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RELATING TREE SERIES TRANSDUCERS AND WEIGHTED TREE AUTOMATA*

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ABSTRACT

Bottom-up tree series transducers (tst) over the semiring $\mathcal{A}$ are implemented with the help of bottom-up weighted tree automata (wta) over an extension of $\mathcal{A}$. Therefore bottom-up $\mathcal{D}$-weighted tree automata ($\mathcal{D}$-wta) with $\mathcal{D}$ a distributive Ω-algebra are introduced. A $\mathcal{D}$-wta is essentially a wta but uses as transition weight an operation symbol of the Ω-algebra $\mathcal{D}$ instead of a semiring element. The given tst is implemented with the help of a $\mathcal{D}$-wta, essentially showing that $\mathcal{D}$-wta are a joint generalization of tst (using IO-substitution) and wta. Then a semiring and a wta are constructed such that the wta computes a formal representation of the semantics of the $\mathcal{D}$-wta. The applicability of the obtained presentation result is demonstrated by deriving a pumping lemma for deterministic finite $\mathcal{D}$-wta from a known pumping lemma for deterministic finite wta. Finally, it is observed that the known decidability results for emptiness cannot be applied to obtain decidability of emptiness for finite $\mathcal{D}$-wta. Thus with help of a weaker version of the derived pumping lemma, decidability of the emptiness problem for finite $\mathcal{D}$-wta is shown under mild conditions on $\mathcal{D}$.

Keywords: weighted tree automaton, distributive Ω-algebra, tree series transducer, semiring, decidability result

1. Introduction

In formal language theory several different accepting and transducing devices were intensively studied [21]. A classical folklore result shows how to implement generalized sequential machines (*see*, *e. g.*, [1]) on weighted automata [22,9,18] with the help of the particular semiring $(\mathfrak{P}(\Sigma^*), \cup, \circ)$ of languages over the alphabet Σ. Naturally, this semiring is not commutative; notwithstanding, the representation

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allows us to transfer results obtained for weighted automata to generalized sequential machines. In this sense, the study of arbitrary weighted automata subsumes the study of generalized sequential machines.

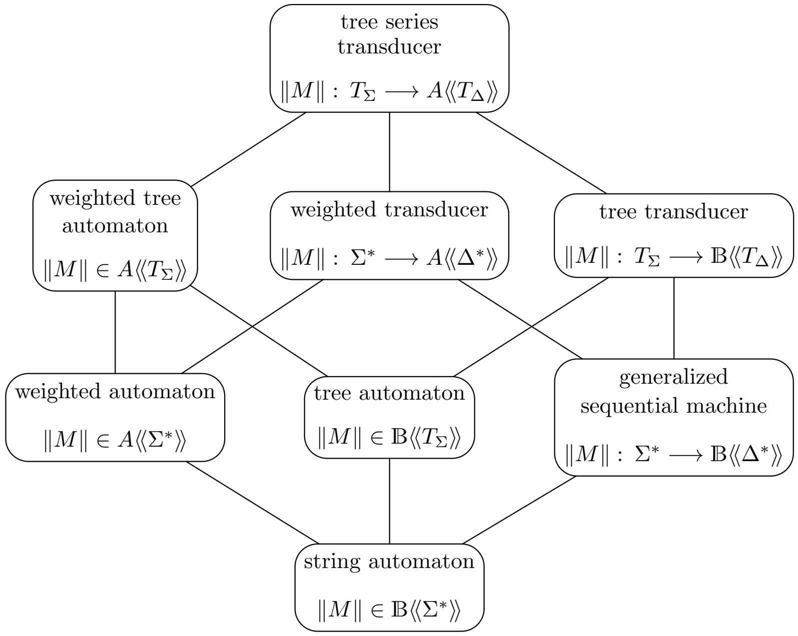

Fig. 1. Generalization hierarchy.

We translate the above representation result to tree languages (*see*, *e. g.*, [8]); *i. e.*, we show how to implement bottom-up tree transducers [20,23] on bottom-up weighted tree automata (wta) [2,5,16]. More generally, we even unearth a relationship between wta and bottom-up tree series transducers (tst) [10,12] using IO-substitution [6,10]. Therefore we first introduce bottom-up $\mathcal{D}$-weighted tree automata ($\mathcal{D}$-wta) with $\mathcal{D}$ a distributive Ω-algebra [11,16]. These devices are essentially wta where the weight of a transition is an operation symbol of an Ω-algebra instead of a semiring element. Such devices can easily simulate both wta and tst by a proper choice of the distributive Ω-algebra (*see* Proposition 2). Next we devise a monoid $\mathcal{A}$ which is capable of emulating the effect of the operation symbols of a distributive Ω-algebra $\mathcal{D}$ (*see* Theorem 1). Then we extend $\mathcal{A}$ to a semiring using the addition of a semiring $\mathcal{B}$ for which $\mathcal{D}$ is a semimodule (*see* Theorem 2). In this way we obtain an abstract addition (of $\mathcal{B}$), which allows us to perform the concrete addition (of $\mathcal{D}$) later. Thereby we obtain a representation result in which a tst or a $\mathcal{D}$-wta is presented as wta, which computes a formal representation of the semantics of the tst or $\mathcal{D}$-wta.

We note that the construction of the semiring preserves many beneficial properties (concerning the addition) of the original distributive Ω-algebra. Hence the study of wta partly subsumes the study of tst. In fact, subsumption holds for deterministic devices; *i. e.*, the study of deterministic wta fully subsumes the study of deterministic tst or $\mathcal{D}$-wta. To illustrate the applicability of the relationship we transfer a pumping lemma [3] for deterministic finite wta to finite $\mathcal{D}$-wta. This is possible, because the semiring addition is irrelevant for deterministic wta and the determinism property is preserved by the constructions. This yields that for a given $\mathcal{D}$-wta M we can construct a wta M' such that $\|M'\| = \|M\|$. Hence the pumping lemma for wta can readily be transfered to $\mathcal{D}$-wta. Since $\mathcal{D}$-wta are a generalization of tst, we implicitly also obtain a pumping lemma for tst. A spelt-out version of this pumping lemma may be found in [19].

However, the results concerning decidability of emptiness for wta [2,3] all require the underlying semiring to be commutative. Not surprisingly, the semiring constructed in order to simulate a tst or $\mathcal{D}$-wta is (usually) not commutative. Thus we reinvestigate decidability of emptiness in the setting of $\mathcal{D}$-wta. Inspired by the derived pumping lemma, we obtain mild restrictions on $\mathcal{D}$, that if imposed, guarantee decidability of emptiness of (not necessarily deterministic) $\mathcal{D}$-wta. It turns out that this way we obtain a generalization of the decidability result of [3], because commutativity of the semiring will trivially imply our restrictions. One of the restrictions we impose is zero-sum freeness. In [2] a statement—similar to a pumping lemma—is derived for wta over fields. However, no interesting field is zero-sum free, so there is still a gap to be bridged.

2. Preliminaries

The set of non-negative integers is denoted by $\mathbb{N}$, and we let $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. In the following, let $k, n \in \mathbb{N}$ and A and B be sets. The interval $[k, n]$ abbreviates $\{\, i \in \mathbb{N} \mid k \leqslant i \leqslant n \,\}$, and we use $[n]$ to stand for $[1, n]$. The cardinality of A is denoted by $\mathrm{card}(A)$. The set of all subsets of A is denoted by $\mathfrak{P}(A)$, and the set of all (total) mappings $f\colon A \longrightarrow B$ is denoted by B^A as customary. Finally, we write A^* for the set of all words over A, $|w|$ for the length of a word $w \in A^*$, and $\cdot$ for concatenation of words.

2.1. Trees and Substitutions

A nonempty set Ω equipped with a mapping $\mathrm{rk}_\Omega\colon \Omega \longrightarrow \mathbb{N}$ is called a *ranked set*, and $\Omega_k = \{\, \omega \in \Omega \mid \mathrm{rk}_\Omega(\omega) = k \,\}$ denotes the set of *operators of rank k*. Finite ranked sets (*i. e.*, Ω is finite) are *ranked alphabets*. Given a ranked set Ω disjoint with B, we write $\Omega \cup B$ to denote the ranked set obtained from Ω by adding the elements of B as nullary symbols. An *Ω-tree* is a partial mapping $t\colon \mathbb{N}_+^* \dashrightarrow \Omega$ such that $\mathrm{dom}(t)$ is finite and prefix-closed, and if $w \in \mathrm{dom}(t)$ and $t(w) \in \Omega_k$, then also $w{\cdot}i \in \mathrm{dom}(t)$ for every $i \in [k]$. The set of all Ω-trees is denoted by T_Ω. We let $\mathrm{height}(t) = 1 + \max\{\, |w| \mid w \in \mathrm{dom}(t) \,\}$ and $\mathrm{size}(t) = \mathrm{card}(\mathrm{dom}(t))$ for every $t \in T_\Omega$. For convenience, we assume a countably infinite set $\mathrm{X} = \{\, \mathrm{x}_i \mid i \in \mathbb{N}_+ \,\}$ of (formal) variables and its subsets $\mathrm{X}_n = \{\, \mathrm{x}_i \mid i \in [n] \,\}$. Let $\mathrm{X}_n \subseteq \Omega_0$ and $t, t'_1, \dots, t'_n \in T_\Omega$. The set $\widehat{T_\Omega}(\mathrm{X}_n)$ contains exactly those trees in which every $x \in \mathrm{X}_n$ occurs exactly once. The expression $t[t'_1, \dots, t'_n]$ denotes the *(parallel) substitution* of t'_i for every occurrence of x_i in t; *i. e.*, $t[t'_1, \dots, t'_n](w \cdot w') = t'_i(w')$ for every $w' \in \mathbb{N}_+^*$ and $w \in \mathrm{dom}(t)$ such that $t(w) = \mathrm{x}_i$ and $t[t'_1, \dots, t'_n](w) = t(w)$ otherwise.

Let $t \in \widehat{T_\Omega}(\mathrm{X}_n)$ with $n \geqslant 1$ and $t' \in \widehat{T_\Omega}(\mathrm{X}_k)$. The *non-identifying substitution* of t' into t, denoted by $t\langle\!|t'|\!\rangle$, yields a tree of $\widehat{T_\Omega}(\mathrm{X}_{k+n-1})$ which is defined by $t\langle\!|t'|\!\rangle = t[t', \mathrm{x}_{k+1}, \dots, \mathrm{x}_{k+n-1}]$. This way no variable of t' is identified with a variable of t. To complete the definition we let $t\langle\!|t'|\!\rangle = t$ whenever $n = 0$. One can compare this with the classical lambda-calculus, where (except for reordering of the arguments) $(\lambda \mathrm{x}_1 \cdots \mathrm{x}_n.t)(\lambda \mathrm{x}_1 \cdots \mathrm{x}_k.t') \Rightarrow \lambda \mathrm{x}_1 \cdots \mathrm{x}_{k+n-1}.t\langle\!|t'|\!\rangle$. Henceforth, we will also use the usual term denotation of trees [13,14] (*i. e.*, $t = \omega(t_1, \dots, t_k)$ for some $\omega \in \Omega_k$ and $t_1, \dots, t_k \in T_\Omega$) and the corresponding induction principle.

2.2. Algebraic Structures

Let A be a set, Ω be a ranked set, and $I = (\, I_k \,)_{k \in \mathbb{N}}$ be a family such that for every $k \in \mathbb{N}$ and $\omega \in \Omega_k$ we have that $I_k(\omega) \colon A^k \longrightarrow A$. The triple (A, Ω, I) is called an *(abstract)* Ω*-algebra* and the particular Ω-algebra $(T_\Omega, \Omega, I_\Omega)$ where $(I_\Omega)_k(\omega)(t_1, \dots, t_k) = \omega(t_1, \dots, t_k)$ for every $k \in \mathbb{N}$, $\omega \in \Omega_k$, and $t_1, \dots, t_k \in T_\Omega$ is called the *initial (term)* Ω*-algebra*. Let X and Ω be disjoint. Every $t \in T_{\Omega \cup \mathrm{X}_n}$ induces a mapping $t^I \colon A^n \longrightarrow A$ defined inductively for every $k \in \mathbb{N}$, $\omega \in \Omega_k$, $t_1, \dots, t_k \in T_{\Omega \cup \mathrm{X}_n}$, and $a_1, \dots, a_n \in A$ by

$$\omega(t_1, \dots, t_k)^I(a_1, \dots, a_n) = I_k(\omega)(t_1^I(a_1, \dots, a_n), \dots, t_k^I(a_1, \dots, a_n)) \tag{1}$$

and $\mathrm{x}_i^I(a_1, \dots, a_n) = a_i$ for every $i \in [n]$. In the sequel we often do not differentiate between the operation symbol ω and the actual operation $I_k(\omega)$ as well as the term $t \in T_{\Omega \cup \mathrm{X}_n}$ and the induced mapping t^I. Usually the context will provide sufficient information as to clarify which meaning is intended. Further, if the set of operations is finite, we commonly list the operations instead of specifying I.

A *monoid* is an algebra $\mathcal{A} = (A, \otimes)$ with an associative operation $\otimes \colon A^2 \longrightarrow A$ and a neutral element $1 \in A$. The neutral element is unique and denoted by $0_{\mathcal{A}}$ or $1_{\mathcal{A}}$ in the sequel. The monoid $\mathcal{A}$ is said to be *commutative*, if $\otimes$ is commutative, and it is said to be *idempotent*, if $a = a \otimes a$ for every $a \in A$. Now let $\mathcal{A} = (A, \otimes)$ be a commutative monoid. We say that $\mathcal{A}$ is *complete*, if it is possible to define an (infinitary) operation $\bigotimes$ such that the following two axioms hold for all index sets I, J and all families $(\, a_i \,)_{i \in I}$ of $a_i \in A$.

(i) $\bigotimes_{i \in \{j\}} a_i = a_j$ and $\bigotimes_{i \in \{j_1, j_2\}} a_i = a_{j_1} \otimes a_{j_2}$ for $j_1 \neq j_2$.
(ii) $\bigotimes_{j \in J} \bigotimes_{i \in I_j} a_i = \bigotimes_{i \in I} a_i$, if $\bigcup_{j \in J} I_j = I$ and $I_{j_1} \cap I_{j_2} = \emptyset$ for $j_1 \neq j_2$.

Henceforth, when we speak about complete monoids we silently assume the operation $\bigotimes$ to be given. The relation $\sqsubseteq \subseteq A^2$ is defined by $a_1 \sqsubseteq a_2$ if and only if there exists $a \in A$ such that $a_1 \otimes a = a_2$. It is easily checked that $\sqsubseteq$ is reflexive and transitive, and if $\sqsubseteq$ is even antisymmetric (and hence a partial order), then $\mathcal{A}$ is said to be *naturally ordered*. Finally, if $\mathcal{A}$ is naturally ordered and complete, then $\mathcal{A}$ is *continuous*, if for every $a \in A$, index set I, and family $(\, a_i \,)_{i \in I}$ of $a_i \in A$

$$\bigotimes_{i \in E} a_i \sqsubseteq a \text{ for all finite } E \subseteq I \quad \iff \quad \bigotimes_{i \in I} a_i \sqsubseteq a \ . \tag{2}$$

Note that idempotent monoids are continuous, if and only if they are *completely idempotent* (*i. e.*, complete and for every nonempty index set I and $a \in A$ we have that $\bigotimes_{i \in I} a = a$).

Let $\mathcal{A} = (A, \oplus, \odot)$ be an algebra made of two monoids $(A, \oplus)$ and $(A, \odot)$ with neutral elements $0_{\mathcal{A}}$ and $1_{\mathcal{A}}$, respectively, of which the former monoid is commutative and the latter has $0_{\mathcal{A}}$ as an *absorbing element* (*i. e.*, $a \odot 0_{\mathcal{A}} = 0_{\mathcal{A}} = 0_{\mathcal{A}} \odot a$ for every $a \in A$). We assume that multiplicative operation symbols have a higher binding priority than additive operation symbols and occasionally drop the multiplicative operation symbol altogether; *i. e.*, simply write $a_1 a_2$ instead of $a_1 \odot a_2$. We say that $\mathcal{A}$ is a *semiring*, if the monoids are connected via the distributivity laws (hence $(a_1 \oplus a_2) \odot (a_3 \oplus a_4) = a_1 a_3 \oplus a_1 a_4 \oplus a_2 a_3 \oplus a_2 a_4$ for every $a_1, a_2, a_3, a_4 \in A$). The semiring $\mathcal{A}$ is called *(additively) idempotent*, if $(A, \oplus)$ is idempotent. Finally, a *complete* semiring consists of a complete monoid $(A, \oplus)$ and satisfies the additional constraint that for every $a, a' \in A$, index set I, and family $(\, a_i \,)_{i \in I}$ of $a_i \in A$

$$\bigoplus_{i \in I} a a_i a' = a \Big(\bigoplus_{i \in I} a_i \Big) a' \ . \tag{3}$$

Let $\mathcal{B} = (B, +)$ be a commutative monoid, $\mathcal{A} = (A, \oplus, \odot)$ be a semiring, and $\cdot \colon A \times B \longrightarrow B$. Then $\mathcal{B}$ is called a *(left) $\mathcal{A}$-semimodule (via $\cdot$)*, if the conditions (i)-(iii) hold for all $a, a_1, a_2 \in A$ and all $b, b_1, b_2 \in B$.

(i) $a \cdot 0_{\mathcal{B}} = 0_{\mathcal{B}}$ and $1_{\mathcal{A}} \cdot b = b$.
(ii) $(a_1 \odot a_2) \cdot b = a_1 \cdot (a_2 \cdot b)$.
(iii) $a \cdot (b_1 + b_2) = a \cdot b_1 + a \cdot b_2$ and $(a_1 \oplus a_2) \cdot b = a_1 \cdot b + a_2 \cdot b$.

Given that $\mathcal{B}$ and $\mathcal{A}$ are complete, $\mathcal{B}$ is called a *complete $\mathcal{A}$-semimodule*, if for all index sets I and J, families $(\, a_i \,)_{i \in I}$ of $a_i \in A$, and families $(\, b_j \,)_{j \in J}$ of $b_j \in B$ additionally the following axiom holds.

$$\Big(\bigoplus_{i \in I} a_i \Big) \cdot \Big(\sum_{j \in J} b_j \Big) = \sum_{i \in I, j \in J} a_i \cdot b_j \tag{4}$$

Clearly each commutative monoid $\mathcal{B} = (B, +)$ is an $\mathbb{N}$-semimodule, where $\mathbb{N}$ is the semiring of non-negative integers $(\mathbb{N}, +, \cdot)$, using $\cdot \colon \mathbb{N} \times B \longrightarrow B$ defined as $n \cdot b = \sum_{i \in [n]} b$ for every $n \in \mathbb{N}$ and $b \in B$. Note that $\sum_{i \in [0]} b = 0_{\mathcal{B}}$. Similarly, every commutative and continuous monoid is a complete $\mathbb{N}_\infty$-semimodule (*see* [16]), where $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot)$. Furthermore, any idempotent and commutative monoid $\mathcal{B}$ is a $\mathbb{B}$-semimodule where $\mathbb{B} = (\{0, 1\}, \vee, \wedge)$ is the *Boolean semiring*, and $\mathcal{B}$ is a complete $\mathbb{B}$-semimodule, if $\mathcal{B}$ is completely idempotent (*see* [16]).

Let (D, Ω) be an Ω-algebra and $(D, +)$ be a commutative monoid with neutral element $0_{\mathcal{D}}$. Then we say that the algera $(D, +, \Omega)$ is a *distributive Ω-algebra* [11,16], if for every $k \in \mathbb{N}$, $\omega \in \Omega_k$, $i \in [k]$, and $d, d_1, \dots, d_k \in D$

(i) $\omega(d_1, \dots, d_{i-1}, 0_{\mathcal{D}}, d_{i+1}, \dots, d_k) = 0_{\mathcal{D}}$,
(ii) $\omega(d_1, \dots, d_{i-1}, d + d_i, d_{i+1}, \dots, d_k) = \omega(d_1, \dots, d, \dots, d_k) + \omega(d_1, \dots, d_k)$.

For convenience, we assume that there exists an operation $0_k \in \Omega_k$ for every $k \in \mathbb{N}$ such that $0_k(d_1, \dots, d_k) = 0_{\mathcal{D}}$ for all $d_1, \dots, d_k \in D$. We say that $\mathcal{D}$ is *complete*, whenever $(D,+)$ is complete and for all $k \in \mathbb{N}$, $\omega \in \Omega_k$, index sets $I_1, \dots, I_k$, and families $(\, d_i \,)_{i \in I_j}$ of $d_i \in D$ for every $j \in [k]$ the equality

$$\omega(\sum_{i_1 \in I_1} d_{i_1}, \dots, \sum_{i_k \in I_k} d_{i_k}) = \sum_{i_1 \in I_1} \cdots \sum_{i_k \in I_k} \omega(d_{i_1}, \dots, d_{i_k}) \tag{5}$$

is satisfied. Accordingly, $\mathcal{D}$ is *continuous*, if $\mathcal{D}$ is complete and $(D,+)$ is continuous. The distributive Ω-algebra $\mathcal{D}$ is an $\mathcal{A}$*-semimodule* for some commutative semiring $\mathcal{A} = (A, \oplus, \odot)$, if $(D,+)$ is an $\mathcal{A}$-semimodule and for every $k \in \mathbb{N}$, $\omega \in \Omega_k$, $a \in A$, $i \in [k]$, and $d_1, \dots, d_k \in D$ the equality

$$\omega(d_1, \dots, d_{i-1}, a \cdot d_i, d_{i+1}, \dots, d_k) = a \cdot \omega(d_1, \dots, d_k) \tag{6}$$

holds. Finally, $\mathcal{D}$ is a *complete* $\mathcal{A}$*-semimodule*, if both $\mathcal{A}$ and $\mathcal{D}$ are by itself complete and for index sets I and J and family $(\, a_i \,)_{i \in I}$ of $a_i \in A$ and $(\, d_j \,)_{j \in J}$ of $d_j \in D$ we have

$$\Big(\bigoplus_{i \in I} a_i\Big) \cdot \Big(\sum_{j \in J} d_j\Big) = \sum_{i \in I, j \in J} a_i \cdot d_j \ . \tag{7}$$

Clearly, every distributive Ω-algebra is an $\mathbb{N}$-semimodule.

2.3. Formal Power Series and Tree Series Substitution

Any mapping $\varphi \colon B \longrightarrow A$ into a commutative monoid $\mathcal{A} = (A, \oplus)$ is also called *(formal) power series*. The set of all power series is denoted by $A\langle\!\langle B \rangle\!\rangle$. We write (φ, b) instead of $\varphi(b)$ for every $b \in B$. The *sum* $\varphi_1 \oplus \varphi_2$ of $\varphi_1, \varphi_2 \in A\langle\!\langle B \rangle\!\rangle$ is defined pointwise by $(\varphi_1 \oplus \varphi_2, b) = (\varphi_1, b) \oplus (\varphi_2, b)$ for every $b \in B$. The *support* $\mathrm{supp}(\varphi)$ of φ is defined by $\mathrm{supp}(\varphi) = \{\, b \in B \mid (\varphi, b) \neq 0_{\mathcal{A}} \,\}$. If the support of φ is finite, then φ is said to be a *polynomial*. The polynomial with empty support is denoted by $\widetilde{0}_{\mathcal{A}}$.

If $B = T_\Omega$ for some ranked set Ω, then φ is also called *tree series*. Let $\mathcal{A} = (A, \oplus, \odot)$ be a complete semiring, and let $n \in \mathbb{N}$, $\mathrm{X}_n \subseteq \Omega_0$, $\varphi \in A\langle\!\langle T_\Omega \rangle\!\rangle$, and $\psi_1, \dots, \psi_n \in A\langle\!\langle T_\Omega \rangle\!\rangle$. We define the *IO tree series substitution* [6, 10] (for short: IO-substitution) of $(\psi_1, \dots, \psi_n)$ into φ, denoted by $\varphi \longleftarrow (\psi_1, \dots, \psi_n)$, as

$$\varphi \longleftarrow (\psi_1, \dots, \psi_n) = \bigoplus_{\substack{t \in T_\Omega, \\ t_1, \dots, t_n \in T_\Omega}} (\varphi, t)(\psi_1, t_1) \cdots (\psi_n, t_n)\, t[t_1, \dots, t_n] \ . \tag{8}$$

Note that irrespective of the number of occurrences of x_i the coefficient (ψ_i, t_i) is taken into account exactly once, even if x_i does not appear in t. Other notions of substitution, like o-IO-substitution [12], OI-substitution [17,6], and [OI]-substitution [7], have been defined for tree series as well, but in this paper we will exclusively deal with IO-substitution.

2.4. Weighted Tree Automata and Tree Series Transducers

An $(I \times J)$*-matrix* over a set S is a mapping $M \colon I \times J \longrightarrow S$. The (i,j)-entry with $i \in I$ and $j \in J$ of M is usually denoted by M_{ij} instead of $M(i,j)$. Let Ω be a ranked set and $\mathcal{A} = (A, \oplus)$ be a commutative monoid. Every family $\mu = (\mu_k)_{k \in \mathbb{N}}$ of $\mu_k \colon \Omega_k \longrightarrow A^{I \times I^k}$ is called *tree representation* over Ω, I, and $\mathcal{A}$. A *deterministic* tree representation additionally fulfills the restriction that for every $\omega \in \Omega_k$ and $i_1, \ldots, i_k \in I$ there exists at most one $i \in I$ such that $\mu_k(\omega)_{i,i_1 \cdots i_k} \neq 0_{\mathcal{A}}$.

A *weighted tree automaton* (wta) [4] is a system $M = (I, \Omega, \mathcal{A}, F, \mu)$ comprising of a nonempty set I of *states*, a ranked alphabet Ω, a semiring $\mathcal{A} = (A, \oplus, \odot)$, a vector $F \in A^I$ of *final weights*, and a tree representation μ over Ω, I, and $(A, \oplus)$. If I is infinite, then $\mathcal{A}$ must be complete, otherwise M is called *finite*. Moreover, M is *deterministic*, if μ is deterministic. Let $\vec{\mu} = (\vec{\mu}_k(\omega))_{k \in \mathbb{N}, \omega \in \Omega_k}$ where $\vec{\mu}_k(\omega) \colon (A^I)^k \longrightarrow A^I$ is defined componentwise for every $i \in I$ and $V_1, \ldots, V_k \in A^I$ by

$$\vec{\mu}_k(\omega)(V_1, \ldots, V_k)_i = \bigoplus_{i_1, \ldots, i_k \in I} \mu_k(\omega)_{i,i_1 \cdots i_k} \odot (V_1)_{i_1} \odot \cdots \odot (V_k)_{i_k} \ . \tag{9}$$

Let $h_\mu \colon T_\Omega \longrightarrow A^I$ be the unique homomorphism from $(T_\Omega, \Omega, I_\Omega)$ to $(A^I, \Omega, \vec{\mu})$. The tree series $\|M\| \in A\langle\langle T_\Omega \rangle\rangle$ *recognized by* M is $(\|M\|, t) = \bigoplus_{i \in I} F_i \odot h_\mu(t)_i$ for every $t \in T_\Omega$.

A *tree series transducer* (tst) [17,10] is a system $M = (I, \Omega, \Delta, \mathcal{A}, F, \mu)$ in which I is a nonempty set of *states*, Ω is a ranked alphabet of *input symbols*, Δ is a ranked alphabet of *output symbols* disjoint with X, $\mathcal{A} = (A, \oplus, \odot)$ is a semiring, $F \in A\langle\langle \widehat{T_\Delta}(\mathrm{X}_1) \rangle\rangle^I$ is a vector of *final outputs*, and μ is a tree representation over Ω, I, and $(A\langle\langle T_{\Delta \cup \mathrm{X}} \rangle\rangle, \oplus)$ such that $\mu_k(\omega) \in A\langle\langle T_{\Delta \cup \mathrm{X}_k} \rangle\rangle^{I \times I^k}$ for every $k \in \mathbb{N}$ and $\omega \in \Omega_k$. If I is finite and each tree series in the range of $\mu_k(\sigma)$ is a polynomial, then M is called *finite*, otherwise $\mathcal{A}$ must be complete. The tst M is *deterministic*, if μ is deterministic. Finite tst over the Boolean semiring $\mathbb{B}$ are also called *tree transducers* [13,14]. Let $\vec{\mu} = (\vec{\mu}_k(\omega))_{k \in \mathbb{N}, \omega \in \Omega_k}$ where $\vec{\mu}_k(\omega) \colon (A\langle\langle T_\Delta \rangle\rangle^I)^k \longrightarrow A\langle\langle T_\Delta \rangle\rangle^I$ is defined componentwise for every $i \in I$ and $V_1, \ldots, V_k \in A\langle\langle T_\Delta \rangle\rangle^I$ by

$$\vec{\mu}_k(\omega)(V_1, \ldots, V_k)_i = \bigoplus_{i_1, \ldots, i_k \in I} \mu_k(\omega)_{i,i_1 \cdots i_k} \longleftarrow ((V_1)_{i_1}, \ldots, (V_k)_{i_k}) \ . \tag{10}$$

Let $h_\mu \colon T_\Omega \longrightarrow A\langle\langle T_\Delta \rangle\rangle^I$ be the unique homomorphism from the algebra $(T_\Omega, \Omega, I_\Omega)$ to $(A\langle\langle T_\Delta \rangle\rangle^I, \Omega, \vec{\mu})$. The *tree-to-tree-series transformation* (t-ts transformation) $\|M\| \colon T_\Omega \longrightarrow A\langle\langle T_\Delta \rangle\rangle$ computed by M is $\|M\|(t) = \bigoplus_{i \in I} F_i \longleftarrow (h_\mu(t)_i)$ for every $t \in T_\Omega$.

3. Establishing the Relationship

Let $\mathcal{D} = (D, +, \Omega)$ be a distributive Ω-algebra. Inspired by the automaton definition of [16] we define $\mathcal{D}$-weighted tree automata ($\mathcal{D}$-wta). Roughly speaking, to each transition of a $\mathcal{D}$-wta an operation symbol of the Ω-algebra is associated and non-determinism is taken care of by the addition.

Definition 1 *Let $\mathcal{D} = (D, +, \Omega)$ be a distributive Ω-algebra. A* $\mathcal{D}$-weighted tree automaton *($\mathcal{D}$-wta) is a system $M = (I, \Sigma, \mathcal{D}, F, \mu)$, where*

- *I is a nonempty set of* states*,*
- *Σ is a ranked alphabet of* input symbols*,*

- $F \in \Omega_1^I$ *is the* final weight vector*, and*
- $\mu = (\mu_k)_{k \in \mathbb{N}}$ *is the* transition table *fulfilling* $\mu_k \colon \Sigma_k \longrightarrow \Omega_k^{I \times I^k}$.

If I is infinite, then $\mathcal{D}$ must be complete. Otherwise, M is called finite*. Finally, M is* deterministic*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $i_1, \ldots, i_k \in I$, there exists at most one $i \in I$ such that $\mu_k(\sigma)_{i,i_1 \cdots i_k} \neq 0_k$.*

Unless stated otherwise let $M = (I, \Sigma, \mathcal{D}, F, \mu)$ be a $\mathcal{D}$-wta over the distributive Ω-algebra $\mathcal{D} = (D, +, \Omega)$ and let $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $i \in I$, and $t = \sigma(t_1, \ldots, t_k) \in T_\Sigma$. Moreover, all function arguments range over their respective domains. Next we define two semantics, namely initial algebra semantics [15] and a semantics based on runs. In the latter the weight of a run is obtained by combining the weights obtained for the direct subtrees with the help of the operation symbol associated to the topmost transition. Nondeterminism is taken care of by adding the weights of all runs on a given input tree.

Definition 2 *Let $\vec{\mu} = (\vec{\mu}_k(\sigma))_{k \in \mathbb{N}, \sigma \in \Sigma_k}$ where $\vec{\mu}_k(\sigma) \colon (D^I)^k \longrightarrow D^I$ is defined componentwise for every $i \in I$ by*

$$\vec{\mu}_k(\sigma)(V_1, \ldots, V_k)_i = \sum_{i_1, \ldots, i_k \in I} \mu_k(\sigma)_{i,i_1 \cdots i_k}((V_1)_{i_1}, \ldots, (V_k)_{i_k}) \ . \tag{11}$$

Let $h_\mu \colon T_\Sigma \longrightarrow D^I$ be the unique homomorphism from $(T_\Sigma, \Sigma, I_\Sigma)$ to $(D^I, \Sigma, \vec{\mu})$. The tree series recognized by *M is defined as $(\|M\|, t) = \sum_{i \in I} F_i(h_\mu(t)_i)$.*

Definition 3 *A* run on *$t \in T_\Sigma$ is a mapping $r \colon \mathrm{dom}(t) \longrightarrow I$. The set of all runs on t is denoted by $R(t)$. The* weight of *r is defined by a mapping $\mathrm{wt}_r \colon \mathrm{dom}(t) \longrightarrow D$ which is defined for $w \in \mathrm{dom}(t)$ with $t(w) \in \Sigma_k$ by*

$$\mathrm{wt}_r(w) = \mu_k(t(w))_{r(w), r(w \cdot 1) \cdots r(w \cdot k)}(\mathrm{wt}_r(w \cdot 1), \ldots, \mathrm{wt}_r(w \cdot k)) \ . \tag{12}$$

The run-based semantics of *M is $(|M|, t) = \sum_{r \in R(t)} F_{r(\varepsilon)}(\mathrm{wt}_r(\varepsilon))$.*

The next proposition states that the initial algebra semantics coincides with the run-based semantics. Intuitively speaking, this reflects the property that nondeterminism can equivalently either be handled locally (initial algebra semantics) or globally (run-based semantics) due to the distributivity of $\mathcal{D}$.

Proposition 1 *For every $\mathcal{D}$-wta $M = (I, \Sigma, \mathcal{D}, F, \mu)$ we have $\|M\| = |M|$.*

Proof. We prove $h_\mu(t)_i = \sum_{r \in R(t), r(\varepsilon) = i} \mathrm{wt}_r(\varepsilon)$.

$$\begin{aligned}
& h_\mu(t)_i \\
= & \sum_{i_1, \ldots, i_k \in I} \mu_k(\sigma)_{i,i_1 \cdots i_k}(h_\mu(t_1)_{i_1}, \ldots, h_\mu(t_k)_{i_k}) \\
& \text{(by definition of } h_\mu\text{)} \\
= & \sum_{i_1, \ldots, i_k \in I} \mu_k(\sigma)_{i,i_1 \cdots i_k}\Big(\sum_{\substack{r_1 \in R(t_1), \\ r_1(\varepsilon) = i_1}} \mathrm{wt}_{r_1}(\varepsilon), \ldots, \sum_{\substack{r_k \in R(t_k), \\ r_k(\varepsilon) = i_k}} \mathrm{wt}_{r_k}(\varepsilon) \Big) \\
& \text{(by induction hypothesis)}
\end{aligned}$$

$$= \sum_{i_1,\ldots,i_k \in I} \sum_{\substack{r_1 \in R(t_1),\\ r_1(\varepsilon)=i_1}} \cdots \sum_{\substack{r_k \in R(t_k),\\ r_k(\varepsilon)=i_k}} \mu_k(\sigma)_{i,i_1\cdots i_k}(\mathrm{wt}_{r_1}(\varepsilon),\ldots,\mathrm{wt}_{r_k}(\varepsilon))$$

(by distributivity of $\mathcal{D}$)

$$= \sum_{r_1 \in R(t_1)} \cdots \sum_{r_k \in R(t_k)} \mu_k(\sigma)_{i,r_1(\varepsilon)\cdots r_k(\varepsilon)}(\mathrm{wt}_{r_1}(\varepsilon),\ldots,\mathrm{wt}_{r_k}(\varepsilon))$$

(by the definition of runs)

$$= \sum_{r \in R(t), r(\varepsilon)=i} \mathrm{wt}_r(\varepsilon) \tag{13}$$

(by definition of wt_r)

Finally, the statement is proved by

$$\begin{aligned} &(\|M\|,t) \\ &= \sum_{i\in I} F_i(h_\mu(t)_i) = \sum_{i \in I} F_i\Big(\sum_{r\in R(t), r(\varepsilon)=i} \mathrm{wt}_r(\varepsilon)\Big) \\ &\quad \text{(by definition of } \|M\| \text{ and (13))} \\ &= \sum_{i\in I} \sum_{r \in R(t), r(\varepsilon)=i} F_i(\mathrm{wt}_r(\varepsilon)) = \sum_{r\in R(t)} F_{r(\varepsilon)}(\mathrm{wt}_r(\varepsilon)) = (|M|,t) \qquad (14) \\ &\quad \text{(by distributivity of } \mathcal{D} \text{ and definition of } |M| \text{) .} \end{aligned}$$

□

In the next proposition we demonstrate the power of $\mathcal{D}$-wta. In fact, weighted tree automata and tree series transducers can be simulated by $\mathcal{D}$-wta. Since tst can easily simulate wta [10], let us only discuss tst. We use an Ω-algebra with all tree series of $\varphi \in A\langle\langle T_{\Delta\cup \mathrm{X}_k}\rangle\rangle$ as k-ary operation symbols, and the effect of applying φ to tree series $\psi_1,\ldots,\psi_k \in A\langle\langle T_\Delta\rangle\rangle$ is just the substitution $\varphi \longleftarrow (\psi_1,\ldots,\psi_k)$. In [10] it was shown that this Ω-algebra together with $\oplus$ as addition forms a distributive Ω-algebra.

Proposition 2 *Let M_1 be a wta and M_2 be a tst.*

(i) There exists a $\mathcal{D}'$-wta M_1' such that $\|M_1'\| = \|M_1\|$.
(ii) There exists a $\mathcal{D}$-wta M_2' such that $\|M_2'\| = \|M_2\|$.

Proof. Since it is clear (*see* [10]), how to simulate a wta with the help of a tst, we only show Statement (ii). Let $M_2 = (I_2,\Sigma,\Delta,\mathcal{A},F_2,\mu_2)$ be the considered tst. We set $\Omega_k = \{\, \underline{\varphi}_k \mid \varphi \in A\langle\langle T_{\Delta\cup\mathrm{X}_k}\rangle\rangle \,\}$ and let $\underline{\varphi}_k \colon A\langle\langle T_\Delta\rangle\rangle^k \longrightarrow A\langle\langle T_\Delta\rangle\rangle$ be defined as $\underline{\varphi}_k(\psi_1,\ldots,\psi_k) = \varphi \longleftarrow (\psi_1,\ldots,\psi_k)$. Then $\mathcal{D} = (A\langle\langle T_\Delta\rangle\rangle,\oplus,\Omega)$ is a distributive Ω-algebra [10], which is complete whenever $\mathcal{A}$ is [16]. Hence we let $M_2' = (I_2,\Sigma,\mathcal{D},F,\mu)$ with $F_i = \underline{F_2(i)}_1$, and for every $i,i_1,\ldots,i_k \in I_2$ we set $\mu_k(\sigma)_{i,i_1\cdots i_k} = \underline{(\mu_2)_k(\sigma)_{i,i_1\cdots i_k}}_k$. We firstly prove $h_\mu(t)_i = h_{\mu_2}(t)_i$.

$$\begin{aligned} &h_\mu(\sigma(t_1,\ldots,t_k))_i \\ &= \sum_{i_1,\ldots,i_k\in I_2} \mu_k(\sigma)_{i,i_1\cdots i_k}(h_\mu(t_1)_{i_1},\ldots,h_\mu(t_k)_{i_k}) \\ &\quad \text{(by definition of } h_\mu) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_1,\ldots,i_k \in I_2} \underline{(\mu_2)_k(\sigma)_{i,i_1\cdots i_k}}_k(h_{\mu_2}(t_1)_{i_1},\ldots,h_{\mu_2}(t_k)_{i_k}) \\
&\qquad \text{(by induction hypothesis and definition of } \mu_k(\sigma)_{i,i_1\cdots i_k}) \\
&= \sum_{i_1,\ldots,i_k \in I_2} (\mu_2)_k(\sigma)_{i,i_1\cdots i_k} \longleftarrow (h_{\mu_2}(t_1)_{i_1},\ldots,h_{\mu_2}(t_k)_{i_k}) \\
&\qquad \text{(by definition of } \underline{(\mu_2)_k(\sigma)_{i,i_1\cdots i_k}}_k) \\
&= h_{\mu_2}(\sigma(t_1,\ldots,t_k))_i \qquad (15) \\
&\qquad \text{(by definition of } h_{\mu_2})
\end{aligned}
$$

We conclude

$$
\begin{aligned}
&(\|M_2'\|, t) \\
&= \sum_{i \in I} F_i(h_\mu(t)_i) = \sum_{i \in I} \underline{F_2(i)}_1(h_{\mu_2}(t)_i) \\
&\qquad \text{(by definition of } \|M_2'\| \text{ and } F_i \text{ and (15))} \\
&= \sum_{i \in I} (F_2)_i \longleftarrow h_{\mu_2}(t)_i = (\|M_2\|, t) \qquad (16) \\
&\qquad \text{(by the definition of } \underline{F_2(i)}_1 \text{ and } h_{\mu_2}) \ . \qquad \square
\end{aligned}
$$

Note that M_2' is deterministic, whenever M_2 is. Next we attack the problem of constructing a semiring from the distributive Ω-algebra $\mathcal{D} = (D, +, \Omega)$. In the following ω ranges over Ω_k. We denote the set $\{\, \omega(\mathrm{x}_1,\ldots,\mathrm{x}_k) \mid k \in \mathbb{N}, \omega \in \Omega_k \,\}$ of shallow trees simply by $\Omega\mathrm{X}$. We can define a monoid which simulates the Ω-algebra (D, Ω) as follows. We use overlining as in $\overline{\omega}(\mathrm{x}_1,\ldots,\mathrm{x}_k)$ whenever we want to refer to the tree obtained by top-concatenation of the overlined symbol with its arguments.

Theorem 1 *For every Ω-algebra (D, Ω) we can construct a monoid $(B, \leftarrow)$ such that $D \cup \Omega\mathrm{X} \subseteq B$ and for all $d_1,\ldots,d_k \in D$*

$$\omega(d_1,\ldots,d_k) = \overline{\omega}(\mathrm{x}_1,\ldots,\mathrm{x}_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k \ . \qquad (17)$$

Proof. Assume that Ω and D are disjoint and let $\Omega' = \Omega \cup D$. Firstly, we define a mapping $h\colon T_{\Omega' \cup \mathrm{X}} \longrightarrow T_{\Omega' \cup \mathrm{X}}$ for every $v \in D \cup \mathrm{X}$ as follows.

$$h(v) = v \qquad (18)$$

$$h(\overline{\omega}(t_1,\ldots,t_k)) = \begin{cases} \omega(h(t_1),\ldots,h(t_k)) & \text{if } h(t_1),\ldots,h(t_k) \in D \ , \\ \overline{\omega}(h(t_1),\ldots,h(t_k)) & \text{otherwise} \end{cases} \qquad (19)$$

The mapping h evaluates terms in an inside-out fashion (*i. e.*, parameters of functions are evaluated before the function definition is applied). Note that $t \in \widehat{T_{\Omega'}}(\mathrm{X}_n)$ implies $h(t) \in \widehat{T_{\Omega'}}(\mathrm{X}_n)$. Secondly, let

$$B = D^* \cup \bigcup_{n \in \mathbb{N}_+} D^* \cdot \widehat{T_{\Omega'}}(\mathrm{X}_n) \ . \qquad (20)$$

Next we define the operation $\leftarrow : B^2 \longrightarrow B$ for every $m, n \in \mathbb{N}_+$, $w \in D^*$, $b \in B$, $t \in \widehat{T_{\Omega'}}(\mathrm{X}_m)$, and $t' \in D \cup \widehat{T_{\Omega'}}(\mathrm{X}_n)$ by

$$w \leftarrow b = w{\cdot}b \tag{21}$$
$$w{\cdot}t \leftarrow \varepsilon = w{\cdot}t \tag{22}$$
$$w{\cdot}t \leftarrow t'{\cdot}b = w{\cdot}(h(t\langle\!\langle t'\rangle\!\rangle)) \leftarrow b\ . \tag{23}$$

Clearly, ε acts as neutral element. An easy inductive proof (on the length of b) shows that $a{\cdot}u \leftarrow b = a{\cdot}(u \leftarrow b)$ for every $a \in D^*$, $u \in D \cup \bigcup_{n\in\mathbb{N}_+} \widehat{T_{\Omega'}}(\mathrm{X}_n)$, and $b \in B$. Let us denote this property by $(\star)$. Next we prove associativity (*i.e.*, $(b_1 \leftarrow b_2) \leftarrow b_3 = b_1 \leftarrow (b_2 \leftarrow b_3)$ for all $b_1, b_2, b_3 \in B$) by induction on the lengths of the participating words (precisely, the length of b_1, b_2, and b_3 in this order).

Induction base: The cases involving ε are trivial.

Induction step: We prove $(a{\cdot}s \leftarrow t{\cdot}b) \leftarrow u{\cdot}c = a{\cdot}s \leftarrow (t{\cdot}b \leftarrow u{\cdot}c)$ for every $a \in D^*$, $s, t, u \in D \cup \bigcup_{n\in\mathbb{N}_+} \widehat{T_{\Omega'}}(\mathrm{X}_n)$, and $b, c \in B$ where $b = \varepsilon$ if $t \notin D$ and $c = \varepsilon$ if $u \notin D$ by case analysis.

Case 1: Let $s \in D$. Then by repeatedly applying $(\star)$ we obtain

$$(a{\cdot}s \leftarrow t{\cdot}b) \leftarrow u{\cdot}c = a{\cdot}s{\cdot}t{\cdot}b \leftarrow u{\cdot}c = a{\cdot}s{\cdot}(t{\cdot}b \leftarrow u{\cdot}c) = a{\cdot}s \leftarrow (t{\cdot}b \leftarrow u{\cdot}c)\ . \tag{24}$$

Case 2: Let $s \notin D$ and $b \neq \varepsilon$. Then

$$\begin{aligned}
&(a{\cdot}s \leftarrow t{\cdot}b) \leftarrow u{\cdot}c = (a{\cdot}(s \leftarrow t) \leftarrow b) \leftarrow u{\cdot}c \\
&= a{\cdot}(s \leftarrow t) \leftarrow (b \leftarrow u{\cdot}c) = (a{\cdot}s \leftarrow t) \leftarrow (b \leftarrow u{\cdot}c) \\
&\qquad \text{(by induction hypothesis because } |a{\cdot}(s \leftarrow t)| = |a{\cdot}s| \text{ and } |b| < |t{\cdot}b|) \\
&= a{\cdot}s \leftarrow (t \leftarrow (b \leftarrow u{\cdot}c)) = a{\cdot}s \leftarrow t{\cdot}(b \leftarrow u{\cdot}c) = a{\cdot}s \leftarrow (t{\cdot}b \leftarrow u{\cdot}c) \\
&\qquad \text{(by induction hypothesis because } |b| < |t{\cdot}b|)\ .
\end{aligned} \tag{25}$$

Case 3: Let $s \notin D$, $b = \varepsilon$, and $h(s\langle\!\langle t\rangle\!\rangle) \in D$. Note that the last assumption yields $t \in D$. Then with the help of $(\star)$

$$(a{\cdot}s \leftarrow t) \leftarrow u{\cdot}c = a{\cdot}(s \leftarrow t) \leftarrow u{\cdot}c = a{\cdot}s \leftarrow t{\cdot}u{\cdot}c = a{\cdot}s \leftarrow (t \leftarrow u{\cdot}c)\ . \tag{26}$$

Case 4: Let $s \notin D$, $b = \varepsilon$, and $h(s\langle\!\langle t\rangle\!\rangle) \notin D$. Then

$$\begin{aligned}
&(a{\cdot}s \leftarrow t) \leftarrow u{\cdot}c = a{\cdot}h\big(h(s\langle\!\langle t\rangle\!\rangle)\langle\!\langle u\rangle\!\rangle\big) \leftarrow c = a{\cdot}h\big(s\langle\!\langle t\rangle\!\rangle\langle\!\langle u\rangle\!\rangle\big) \leftarrow c \\
&= a{\cdot}h\big(s\langle\!\langle t\langle\!\langle u\rangle\!\rangle\rangle\!\rangle\big) \leftarrow c = a{\cdot}h\big(s\langle\!\langle h(t\langle\!\langle u\rangle\!\rangle)\rangle\!\rangle\big) \leftarrow c = (a{\cdot}s \leftarrow h(t\langle\!\langle u\rangle\!\rangle)) \leftarrow c \\
&= a{\cdot}s \leftarrow (h(t\langle\!\langle u\rangle\!\rangle) \leftarrow c) = a{\cdot}s \leftarrow (t \leftarrow u{\cdot}c) \\
&\qquad \text{(by induction hypothesis because } |h(t\langle\!\langle u\rangle\!\rangle)| = |t| \text{ and } |c| < |u{\cdot}c|)\ .
\end{aligned} \tag{27}$$

Hence $(B, \leftarrow)$ is a monoid with neutral element ε. It remains to prove (17).

$$\begin{aligned}
\overline{\omega}(\mathrm{x}_1, \ldots, \mathrm{x}_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k &= h(\overline{\omega}(d_1, \mathrm{x}_2, \ldots, \mathrm{x}_k)) \leftarrow d_2 \leftarrow \cdots \leftarrow d_k \\
&= h(\overline{\omega}(d_1, \ldots, d_k)) = \omega(d_1, \ldots, d_k)\ .
\end{aligned} \tag{28}$$

□

Roughly speaking, one can understand $\leftarrow$ as function composition where the arguments are lambda-terms and the evaluation (which is done via h) is call-by-value. Next we would like to extend this monoid to a semiring by introducing the addition of the distributive Ω-algebra $\mathcal{D} = (D, +, \Omega)$. However, the addition should also be able to sum up terms, hence we first use an abstract addition coming from a semiring for which $\mathcal{D}$ is a complete semimodule.

Let $\mathcal{A} = (A, \oplus, \odot)$ be a semiring. We lift the operation $\leftarrow\colon B^2 \longrightarrow B$ to an operation $\leftarrow\colon A\langle\langle B\rangle\rangle^2 \longrightarrow A\langle\langle B\rangle\rangle$ by

$$\psi_1 \leftarrow \psi_2 = \bigoplus_{b_1, b_2 \in B} \big((\psi_1, b_1) \odot (\psi_2, b_2)\big)\, (b_1 \leftarrow b_2) \ . \tag{29}$$

Let $(D, +, \Omega)$ be a complete $\mathcal{A}$-semimodule via $\cdot\colon A \times D \longrightarrow D$. Then we define the *sum (summed in D) of* $\varphi \in A\langle\langle D\rangle\rangle$ by the mapping $\sum\colon A\langle\langle D\rangle\rangle \longrightarrow D$ with $\sum \varphi = \sum_{d \in D} (\varphi, d) \cdot d$. For a vector $V \in A\langle\langle D\rangle\rangle^I$ we let $(\sum V)_i = \sum V_i$. We identify $1_{\mathcal{A}}\ d$ with d.

Proposition 3 *Let $\mathcal{D}$ be a complete $\mathcal{A}$-semimodule where $\mathcal{A} = (A, \oplus, \odot)$ is a semiring. Furthermore, let $\varphi_1, \dots, \varphi_k \in A\langle\langle D\rangle\rangle$. Then*

(i) $\sum(\bigoplus_{i \in I} \varphi_i) = \sum_{i \in I} \sum \varphi_i$ for every family $(\,\varphi_i\,)_{i \in I}$ of $\varphi_i \in A\langle\langle D\rangle\rangle$ and
(ii) $\omega(\sum \varphi_1, \dots, \sum \varphi_k) = \sum\big(\overline{\omega}(x_1, \dots, x_k) \leftarrow \varphi_1 \leftarrow \cdots \leftarrow \varphi_k\big)$.

Proof. We prove the items separately.

(i) Let $(\,\varphi_i\,)_{i \in I}$ be a family of $\varphi_i \in A\langle\langle D\rangle\rangle$.

$$\begin{aligned} \sum\Big(\bigoplus_{i \in I} \varphi_i\Big) &= \sum_{d \in D} \Big(\bigoplus_{i \in I} \varphi_i, d\Big) \cdot d = \sum_{d \in D} \Big(\bigoplus_{i \in I} (\varphi_i, d)\Big) \cdot d \\ &= \sum_{d \in D} \Big(\sum_{i \in I} (\varphi_i, d) \cdot d\Big) = \sum_{i \in I} \Big(\sum_{d \in D} (\varphi_i, d) \cdot d\Big) = \sum_{i \in I} \sum \varphi_i \end{aligned} \tag{30}$$

(ii) Now we prove $\omega(\sum \varphi_1, \dots, \sum \varphi_k) = \sum\big(\overline{\omega}(x_1, \dots, x_k) \leftarrow \varphi_1 \leftarrow \cdots \leftarrow \varphi_k\big)$.

$$\begin{aligned} &\omega(\sum \varphi_1, \dots, \sum \varphi_k) \\ &= \omega\Big(\sum_{d_1 \in D} (\varphi_1, d_1) \cdot d_1, \dots, \sum_{d_k \in D} (\varphi_k, d_k) \cdot d_k\Big) \\ &= \sum_{d_1, \dots, d_k \in D} \omega((\varphi_1, d_1) \cdot d_1, \dots, (\varphi_k, d_k) \cdot d_k) \\ &= \sum_{d_1, \dots, d_k \in D} \Big(\bigodot_{j \in [k]} (\varphi_j, d_j)\Big) \cdot \omega(d_1, \dots, d_k) \\ &= \sum \bigoplus_{d_1, \dots, d_k \in D} \Big(\bigodot_{j \in [k]} (\varphi_j, d_j)\Big)\, \omega(d_1, \dots, d_k) \\ &= \sum \bigoplus_{d_1, \dots, d_k \in D} \Big(\bigodot_{j \in [k]} (\varphi_j, d_j)\Big)\, \overline{\omega}(x_1, \dots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k \\ &= \sum \overline{\omega}(x_1, \dots, x_k) \leftarrow \varphi_1 \leftarrow \cdots \leftarrow \varphi_k \end{aligned} \tag{31}$$

□

Thus combining Theorem 1 and Proposition 3, we can construct a semiring with the following properties. In fact, we use this semiring for our presentation results.

Theorem 2 *For every continuous $\mathcal{D}$ there exists a semiring $(C, \oplus, \leftarrow)$ such that $D \cup \Omega \mathrm{X} \subseteq C$ and for all $d_1, \ldots, d_k \in D$*

(i) $\omega(d_1, \ldots, d_k) = \overline{\omega}(\mathrm{x}_1, \ldots, \mathrm{x}_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k$ *and*
(ii) $\sum(\bigoplus_{i \in I} d_i) = \sum_{i \in I} d_i$.

Proof. Let $\mathcal{A} = (A, \oplus, \odot)$ be a semiring such that $\mathcal{D}$ is a complete $\mathcal{A}$-semimodule. For example, $\mathcal{A}$ can always be chosen to be $\mathbb{N}_\infty$. By Theorem 1 there exists a monoid $(B, \leftarrow)$ such that Statement (i) holds for elements of B. Consequently, let $C = A\langle\langle B \rangle\rangle$ and $\leftarrow : C^2 \longrightarrow C$ be the extension of $\leftarrow$ on B. Clearly, $(C, \oplus, \leftarrow)$ is a semiring and by Theorem 1 and Proposition 3 the Statements (i) and (ii) hold. $\square$

The semiring $(A\langle\langle B \rangle\rangle, \oplus, \leftarrow)$ constructed in Theorem 2 will be denoted by $G_{\mathcal{A}}(\mathcal{D})$ in the sequel. We note that $G_{\mathcal{A}}(\mathcal{D})$ is complete, because $\mathcal{A}$ is complete (*see* [16]). Hence we are ready to state our main representation theorem.

Theorem 3 *Let $M_1 = (I_1, \Sigma, \mathcal{D}, F_1, \mu_1)$ be a $\mathcal{D}$-wta and M_2 be a tst.*

- *There exists a wta $M_1' = (I_1, \Sigma, G_{\mathcal{A}}(\mathcal{D}), F, \mu)$ such that $\|M_1\| = \sum \|M_1'\|$.*
- *There exists a wta M_2' such that $\|M_2\| = \sum \|M_2'\|$.*

Proof. The second statement follows from the first and Proposition 2, so it remains to prove the first statement. Let $F_i = \overline{(F_1)_i}(x_1)$ and

$$\mu_k(\sigma)_{i,i_1 \cdots i_k} = \overline{(\mu_1)_k(\sigma)_{i,i_1 \cdots i_k}}(\mathrm{x}_1, \ldots, \mathrm{x}_k) \ . \tag{32}$$

We first prove $\sum h_\mu(t)_i = h_{\mu_1}(t)_i$ by induction.

$$\begin{aligned}
& h_{\mu_1}(\sigma(t_1, \ldots, t_k))_i \\
= & \sum_{i_1, \ldots, i_k \in I_1} (\mu_1)_k(\sigma)_{i,i_1 \cdots i_k}(h_{\mu_1}(t_1)_{i_1}, \ldots, h_{\mu_1}(t_k)_{i_k}) \\
& \quad \text{(by definition of } h_{\mu_1}\text{)} \\
= & \sum_{i_1, \ldots, i_k \in I_1} (\mu_1)_k(\sigma)_{i,i_1 \cdots i_k} \big(\sum h_\mu(t_1)_{i_1}, \ldots, \sum h_\mu(t_k)_{i_k}\big) \\
& \quad \text{(by induction hypothesis)} \\
= & \sum_{i_1, \ldots, i_k \in I_1} \sum \overline{(\mu_1)_k(\sigma)_{i,i_1 \cdots i_k}}(\mathrm{x}_1, \ldots, \mathrm{x}_k) \leftarrow h_\mu(t_1)_{i_1} \leftarrow \cdots \leftarrow h_\mu(t_k)_{i_k} \\
& \quad \text{(by Proposition 3(ii))} \\
= & \sum \bigoplus_{i_1, \ldots, i_k \in I_1} \overline{(\mu_1)_k(\sigma)_{i,i_1 \cdots i_k}}(\mathrm{x}_1, \ldots, \mathrm{x}_k) \leftarrow h_\mu(t_1)_{i_1} \leftarrow \cdots \leftarrow h_\mu(t_k)_{i_k} \\
& \quad \text{(by Proposition 3(i))} \\
= & \sum \bigoplus_{i_1, \ldots, i_k \in I_1} \mu_k(\sigma)_{i,i_1 \cdots i_k} \leftarrow h_\mu(t_1)_{i_1} \leftarrow \cdots \leftarrow h_\mu(t_k)_{i_k} \\
& \quad \text{(by definition of } \mu_k(\sigma)_{i,i_1 \cdots i_k}\text{)} \\
= & \sum h_\mu(\sigma(t_1, \ldots, t_k))_i
\end{aligned} \tag{33}$$

Using this property we can easily show

$$
\begin{aligned}
&(\|M_1\|, t) \\
&= \sum_{i \in I_1} (F_1)_i (h_{\mu_1}(t)_i) = \sum_{i \in I_1} (F_1)_i \big(\sum h_\mu(t)_i\big) \\
&\quad \text{(by definition of } \|M_1\| \text{ and (33))} \\
&= \sum_{i \in I_1} \sum \overline{(F_1)_i}(\mathrm{x}_1) \leftarrow h_\mu(t)_i = \sum \bigoplus_{i \in I_1} F_i \leftarrow h_\mu(t)_i = \sum (\|M_1'\|, t) \qquad (34) \\
&\quad \text{(by Proposition 3 and definition of } F_i \text{ and } \|M_1'\|) \ . \qquad \square
\end{aligned}
$$

Note that again M_1' can be chosen to be deterministic, whenever M_1 is deterministic. The main reason for the remaining summation is the fact that we do not know how to define sums like $\overline{\omega}(\mathrm{x}_1, \ldots, \mathrm{x}_k) + \overline{\omega}'(\mathrm{x}_1, \ldots, \mathrm{x}_k)$ for $\omega, \omega' \in \Omega_k$. Hence, we finally consider deterministic $\mathcal{D}$-wta, because there the summation is irrelevant.

Corollary 1 *Let M be a deterministic $\mathcal{D}$-wta. Then there exists a wta M' such that $\|M\| = \|M'\|$.*

Proof. Let $\mathcal{B} = (B, \leftarrow)$ be the monoid constructed from $\mathcal{D}$ in Theorem 1. We embed $\mathcal{B}$ into the following semiring $\mathcal{A} = (A, \oplus, \odot)$. Let $A = B \cup \{\perp\}$ for some $\perp \notin B$ and define $\oplus, \odot \colon A \times A \longrightarrow A$ for every $a_1, a_2 \in A$ by

$$
a_1 \oplus a_2 = \begin{cases} \perp & \text{if } a_1, a_2 \in B \ , \\ a_1 & \text{if } a_2 = \perp \ , \\ a_2 & \text{otherwise} \ , \end{cases} \quad \text{and} \quad a_1 \odot a_2 = \begin{cases} a_1 \leftarrow a_2 & \text{if } a_1, a_2 \in B \ , \\ \perp & \text{otherwise} \ . \end{cases} \tag{35}
$$

Then $\mathcal{A}$ is a semiring with $\perp$ and ε as additive and multiplicative neutral element, respectively. Let $M = (I, \Sigma, \mathcal{D}, F, \mu)$ be the deterministic $\mathcal{D}$-wta. We construct $M' = (I, \Sigma, \mathcal{A}, F', \mu')$ as follows. Let $F_i' = \overline{F_i}(\mathrm{x}_1)$ whenever $F_i \neq 0_1$ and $F_i' = \perp$ otherwise. Moreover, let

$$
\mu_k'(\sigma)_{i, i_1 \cdots i_k} = \begin{cases} \overline{\mu_k(\sigma)_{i, i_1 \cdots i_k}}(\mathrm{x}_1, \ldots, \mathrm{x}_k) & \text{if } \mu_k(\sigma)_{i, i_1 \cdots i_k} \neq 0_k \ , \\ \perp & \text{otherwise} \ . \end{cases} \tag{36}
$$

We can then easily prove $h_{\mu'}(t)_i = h_\mu(t)_i$ and $\|M'\| = \|M\|$ by induction. $\square$

In [19] we have shown that for tst over completely idempotent semirings we can refine the semiring and obtain a wta which computes exactly the semantics of the tst. This allowed us to conclude that all tree transducers (which are finite tst over the Boolean semiring) can be simulated by wta. Thus we successfully generalized the classical result which shows that generalized sequential machines are weighted automata over the semiring $(\mathfrak{P}(\Sigma^*), \cup, \circ)$.

4. Decidability of the Emptiness Problem

In [19] we showed how to apply the obtained characterization of tst and $\mathcal{D}$-wta to a pumping lemma [3] for deterministic wta. From that we obtained a pumping lemma for deterministic $\mathcal{D}$-wta, and using this pumping lemma we can derive decidability of the emptiness problem for deterministic $\mathcal{D}$-wta. However, we generalize this result and obtain decidability of the emptiness problem for $\mathcal{D}$-wta, which are not necessarily deterministic.

In this section, let $\mathcal{A} = (A, \oplus, \odot)$ be a semiring and $\mathcal{D} = (D, +, \Omega)$ be a distributive Ω-algebra. Finally, let $\mathcal{S}_\Sigma(\mathcal{D})$ (respectively, $\mathcal{S}^{\mathrm{d}}_\Sigma(\mathcal{D})$) be the class of recognizable (by a $\mathcal{D}$-wta) tree series (respectively, deterministically recognizable tree series); *i. e.*, for every $S \in \mathcal{S}_\Sigma(\mathcal{D})$ (respectively, $S \in \mathcal{S}^{\mathrm{d}}_\Sigma(\mathcal{D})$) there exists a finite (respectively, determinisitic finite) $\mathcal{D}$-wta $M = (Q, \Sigma, \mathcal{D}, F, \mu)$ such that $S = \|M\|$.

Theorem 4 (Theorem 13 in [19]) *Let $S \in \mathcal{S}^{\mathrm{d}}_\Sigma(\mathcal{D})$ and let $\Omega' = \Omega \cup D$. There exists an $m \in \mathbb{N}$ such that for every $t \in \mathrm{supp}(S)$ with $\mathrm{height}(t) \geqslant m+1$ there exist $C, C' \in \widehat{T_\Sigma}(\mathrm{X}_1)$, $t' \in T_\Sigma$, and $a, a' \in \widehat{T_{\Omega'}}(\mathrm{X}_1)$, and $d \in D$ such that*

- $t = C'[C[t']]$,
- $\mathrm{height}(C[t']) \leqslant m+1$ *and* $C \neq \mathrm{x}_1$, *and*
- $(S, C'[C^n[t']]) = a' \leftarrow a^n \leftarrow d$ *for every* $n \in \mathbb{N}$.

It follows from the proof of Corollary 5.8 of [3] that m can be chosen to be the number of states of a deterministic $\mathcal{D}$-wta recognizing S. Let $S \in \mathcal{S}^{\mathrm{d}}_\Sigma(\mathcal{D})$ and $M = (Q, \Sigma, \mathcal{D}, F, \mu)$ be a deterministic $\mathcal{D}$-wta recognizing S with $m = \mathrm{card}(Q)$. The pumping lemma suggests that $S = \widetilde{0}_\mathcal{D}$ if and only if $\sum_{t \in T_\Sigma, \mathrm{height}(t) \leqslant m} (S,t)\, t = \widetilde{0}_\mathcal{D}$. In the following proposition we prove this fact for mildly restricted distributive Ω-algebras $\mathcal{D}$ and (not necessarily deterministic) $\mathcal{D}$-wta M.

First let us define the restrictions. We say that $\mathcal{D}$ is *zero-sum free*, if $d_1 + d_2 = 0_\mathcal{D}$ implies $d_1 = 0_\mathcal{D} = d_2$ for all $d_1, d_2 \in D$. Note that whenever $\mathcal{D}$ is complete, then it is also zero-sum free. In [3] commutativity of the semiring is required to show decidability of the emptiness problem. The next notion presents a suitable restriction for Ω-algebras. In fact, it is weaker than commutativity and covers only the essence needed to make the proof of [3] work for semirings. We say that $\mathcal{D}$ has *Property (P)*, if for every $\varrho_1, \varrho_2 \in \widehat{T_\Omega}(\mathrm{X}_1)$ and $d \in D$ the fact $\varrho_1^I(\varrho_2^I(d)) \neq 0_\mathcal{D}$ implies $\varrho_1^I(d) \neq 0_\mathcal{D}$.

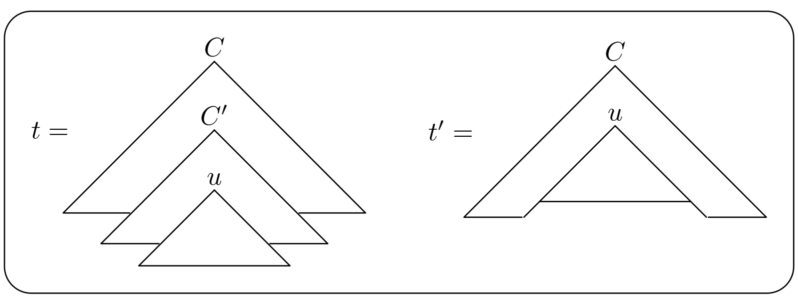

Fig. 2. The decomposition employed in Proposition 4.

Proposition 4 *Let $\mathcal{D}$ be zero-sum free distributive Ω-algebra with Property (P), and let $M = (Q, \Sigma, \mathcal{D}, F, \mu)$ be a finite $\mathcal{D}$-wta. Then*

$$\widetilde{0}_\mathcal{D} = \|M\| \quad \Longleftrightarrow \quad \widetilde{0}_\mathcal{D} = \sum_{t \in T_\Sigma, \mathrm{height}(t) \leqslant \mathrm{card}(Q)} (\|M\|, t)\, t \ . \tag{37}$$

Proof. The direction $\Rightarrow$ is clear, so it remains to prove the direction $\Leftarrow$. The proof is by contradiction; hence assume that the right hand side of (37) holds while $\|M\| \neq \widetilde{0}_{\mathcal{D}}$. Consequently, there exists a minimal (with respect to the size) $t \in T_\Sigma$ such that $(\|M\|, t) \neq 0_{\mathcal{D}}$. From the assumption it is immediate that $\mathrm{height}(t) > \mathrm{card}(Q)$. Moreover, by Proposition 1 we know that $\|M\| = |M|$, which allows us to concentrate on runs in the remaining proof.

Since $(|M|, t) = \sum_{r \in R(t)} F_{r(\varepsilon)}(\mathrm{wt}_r(\varepsilon))$, there exists a run $r \in R(t)$ such that $F_{r(\varepsilon)}(\mathrm{wt}_r(\varepsilon)) \neq 0_{\mathcal{D}}$. Due to the fact that $\mathrm{height}(t) > \mathrm{card}(Q)$ there exist $n \in \mathbb{N}$ and $w_1 \cdots w_n \in \mathrm{dom}(t)$ such that $n \geqslant \mathrm{card}(Q)$. By the pigeon-hole principle there exist $j_1, j_2 \in [0, n]$ such that $j_1 < j_2$ and $r(w_1 \cdots w_{j_1}) = r(w_1 \cdots w_{j_2})$. Now consider the trees $C, C' \in \widehat{T_\Sigma}(\mathrm{X}_1)$ and $u \in T_\Sigma$ defined by

$$C(w) = \begin{cases} t(w) & \text{if } w_1 \cdots w_{j_1} \text{ is not a prefix of } w \ , \\ \mathrm{x}_1 & \text{if } w = w_1 \cdots w_{j_1} \ , \\ \text{undefined} & \text{otherwise} \ , \end{cases} \tag{38}$$

$$C'(w) = \begin{cases} t(w_1 \cdots w_{j_1} \cdot w) & \text{if } w_{j_1+1} \cdots w_{j_2} \text{ is not a prefix of } w \ , \\ \mathrm{x}_1 & \text{if } w = w_{j_1+1} \cdots w_{j_2} \ , \\ \text{undefined} & \text{otherwise} \ , \end{cases} \tag{39}$$

and $u(w) = t(w_1 \cdots w_{j_2} \cdot w)$ for every $w \in \mathbb{N}_+^*$. Clearly, $t = C[C'[u]]$, and we let $t' = C[u]$. It is easily checked that there is an $r' \in R(t')$ defined by

$$r'(w) = \begin{cases} r(w') & \text{if } w = w_1 \cdots w_{j_1} \cdot w' \text{ for some } w' \in \mathbb{N}_+^* \ , \\ r(w) & \text{otherwise} \ . \end{cases} \tag{40}$$

Let $\mathrm{eval}\colon T_{\Sigma \cup \mathrm{X}} \times \mathrm{dom}(t) \longrightarrow T_{\Omega \cup \mathrm{X}}$ be defined for every $w \in \mathrm{dom}(t)$, $x \in \mathrm{X}$, $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $C_1, \ldots, C_k \in T_{\Sigma \cup \mathrm{X}}$ by

$$\mathrm{eval}(x, w) = x \tag{41}$$

$$\mathrm{eval}(\sigma(C_1, \ldots, C_k), w) = \overline{\mu_k(\sigma)_{r(w), r(w\cdot 1) \cdots r(w \cdot k)}}(\mathrm{eval}(C_1, w\cdot 1), \ldots, \mathrm{eval}(C_k, w\cdot k)) \ . \tag{42}$$

Hence the trees C, C' induce trees $\varrho, \varrho' \in \widehat{T_\Omega}(\mathrm{X}_1)$ by setting $\varrho = \mathrm{eval}(C, \varepsilon)$ and $\varrho' = \mathrm{eval}(C', w_1 \cdots w_{j_1})$. Finally, we let $d = \mathrm{eval}(t, w_1 \cdots w_{j_2})$. Now we observe $\mathrm{wt}_r(\varepsilon) = \varrho(\varrho'(d))$ and $\mathrm{wt}_{r'}(\varepsilon) = \varrho(d)$. Due to the fact that $r(\varepsilon) = r'(\varepsilon)$ we conclude that $F_{r'(\varepsilon)}(\mathrm{wt}_{r'}(\varepsilon)) = F_{r(\varepsilon)}(\varrho(d)) \neq 0_{\mathcal{D}}$ using Property (P). By zero-sum freeness this allows us to derive $(|M|, t') \neq 0_{\mathcal{D}}$. However, $\mathrm{size}(t') < \mathrm{size}(t)$ and $\mathrm{card}(Q) < \mathrm{height}(t')$ because $(|M|, t') \neq 0_{\mathcal{D}}$. This constitutes a contradiction because t was chosen minimal with the property that $(\|M\|, t) \neq 0_{\mathcal{D}}$. Hence there is no minimal tree t such that $(\|M\|, t) \neq 0_{\mathcal{D}}$; and thus $\|M\| = \widetilde{0}_{\mathcal{D}}$. □

The previous proposition essentially shows that the emptiness problem for finite $\mathcal{D}$-wta M is decidable, because in order to show that the recognized tree series is empty (*i. e.*, $\|M\| = \widetilde{0}_{\mathcal{D}}$), we only have to check finitely many small trees (*i. e.*, trees whose height is at most the number of states of M). The next theorem makes the required assumptions explicit. It turns out that only the operations (in fact, the operations that occur in the transition table of M are sufficient) need to be recursive. By that we mean that there exists a suitable coding of the elements of D and for each $k \in \mathbb{N}$ and $\omega \in \Omega_k$ there exists a TURING machine, which presented the coding of $d_1, \ldots, d_k \in D$ on k input tapes eventually stops with the coding of $\omega(d_1, \ldots, d_k)$ on the output tape. Moreover, we assume that the input tree series $S \in \mathcal{S}_\Sigma(\mathcal{D})$ is finitely presented. For the sake of this paper, we assume that we are given a finite $\mathcal{D}$-wta recognizing S.

Theorem 5 *Let $\mathcal{D}$ be a zero-sum free distributive Ω-algebra with Property (P). Moreover, let the operations of Ω be recursive functions and $S \in \mathcal{S}_\Sigma(\mathcal{D})$. Then emptiness of S (i. e., $S = \widetilde{0}_\mathcal{D}$) is decidable.*

Proof. Let $M = (Q, \Sigma, \mathcal{D}, F, \mu)$ be a finite $\mathcal{D}$-wta recognizing S. By Proposition 4 it suffices to decide $\widetilde{0}_\mathcal{D} = \sum_{t \in T_\Sigma, \mathrm{height}(t) \leqslant \mathrm{card}(Q)} (\|M\|, t)\, t$. By Proposition 1 we have $\|M\| = |M|$, so we again use the run-based semantics for the remainder of the proof. Since there are only finitely many trees $t \in T_\Sigma$ with a height at most $\mathrm{card}(Q)$ and at most finitely many runs on t, we have to check for finitely many runs $r \in R(t)$ whether $F_{r(\varepsilon)}(\mathrm{wt}_r(\varepsilon)) \neq 0_\mathcal{D}$. Since the operations of Ω are recursive, we can decide this inequality. With the help of zero-sum freeness we can conclude that the existence of one such run ensures $\|M\| \neq 0_\mathcal{D}$. On the other hand, if there is no such run, then clearly $\|M\| = \widetilde{0}_\mathcal{D}$. □

Note that zero-sum freeness is sufficient for the proof; in particular, the addition of $\mathcal{D}$ need not be recursive. Let us instantiate the last theorem to tst and wta. A semiring $\mathcal{A} = (A, \oplus, \odot)$ is said to be *zero-sum free*, if for every $a_1, a_2 \in A$ the fact $a_1 \oplus a_2 = 0_\mathcal{A}$ implies $a_1 = 0_\mathcal{A} = a_2$. Recall the distributive Ω-algebra $\mathcal{D} = (D, +, \Omega)$ used in Proposition 2 to simulate a tst $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$.

Corollary 2 *Let $\mathcal{A} = (A, \oplus, \odot)$ be a commutative and zero-sum free semiring such that $\odot$ and $\oplus$ are recursive and $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a finite tst. Then it is decidable whether $\|M\|(t) = \widetilde{0}_\mathcal{A}$ for every $t \in T_\Sigma$.*

Proof. According to Proposition 2 we have seen that there exists a distributive Ω-algebra $\mathcal{D} = (A, \oplus, \Omega)$ and a Ω-wta M' such that $\|M\| = \|M'\|$. Recall that $\Omega_k = \{\, \underline{\varphi}_k \mid \varphi \in A\langle\langle T_{\Delta \cup X_k} \rangle\rangle \,\}$ with $\underline{\varphi}_k(\psi_1, \ldots, \psi_k) = \varphi \longleftarrow (\psi_1, \ldots, \psi_k)$. Clearly, the substitution operation is recursive, if all participating tree series are polynomial. In [10] it is shown that finite tst compute on polynomial tree series solely.

Moreover, we note that $\mathcal{D}$ is zero-sum free, where the zero of $\mathcal{D}$ is $\widetilde{0}_\mathcal{A}$. Due to commutativity of $\mathcal{A}$, also Property (P) holds for $\mathcal{D}$. Hence we can apply Theorem 5 to decide emptiness of $\|M'\|$ and thereby emptiness of $\|M\|$. □

Actually, by refining the proof one can even show that $\oplus$ need not be recursive. Note that every idempotent semiring is zero-sum free as well as every naturally ordered or complete semiring. Consequently, the above corollary applies to a very large class of semirings.

Corollary 3 *Let $\mathcal{A} = (A, \oplus, \odot)$ be a commutative and zero-sum free semiring such that $\odot$ is recursive and $M = (Q, \Sigma, \mathcal{A}, F, \mu)$ be a finite wta. Then it is decidable whether $\|M\| = \widetilde{0}_{\mathcal{A}}$.*

Proof. The proof is analogous to the proof of Corollary 2. $\square$

The statement in [2], which is similar to a pumping lemma, holds for wta over fields. Naturally, no interesting field is zero-sum free, so our results cannot be applied. This leaves a gap to be explored. Identifying the necessary conditions for the construction of [2] would have a two-fold benefit. Firstly, we may be able to close the aforementioned gap, and secondly we may be able to generalize the statement to tst or $\mathcal{D}$-wta.

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