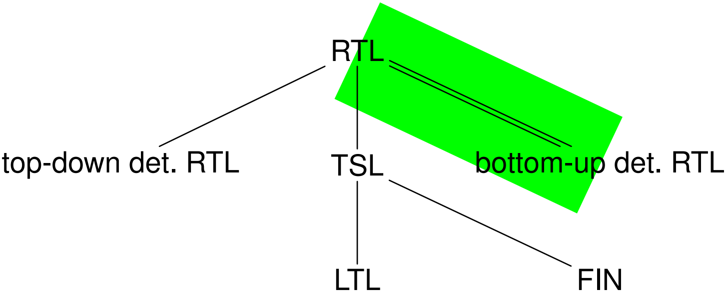

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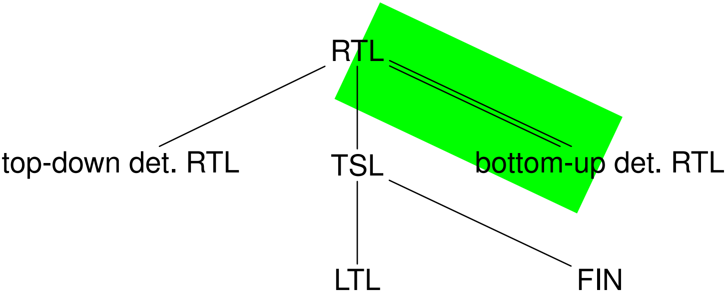


Remark

finite tree languages $\not\subseteq$ top-down deterministic RTL

Tree Automaton

Theorem

Regular tree languages are closed under

- all BOOLEAN operations
- substitution (quotients) and iteration
- (non-deterministic) relabelings
- linear homomorphisms
- inverse homomorphisms

Tree Automaton

Theorem

Regular tree languages are closed under substitution

Definition

$L, L' \subseteq T_\Sigma$ tree languages and $\alpha \in \Sigma$

$$L[\alpha \leftarrow L']$$

contains all trees obtained from a tree of L
by replacing each leaf labeled α by a tree of L'

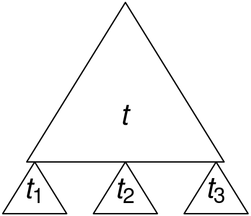
different occurrences can be differently replaced

Tree Automaton

Theorem

Regular tree languages are closed under substitution

$$L[\alpha \leftarrow L']$$



$t \in L$

$t_1, t_2, t_3 \in L'$

Tree Automaton

DTA = bottom-up deterministic tree automaton

Definition

A TA is minimal in $\mathcal{C}$ if all equivalent TAs of $\mathcal{C}$ have at least as many states

Theorem

Complexity of minimization problems:

outp. $\mathcal{C}$ \ inp. model	DTA	TA
DTA	PTime	ExpTime
TA	PSpace-hard in ExpTime	ExpTime

Tree Automaton

Definition

Height of a tree $t \in T_\Sigma(Q)$:

$$\begin{aligned} \text{height}(q) &= 0 \\ \text{height}(\sigma(t_1, \ldots, t_k)) &= 1 + \max \{\text{height}(t_i) \mid 1 \leq i \leq k\} \end{aligned}$$

Tree Automaton

Definition

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Intuition

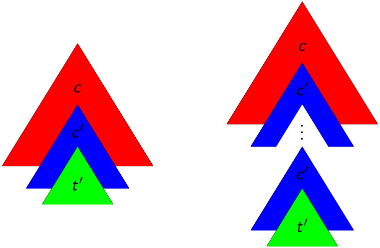
Height of t
$=$ number of edges in a maximal path from the root to a leaf

Tree Automaton

Theorem (Pumping Lemma)

For every regular tree language $L \subseteq T_\Sigma$ there exists $n \in \mathbb{N}$ such that for every $t \in L$ with $\mathrm{height}(t) \geq n$ there exist

- *contexts $c, c' \in C_\Sigma$ and $t' \in T_\Sigma$ such that $t = c[c'[t']]$*
- $c' \neq \Box$
- *$c[c'[c' \cdots c'[t'] \cdots]] \in L$ for any number of c'*



Tree Automaton

Problem	Complexity	
	DTA	TA
Emptiness	PTime	PTime
Finiteness	PTime	PTime
Universality	PTime	ExpTime
Inclusion	PTime	ExpTime
Equivalence	PTime	ExpTime
Membership	in logDCFL	logCFL

Literature

Textbooks

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