HASSE diagrams for classes of deterministic bottom-up tree-to-tree-series transformations

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Abstract

The relationship between classes of tree-to-tree-series and o-tree-to-tree-series transformations, which are computed by restricted deterministic bottom-up weighted tree transducers, is investigated. Essentially, these transducers are deterministic bottom-up tree series transducers, except that the former are defined over monoids whereas the latter are defined over semirings and only use the multiplicative monoid thereof. In particular, the common restrictions of nondeletion, linearity, totality, and homomorphism can equivalently be defined for deterministic bottom-up weighted tree transducers.

Using well-known results of classical tree transducer theory and also new results on deterministic weighted tree transducers, classes of tree-to-tree-series and o-tree-to-tree-series transformations computed by restricted deterministic bottom-up weighted tree transducers are ordered by set inclusion. More precisely, for every commutative monoid and all sensible combinations of the above mentioned restrictions, the inclusion relation of the classes of tree-to-tree-series and o-tree-to-tree-series transformations is completely conveyed by means of HASSE diagrams.

Keywords: Tree Transducer, Semiring, Tree Series, Hasse Diagram

1. Introduction

Bottom-up tree series transducers [14, 21, 31, 20] were introduced as a generalization of bottom-up tree transducers [35, 38, 12] and bottom-up weighted tree automata [37, 29, 6, 5].

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The latter have been applied to code selection in compilers [18, 4] and tree pattern matching [36]. Moreover, a rich theory of bottom-up tree transducers was developed (cf. [12, 1, 13, 23, 24, 34, 8] as seminal or survey papers and monographs) during the seventies, whereas bottom-up weighted tree automata just recently received more attention (e.g., [36, 37, 29, 6, 5, 3, 10, 11, 17]).

In [14, 21, 22, 20] several generalizations of well-known theorems of the theory of tree transducers have been proved for bottom-up tree series transducers, e.g.,

- the generalization of the decomposition of the class of bottom-up tree transformations (cf. Theorem 5.7 of [14] and page 220 of [12]); in its turn the result of [12] generalizes the decomposition of gsm-mappings as proved in [33];
- the generalization of (some) composition hierarchy results for bottom-up tree transformation classes (cf. Theorem 6.24 of [20] and Corollary 8.13(iii) of [23]);
- the generalization of the equivalence of a rewrite semantics and the initial algebra semantics for bottom-up tree transducers (cf. Theorem 5.10 of [22] and Lemma 5.6 of [12]).

Roughly speaking, a bottom-up tree series transducer is a bottom-up tree transducer in which the transitions carry a weight; the weight is an element of some semiring. The rewrite semantics works as follows. Suppose that the transducer has processed all direct subtrees of some input tree, i.e., it (nondeterministically) computed output trees and their corresponding weights. Then, according to the states in which the computation of the output trees ended, it selects a tree and corresponding weight from its transition table. The selected tree and the output trees are combined with the help of substitution and the weights are combined by means of the semiring multiplication. If for some pair of input and output trees there is more than one computation ending in a final state, then the weights of these computations are combined by means of the semiring addition.

In this paper, we deal with deterministic bottom-up tree series transducers. In this case, for every input tree there is at most one successful computation (cf. Proposition 3.12 of [14]), i.e., at most one computed output tree and its corresponding weight. Thus the semiring addition is irrelevant and we base our investigations on so-called deterministic bottom-up weighted tree transducers (for short: deterministic bu-w-tt) over some multiplicative monoid. Essentially, these are deterministic bottom-up tree series transducers over some semiring, of which only the multiplicative part is used.

Specifically, we deal with two modes of tree series substitution. The first is called pure tree series substitution [7, 14] (for short: pure substitution) and represents a computational approach, i.e., the output trees represent values of computations, and the weight associated to an output tree can be viewed as the cost of computing this value. When combining output trees, their weights are simply multiplied to obtain the weight of the combined output tree. This is irrespective of the number of uses of an output tree, i.e., an output may be copied without penalty, which represents the computational approach in the sense that a value is available and can be reused without recomputation. On the other hand, we also investigate tree series substitution respecting occurrences [21] (for short: o-substitution), which represents a more material approach. There the weights of the output trees are taken to the n-th power, if the corresponding output tree is used in n copies. In this approach, an output tree stands for a composite, and the weight of an output tree reflects the (monetary) cost of creating or obtaining this particular composite. When combining composites into a new composite, its cost is obtained by multiplying the costs of its components; each component taken as often as needed to assemble the composite.

In the same way as for deterministic bottom-up tree transducers or deterministic bottom-up tree series transducers, we can also define restrictions for deterministic bu-w-tt, e.g., the restrictions of nondeletion, linearity, totality, and homomorphism (cf., e.g., [12]). The class of tree-to-tree-series transformations, which is computed by deterministic bu-w-tt obeying the restrictions π (e.g., being a nondeleting homomorphism) over the monoid $\mathcal{A}$, is denoted by π–$\mathrm{BOT}^{\mathrm{mod}}(\mathcal{A})$ where mod is either ε (the empty word) or o. In the former case, the semantics is defined using pure substitution, whereas o-substitution is used in the latter case. We abbreviate each restriction by its first letter, e.g., h abbreviates homomorphism, and use juxtaposition of the letters to denote a combination of restrictions, e.g., hn for nondeleting homomorphism.

The monoids $(A, \odot, \mathbf{1})$ we employ have an absorbing element $\mathbf{0} \in A$ and are denoted by $(A, \odot, \mathbf{1}, \mathbf{0})$. Our main results are present in the Hasse diagrams contained in Section 4 (cf. Theorem 4.8, Theorem 4.16, Theorem 4.19, Theorem 4.23, and Theorem 4.25). Specifically, we conclude that

- the monoids $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are (up to isomorphism) the only monoids $\mathcal{A}$ such that, for every combination π of restrictions, π–$\mathrm{BOT}^{o}(\mathcal{A}) = \pi$–$\mathrm{BOT}(\mathcal{A})$ holds (cf. Corollary 4.6), and
- only in idempotent monoids $\mathcal{A}$ the equality hn–$\mathrm{BOT}^{o}(\mathcal{A}) =$ hn–$\mathrm{BOT}(\mathcal{A})$ holds (cf. Corollary 4.22).

Let us discuss the first item in some detail. It is rather clear that for $\mathbb{Z}_1$ and $\mathbb{Z}_2$ pure and o-substitution coincide, and for all other monoids $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ there is at least one element a different from both $\mathbf{0}$ and $\mathbf{1}$. Consider an output tree weighted a and another one weighted $\mathbf{1}$. The property, which separates pure and o-substitution in this case, is that pure substitution may tell those two different output trees apart even when deleting them. This is due to the fact that, when using pure substitution, the weight of the deleted output tree is still accounted, which is not the case for o-substitution.

Considering the second item, it is again straightforward to observe the equality, because $a^n = a$ for all elements a of the idempotent monoid and $n \geq 1$. In a non-idempotent monoid the property $a \neq a^2$ can be used to separate pure and o-substitution with the help of a copying homomorphism bu-w-tt. Therefore, imagine an output tree with weight a. If this output is used in a transition which copies it, then pure substitution accounts a just once while o-substitution accounts a twice.

In the following let us consider combinations π of restrictions which do not contain the homomorphism restriction. It turns out that

- π–$\mathrm{BOT}^{o}(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}(\mathcal{A})$ for every periodic and commutative monoid $\mathcal{A}$, whenever the nondeletion restriction is present in π (cf. Lemma 4.12),
- π–$\mathrm{BOT}(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}^{o}(\mathcal{A})$ for every periodic and commutative monoid $\mathcal{A}$, whenever the linearity restriction is present in π (cf. Lemma 4.12),
- π–$\mathrm{BOT}(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}^{o}(\mathcal{A})$ for every periodic, commutative, and regular monoid $\mathcal{A}$ (cf. Lemma 4.17), and
- π–$\mathrm{BOT}^{o}(\mathcal{A}) = \pi$–$\mathrm{BOT}(\mathcal{A})$ for every periodic and commutative group $\mathcal{A}$ (cf. Lemma 4.24).

All four results build on the properties of periodicity and commutativity, of which the former allows us to keep track of the weights in the states (because there are only finitely many different powers of any element), and the latter allows us to reorder the factors. Furthermore, the results mentioned above do not hold for π containing the homomorphism restriction, because of the additional states required for the book-keeping.

In the situation encountered in the first item, the weight a of an output tree is taken to the n-th power by means of o-substitution where $n \geq 1$. Pure substitution does account for the weight a of the output tree exactly once, but the remaining a^{n-1} can be remembered in the state and applied to the transition weight. The nondeletion property is necessary, because otherwise a might be raised to the 0-th power by o-substitution, thereby essentially neglecting a. However, pure substitution again accounts a once, and in general, it is not possible to "divide" by a. Given a group, the mentioned division is possible, which is explains why π–$\mathrm{BOT}^o(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}(\mathcal{A})$ in the fourth result.

The situation is quite similar for the second result. Pure substitution accounts the weight a of an output tree exactly once and o-substitution may account a once or not at all, because of the linearity restriction. Due to periodicity and commutativity we can keep track of the missing factor a and apply it to the transition weight, in case a is not accounted by o-substitution. Finally, if the linearity condition is absent, then o-substitution may account the weight a more often than pure substitution. In general there is no way to get rid of this additional factor unless the monoid is regular, which explains the third result and the direction π–$\mathrm{BOT}(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}^o(\mathcal{A})$ in the last result.

Moreover, for every monoid $\mathcal{A}$ we have π–$\mathrm{BOT}^o(\mathcal{A}) = \pi$–$\mathrm{BOT}(\mathcal{A})$, if both the nondeletion and linearity restriction are present in π (cf. Theorem 5.5 of [21] and Proposition 4.4). In the remaining cases for commutative monoids $\mathcal{A}$ and combinations π of restrictions we have that π–$\mathrm{BOT}^o(\mathcal{A})$ and π–$\mathrm{BOT}(\mathcal{A})$ are incomparable with respect to set inclusion. In particular, if the monoid $\mathcal{A}$ is non-periodic, then, for every combination π of restrictions not containing both the nondeletion and linearity restriction, we obtain the incomparability of π–$\mathrm{BOT}^o(\mathcal{A})$ and π–$\mathrm{BOT}(\mathcal{A})$ (cf. Lemma 4.7).

This paper is structured as follows. Section 2 reviews the relevant basic mathematical notions and notations, in particular partial orders, trees and bottom-up tree transducers, monoids and semirings, and substitutions of formal tree series. Section 3 recalls the definition of deterministic bottom-up tree series transducers from [14] and introduces deterministic bu-w-tt along with the aforementioned restrictions. Moreover, we relate the notions of deterministic bottom-up tree series transducer, deterministic bu-w-tt, and deterministic tree transducer. Finally, Section 4 details the Hasse diagrams obtained for the various subclasses of tree-to-tree-series and o-tree-to-tree-series transformations computed by restricted deterministic bu-w-tt. The Hasse diagrams will be complete in the sense that we present a Hasse diagram for every commutative monoid with an absorbing element $\mathbf{0}$.

2. Preliminaries

In this section we present some basic notions and notations required in the sequel. The first subsection recalls partial orders [9] and associated notions. Words, trees, and tree transducers [32, 23, 24] are considered in the second subsection, whereas the third subsection is dedicated to algebraic structures and, in particular, monoids [27, 28] and semirings [30, 26, 25]. Finally, the section is concluded by the presentation of formal tree series [2, 30, 7] and tree series substitution [7, 14, 21].

The set $\{0, 1, 2, \ldots\}$ of all non-negative integers is denoted by $\mathbb{N}$, and the set $\{1, 2, \ldots\}$ of all positive integers is denoted by $\mathbb{N}_+$. For every $i, j \in \mathbb{N}$ the interval $\{k \in \mathbb{N} \mid i \leq k \leq j\}$ is abbreviated by $[i, j]$. In particular, we use the shorthand $[j]$ instead of $[1, j]$. Recall that $\mathrm{card}(S)$ denotes the *cardinality*, i.e., the number of elements, of a finite set S, hence $\mathrm{card}([j]) = j$. The *power set* of a set S is the set of all its subsets, i.e., $\mathcal{P}(S) = \{ S' \mid S' \subseteq S \}$, and the set of all finite subsets is $\mathcal{P}_{\mathrm{f}}(S) = \{ S' \subseteq S \mid S' \text{ is finite} \}$. We write $f : S_1 \longrightarrow S_2$ for a total mapping from the nonempty set S_1 into the nonempty set S_2. The *range of* f is then defined to be the set $\{ f(s_1) \mid s_1 \in S_1 \}$.

2.1. Partial orders

Given a nonempty set S, a binary relation $\leq \subseteq S \times S$ is called *partial order (on S)*, if $\leq$ is (i) *reflexive*, i.e., for every $s \in S$ we have $s \leq s$, (ii) *antisymmetric*, i.e., for every $s_1, s_2 \in S$ the facts $s_1 \leq s_2$ and $s_2 \leq s_1$ imply $s_1 = s_2$, and (iii) *transitive*, i.e., for every $s_1, s_2, s_3 \in S$ with $s_1 \leq s_2$ and $s_2 \leq s_3$ also $s_1 \leq s_3$ holds.

A partial order $\leq \subseteq S \times S$, which fulfils for every $s_1, s_2 \in S$ the condition that $s_1 \leq s_2$ or $s_2 \leq s_1$, is said to be a *total order*. Contrary, whenever neither $s_1 \leq s_2$ nor $s_2 \leq s_1$, then s_1 and s_2 are said to be *incomparable*. As usual, the *strict order* $< \subseteq S \times S$ is derived from $\leq$ by setting $s_1 < s_2$, if and only if $s_1 \leq s_2$ and $s_1 \neq s_2$. Moreover, we define the *covering relation* $\lessdot \subseteq S \times S$ derived from $\leq$ by setting $s_1 \lessdot s_2$, if $s_1 < s_2$ and for every $s \in S$ the condition $s_1 \leq s < s_2$ implies $s = s_1$.

Finite partial orders can be visualized by means of Hasse *diagrams* [9]. A Hasse diagram is a (directed, acyclic, and unlabelled) graph $G = (S, \lessdot)$ with the set S of vertices and the set $\lessdot$ of edges, i.e., there is a directed edge from vertex $s_1 \in S$ to vertex $s_2 \in S$, if and only if $s_1 \lessdot s_2$. In pictorial expressions, the vertices are displayed by naming the element of S, and the edges are drawn as line segments connecting vertices. We generally assume that all edges are directed upwards, and a line segment is only supposed to intersect with a vertex, if the vertex is either its starting or ending point.

Finally, a binary relation $\equiv \subseteq S \times S$ is said to be an *equivalence relation*, if $\equiv$ is (i) reflexive, (ii) transitive, and (iii) *symmetric*, i.e., for every $s_1, s_2 \in S$ the property $s_1 \equiv s_2$ implies $s_2 \equiv s_1$. The *equivalence class* of $s \in S$ (with respect to $\equiv$) is the set $[s]_{\equiv} = \{ s' \in S \mid s \equiv s' \}$.

2.2. Words, trees, and bottom-up tree transducers

By a *word of length* $n \in \mathbb{N}$ we mean an element of the n-fold Cartesian product $S^n = S \times \cdots \times S$ of a set S. The set of all words over S is denoted by S^*, where the particular element $() \in S^0$, called the *empty word*, is displayed as ε, and the *length of a word* $w \in S^*$ is denoted by $|w|$; thus $|\varepsilon| = 0$.

Every nonempty and finite set S is called *alphabet*, of which elements are termed *symbols*. A *ranked alphabet* is defined to be a pair (Σ, rk), of which Σ is an alphabet and $\mathrm{rk} : \Sigma \longrightarrow \mathbb{N}$ associates to every symbol of Σ its *rank*. For every $n \in \mathbb{N}$ we use $\Sigma^{(n)}$ to denote the set of symbols having rank n, i.e., $\Sigma^{(n)} = \{ \sigma \in \Sigma \mid \mathrm{rk}(\sigma) = n \}$. In the following, we usually assume rk to be implicitly given, identify (Σ, rk) with Σ, and specify the ranked alphabet by listing the elements of Σ with their ranks put in parentheses as superscripts as, for example, in $\{\sigma^{(2)}, \alpha^{(0)}\}$.

Henceforth, let Σ be a ranked alphabet and $X = \{ x_i \mid i \in \mathbb{N}_+ \}$ be a fixed countable set of (formal) variables. The set of (finite, labelled, and ordered) Σ*-trees indexed by* $V \subseteq X$, denoted by $T_\Sigma(V)$, is inductively defined to be the smallest set T such that (i) $V \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \dots, s_k \in T$ also $\sigma(s_1, \dots, s_k) \in T$. Since we generally assume that $\Sigma \cap X = \emptyset$, we write α instead of $\alpha()$ for every $\alpha \in \Sigma^{(0)}$. The set T_Σ of *ground* trees is an abbreviation for $T_\Sigma(\emptyset)$. Moreover, given $s \in T_\Sigma(V)$ and unary $\gamma \in \Sigma^{(1)}$, we abbreviate

$$\underbrace{\gamma(\gamma(\cdots(\gamma(s))\cdots))}_{n\text{-times } \gamma}$$

simply by $\gamma^n(s)$. Note that $\gamma^0(s) = s$.

The number of occurrences of a given variable or symbol $z \in V \cup \Sigma$ in $s \in T_\Sigma(V)$ is denoted by $|s|_z$. For every $n \in \mathbb{N}$ we denote $\{x_1, \ldots, x_n\}$ by the shorthand X_n (note that $X_0 = \emptyset$). Given $n \in \mathbb{N}$, $s \in T_\Sigma(X_n)$, and $t_1, \ldots, t_n \in T_\Sigma(V)$, the expression $s[t_1, \ldots, t_n]$ denotes the result of replacing (in parallel) for every $i \in [n]$ every occurrence of x_i in s by t_i, i.e., $x_i[t_1, \ldots, t_n] = t_i$ for every $i \in [n]$ and

$$\sigma(s_1, \ldots, s_k)[t_1, \ldots, t_n] = \sigma(s_1[t_1, \ldots, t_n], \ldots, s_k[t_1, \ldots, t_n])$$

for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma(X_n)$. Moreover, for tree languages $L \subseteq T_\Sigma(X_k)$ and $L_1, \ldots, L_k \subseteq T_\Sigma$ we use

$$L[L_1, \ldots, L_k] = \{\, s[t_1, \ldots, t_k] \mid s \in L, t_1 \in L_1, \ldots, t_k \in L_k \,\} \ .$$

Let $Y \subset X$ be finite and let $s \in T_\Sigma(X)$. The tree s is called *nondeleting in* Y (respectively, *linear in* Y), if every $y \in Y$ occurs at least once, i.e., $|s|_y \geq 1$, (respectively, at most once, i.e., $|s|_y \leq 1$) in s. We recursively define $\mathrm{size}, \mathrm{height} : T_\Sigma(V) \longrightarrow \mathbb{N}_+$ by the following equalities:

- for every $v \in V$ we have $\mathrm{size}(v) = 1 = \mathrm{height}(v)$,
- for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma(V)$ we have

$$\begin{aligned} \mathrm{size}(\sigma(s_1, \ldots, s_k)) &= 1 + \sum_{i \in [k]} \mathrm{size}(s_i) \\ \mathrm{height}(\sigma(s_1, \ldots, s_k)) &= 1 + \max_{i \in [k]} \mathrm{height}(s_i) \ . \end{aligned}$$

Let Σ be a ranked alphabet in which just one symbol is non-nullary, i.e., $\bigcup_{n \in \mathbb{N}_+} \Sigma^{(n)} = \{\sigma\}$. The set of *fully balanced (and symmetric) trees (over* Σ*)* is defined to be the smallest subset $T \subseteq T_\Sigma$ such that $\Sigma^{(0)} \subseteq T$, and given a fully balanced tree $s \in T$, the tree $\sigma(s, \ldots, s) \in T$ is fully balanced. Note that if $\mathrm{card}(\Sigma^{(0)}) = 1$, then the height of a fully balanced tree already characterizes the tree uniquely.

Finally, we shortly recall the concept of a deterministic bottom-up tree transducer [35, 38, 12, 23] (splitting up a rule into its state behavior and the computed output in an obvious way). A *deterministic bottom-up tree transducer* is a tuple $M = (Q, \Sigma, \Delta, F, \delta, \mu)$, where Q and $F \subseteq Q$ are finite sets of *states* and *final states*, respectively, Σ and Δ are the *input* and *output ranked alphabet*, respectively, $\delta = (\,\delta^k_\sigma : Q^k \longrightarrow Q\,)_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}}$ is a family of *transition mappings*, and $(\,\mu^k_\sigma : Q^k \longrightarrow \mathcal{P}_\mathrm{f}(T_\Delta(X_k))\,)_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}}$ is a family of *output mappings*. Additionally, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q$ we require $\mathrm{card}(\mu^k_\sigma(q_1, \ldots, q_k)) \leq 1$. The semantics of deterministic bottom-up tree transducers is defined inductively as follows. Let $\widehat{\delta} : T_\Sigma \longrightarrow Q$ be the mapping with $\widehat{\delta}(\sigma(s_1, \ldots, s_k)) = \delta^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k))$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$. Further, let $\widehat{\mu} : T_\Sigma \longrightarrow \mathcal{P}_\mathrm{f}(T_\Delta)$ with

$$\widehat{\mu}(\sigma(s_1, \ldots, s_k)) = \mu^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k))[\widehat{\mu}(s_1), \ldots, \widehat{\mu}(s_k)] \ .$$

The *tree transformation computed by M* is $\tau_M : T_\Sigma \longrightarrow \mathcal{P}_{\mathrm{f}}(T_\Delta)$ defined by

$$\tau_M(s) = \{\, t \in \widehat{\mu}(s) \mid \widehat{\delta}(s) \in F \,\} \ .$$

Note that $\mathrm{card}(\tau_M(s)) \leq 1$ for every $s \in T_\Sigma$. The class of tree transformations computable by deterministic bottom-up tree transducers is denoted by d–$\mathrm{BOT_{tt}}$.

2.3. Monoids and semirings

A *monoid* is an algebraic structure $\mathcal{A} = (A, \otimes, \mathbf{1})$ consisting of a *carrier (set) A* together with a binary operation $\otimes : A^2 \longrightarrow A$ and a constant element $\mathbf{1} \in A$, such that the operation $\otimes$ is *associative*, i.e., for every $a_1, a_2, a_3 \in A$ the equality $a_1 \otimes (a_2 \otimes a_3) = (a_1 \otimes a_2) \otimes a_3$ is satisfied, and $\mathbf{1}$ is the *unit element* with respect to $\otimes$, i.e., for every $a \in A$ we demand $\mathbf{1} \otimes a = a = a \otimes \mathbf{1}$. A monoid $(B, \odot, \mathbf{1})$ is a *submonoid* of $\mathcal{A}$, if $B \subseteq A$ and for every $b_1, b_2 \in B$ it holds that $b_1 \odot b_2 = b_1 \otimes b_2$. The *submonoid generated by* $A' \subseteq A$, denoted by $\langle A' \rangle_\otimes$, is the smallest submonoid $(B, \odot, \mathbf{1})$ of $\mathcal{A}$ such that $A' \subseteq B$. Further, $\mathcal{A}$ is said to be *commutative*, if for every $a_1, a_2 \in A$ the equality $a_1 \otimes a_2 = a_2 \otimes a_1$ is fulfilled. The monoid $\mathcal{A}$ possesses an *absorbing element* $\mathbf{0} \in A$, if for every $a \in A$ the equality $a \otimes \mathbf{0} = \mathbf{0} = \mathbf{0} \otimes a$ holds. If an absorbing element exists, then it is necessarily unique. Moreover, it can be adjoined to every monoid not possessing an absorbing element. To show this, let $(A, \otimes, \mathbf{1})$ be a monoid and $\mathbf{0} \notin A$. Then $(A \cup \{\mathbf{0}\}, \odot, \mathbf{1})$ with $a_1 \odot a_2 = a_1 \otimes a_2$, if $a_1, a_2 \in A$, and otherwise $a_1 \odot a_2 = \mathbf{0}$ is a monoid with an absorbing element, namely $\mathbf{0}$. We denote a monoid $(A, \odot, \mathbf{1})$ possessing the absorbing element $\mathbf{0}$ by $(A, \odot, \mathbf{1}, \mathbf{0})$. For the sake of simplicity, we assume that, for no monoid considered, the element $\mathbf{1}$ is an absorbing element, i.e., we ignore the trivial monoid with the singleton carrier set.

Let $\mathcal{A} = (A, \otimes, \mathbf{1})$ be a monoid. As usual, for every $a \in A$ and $n \in \mathbb{N}$ we denote by a^n the n-fold product $a \otimes \cdots \otimes a$ and set $a^0 = \mathbf{1}$. Further, given $n \in \mathbb{N}$ and a family $(\, a_i \,)_{i \in [n]}$ of $a_i \in A$, we also use the *product (notation)* $\prod_{i \in [n]} a_i = a_1 \otimes \cdots \otimes a_n$, where the order is determined by the total order $1 < 2 < \cdots$ on the index set. Note that $\prod_{i \in [0]} a_i = \mathbf{1}$. Next we define some common properties of monoids. The monoid $\mathcal{A}$ is said to be

- *finite*, if A is finite,
- *idempotent*, if for every $a \in A$ we have $a \otimes a = a$,
- *periodic*, if for every $a \in A$ there exist $i, j \in \mathbb{N}$ such that $i \neq j$ and $a^i = a^j$.
- *regular*, if for every $a \in A$ there exists an $a' \in A$, also called *a weak inverse of a*, such that $a \otimes a' \otimes a = a$, and
- *a group*, if for every $a \in A$ there exists an $a' \in A$, also called *the inverse of a*, such that $a \otimes a' = \mathbf{1} = a' \otimes a$.

We denote groups by $(A, \otimes, (\cdot)^{-1}, \mathbf{1})$, where $(\cdot)^{-1} : A \longrightarrow A$ maps each element to its (unique) inverse. Furthermore, we say that a monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ with an absorbing $\mathbf{0}$ is a group (with an absorbing zero) and denote this by $(A, \odot, (\cdot)^{-1}, \mathbf{1}, \mathbf{0})$, if for every $a \in A \setminus \{\mathbf{0}\}$ there exists an inverse element. The following proposition collects some trivial interrelations between the aforementioned properties.

Proposition 2.1. *Let* $\mathcal{A} = (A, \otimes, \mathbf{1})$ *be a monoid. We observe the following implications between properties of* $\mathcal{A}$.

Table 1: Various monoids and their properties.

monoid	commutative	finite	idempotent	periodic	regular	group
$\mathbb{N}$	yes	NO	NO	NO	NO	NO
$\mathbb{Z}_\infty$	yes	NO	NO	NO	yes	yes
$\mathbb{Z}_2$	yes	yes	yes	yes	yes	yes
$\mathbb{Z}_3$	yes	yes	NO	yes	yes	yes
$\mathbb{Z}_4$	yes	yes	NO	yes	NO	NO
$\mathbb{Z}_6$	yes	yes	NO	yes	yes	NO
$\mathbb{R}_{\max}$	yes	NO	yes	yes	yes	NO
$\mathbb{L}_S$	NO	NO	NO	NO	NO	NO

(i) Finiteness implies periodicity.

(ii) Idempotency implies periodicity and regularity.

(iii) If $\mathcal{A}$ is a group, then $\mathcal{A}$ is also regular and for every $a \in A$ the equality $a = a^2$ implies $a = \mathbf{1}$.

Important monoids possessing an absorbing element include

- the multiplicative *monoid of the non-negative integers* $\mathbb{N} = (\mathbb{N}, \cdot, 1, 0)$ with the common operation of multiplication,
- the additive *group of the integers* $\mathbb{Z}_\infty = (\mathbb{Z} \cup \{+\infty\}, +, 0, (+\infty))$ with the usual addition on integers $\mathbb{Z}$ extended to $(+\infty)$ such that $(+\infty)$ is an absorbing element,
- the multiplicative group $\mathbb{Z}_2 = ([0, 1], \cdot, 1, 0)$,
- the multiplicative group $\mathbb{Z}_3 = ([0, 2], \cdot, 1, 0)$ with multiplication modulo 3,
- the multiplicative monoid $\mathbb{Z}_4 = ([0, 3], \cdot, 1, 0)$ with multiplication modulo 4,
- the multiplicative monoid $\mathbb{Z}_6 = ([0, 5], \cdot, 1, 0)$ with multiplication modulo 6,
- the max-*monoid over the reals* $\mathbb{R}_{\max} = (\mathbb{R} \cup \{+\infty, -\infty\}, \max, (-\infty), (+\infty))$ with the standard maximum operation on the reals $\mathbb{R}$, and
- the *language monoid* $\mathbb{L}_S = (\mathcal{P}(S^*), \circ, \{\varepsilon\}, \emptyset)$ for some alphabet S with concatenation of words lifted to sets of words as multiplication.

The properties of the introduced monoids are summarized in Table 1, where we assume that S is a non-trivial alphabet, i.e., $\mathrm{card}(S) > 1$, otherwise $\mathbb{L}_S$ is commutative.

A *semiring (with one and absorbing zero)* is an algebraic structure $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with the operations of *addition* $\oplus : A^2 \longrightarrow A$ and *multiplication* $\odot : A^2 \longrightarrow A$, of which $(A, \oplus, \mathbf{0})$, also called the *additive monoid*, and $(A, \odot, \mathbf{1}, \mathbf{0})$, also called the *multiplicative monoid*, are monoids. Additionally, the former monoid is required to be commutative, the latter possesses $\mathbf{0}$ as an absorbing element, and the monoids are connected via the *distributivity laws*, i.e., for every $a_1, a_2, a_3 \in A$ the equalities $a_1 \odot (a_2 \oplus a_3) = (a_1 \odot a_2) \oplus (a_1 \odot a_3)$ and $(a_1 \oplus a_2) \odot a_3 = (a_1 \odot a_3) \oplus (a_2 \odot a_3)$ hold. A *commutative semiring* $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is defined to be a semiring, in which the monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ is commutative.

In semirings we use the product notation of the multiplicative monoid and the *sum (notation)* $\sum_{i\in I} a_i$ for every index set I such that only finitely many $a_i \in A$ with $i \in I$ are different from $\mathbf{0}$. Note that the order is obviously irrelevant due to commutativity, and note further that $\sum_{i\in[0]} a_i = \mathbf{0}$. By convention, we assume that multiplication has a higher (binding) priority than addition, e.g., we read $a_1 \oplus a_2 \odot a_3$ as $a_1 \oplus (a_2 \odot a_3)$. Examples of semirings can be found, for example, in [26, 25].

Proposition 2.2. *There exists a monoid* $(A, \odot, \mathbf{1}, \mathbf{0})$ *with an absorbing* $\mathbf{0}$ *such that there does not exist a semiring* $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$.

Proof. We firstly provide the operation table of such a monoid $(\{\mathbf{0}, \mathbf{1}, a, b\}, \odot, \mathbf{1}, \mathbf{0})$, which is even commutative.

$\odot$	$\mathbf{0}$	$\mathbf{1}$	a	b
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	a	b
a	$\mathbf{0}$	a	a	b
b	$\mathbf{0}$	b	b	a

Suppose there exists a commutative monoid $(\{\mathbf{0}, \mathbf{1}, a, b\}, \oplus, \mathbf{0})$ such that $(\{\mathbf{0}, \mathbf{1}, a, b\}, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring. Consider the sum $\mathbf{1} \oplus b$.

1. Let $\mathbf{1} \oplus b \in \{\mathbf{1}, a\}$. Then by distributivity $a \odot (\mathbf{1} \oplus b) = a \oplus b = a$, but $b \odot (\mathbf{1} \oplus b) = b \oplus a = b$. Hence $a \oplus b \neq b \oplus a$ which is contradictory.
2. Let $\mathbf{1} \oplus b = b$. Then again by distributivity $a \odot (\mathbf{1} \oplus b) = a \oplus b = b$, but $b \odot (\mathbf{1} \oplus b) = b \oplus a = a$. Hence $a \oplus b \neq b \oplus a$ which is contradictory.
3. Let $\mathbf{1} \oplus b = \mathbf{0}$. Then

$$(\mathbf{1} \oplus b) \oplus a = a \neq \mathbf{1} = \mathbf{1} \oplus a \odot \mathbf{0} = \mathbf{1} \oplus a \odot (\mathbf{1} \oplus b) = \mathbf{1} \oplus a \oplus b = \mathbf{1} \oplus (b \oplus a) \ ,$$

which is a contradiction to associativity. □

However, we can always embed the multiplicative monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ into a semiring as follows. Let $\perp \notin A$ and let $A' = A \cup \{\perp\}$. Further, define $\oplus, \otimes : A' \times A' \longrightarrow A'$ for every $a_1, a_2 \in A'$ by

$$a_1 \oplus a_2 = \begin{cases} \mathbf{0} & \text{, if } a_1, a_2 \in A \\ a_1 & \text{, if } a_2 = \perp \\ a_2 & \text{, otherwise} \end{cases} \qquad \text{and} \qquad a_1 \otimes a_2 = \begin{cases} a_1 \odot a_2 & \text{, if } a_1, a_2 \in A \\ \perp & \text{, otherwise .} \end{cases}$$

Then $(A', \oplus, \otimes, \perp, \mathbf{1})$ is a semiring (with a new zero).

2.4. Formal tree series

Let Δ be a ranked alphabet and additionally $V \subseteq X$. Every $\varphi : T_\Delta(V) \longrightarrow A$ into a nonempty set A is called *formal tree series (over* Δ*,* V*, and* A*)*. We use $A\langle\langle T_\Delta(V)\rangle\rangle$ to denote the set of all formal tree series over Δ, V, and A. Given $t \in T_\Delta(V)$, we usually write (φ, t), termed the *coefficient* of t, instead of $\varphi(t)$ and $\sum_{t\in T_\Delta(V)} (\varphi, t)\, t$ instead of φ, in order to follow the established conventions. For example, $\sum_{t\in T_\Delta(V)} \operatorname{size}(t)\, t$ is the tree series which associates to every tree its size. In addition, if there is an $a \in A$ such that for every $t \in T_\Delta(V)$ the coefficient $(\varphi, t) = a$ is constant, then φ is said to be *constant*, and we use $\widetilde{a}$ to abbreviate such φ.

Let $(A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid with an absorbing $\mathbf{0}$ and $\varphi \in A\langle\langle T_\Delta(V)\rangle\rangle$. The *support* of φ is defined to be the set $\mathrm{supp}(\varphi) = \{\, t \in T_\Delta(V) \mid (\varphi, t) \neq \mathbf{0} \,\}$. Whenever $\mathrm{supp}(\varphi)$ is finite, we say that φ is a *polynomial*, and moreover, a polynomial φ is said to be a *monomial*, if $\mathrm{card}(\mathrm{supp}(\varphi)) \leq 1$. Clearly, a monomial φ obeys $\varphi = a\, t$ for some $a \in A$ and $t \in T_\Delta(V)$. The set of all monomial (respectively, polynomial) formal tree series (over Δ, V, and A) is denoted by $A[T_\Delta(V)]$ (respectively, $A\langle T_\Delta(V)\rangle$). A tree series $\varphi \in A\langle\langle T_\Delta(V)\rangle\rangle$ is said to be *boolean*, if for every $t \in T_\Delta(V)$ the coefficient obeys $(\varphi, t) \in \{\mathbf{0}, \mathbf{1}\}$. Provided $L \subseteq T_\Delta(V)$, we define the *characteristic tree series of* L by

$$(\mathrm{char}(L), t) = \begin{cases} \mathbf{1} & \text{, if } t \in L \\ \mathbf{0} & \text{, otherwise} \end{cases}$$

for every $t \in T_\Delta(V)$. Note that $\mathrm{char}(L)$ is boolean and $\mathrm{char}(L) \in A\langle T_\Delta(V)\rangle$ if and only if $L \in \mathcal{P}_\mathrm{f}(T_\Delta(V))$. Moreover, $\mathrm{char}(L) \in A[T_\Delta(V)]$ if and only if $L \in \mathcal{P}_\mathrm{f}(T_\Delta(V))$ and $\mathrm{card}(L) \leq 1$.

Provided that $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, we define the sum of $\psi_1, \psi_2 \in A\langle\langle T_\Delta(V)\rangle\rangle$ pointwise by $(\psi_1 \oplus \psi_2, t) = (\psi_1, t) \oplus (\psi_2, t)$ for every $t \in T_\Delta(V)$. Tree substitution can then be generalized to tree languages as well as to tree series over semirings. Let $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring, $n \in \mathbb{N}$, $\varphi \in A\langle T_\Delta(X_n)\rangle$, and $\psi_1, \ldots, \psi_n \in A\langle T_\Delta(V)\rangle$. In [7, 14] the authors define an IO-substitution [16, 15], i.e., for two occurrences of a variable $x \in X$ the same tree is to be substituted, on tree series. *(Pure) substitution* of $(\psi_1, \ldots, \psi_n)$ *into* φ, denoted by $\varphi \longleftarrow (\psi_1, \ldots, \psi_n)$, is defined by

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \mathrm{supp}(\varphi), \\ (\forall i \in [n]):\, t_i \in \mathrm{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i)\Big)\, t[t_1, \ldots, t_n] \ .$$

Irrespective of the number of occurrences of x_i for some $i \in [n]$, the coefficient (ψ_i, t_i) is taken into account exactly once, even if x_i does not appear at all in t. This particularity led to the introduction of a different notion of substitution, which is also an IO-substitution, defined in [21] as follows.

$$\varphi \overset{o}{\longleftarrow} (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \mathrm{supp}(\varphi), \\ (\forall i \in [n]):\, t_i \in \mathrm{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i)^{|t|_{x_i}}\Big)\, t[t_1, \ldots, t_n]$$

This notion of substitution, called *o-substitution*, takes (ψ_i, t_i) into account as often as the corresponding x_i appears in t. However, both notions are defined only for formal tree series over semirings. Next, we restrict the substitutions to monomials and thereby obtain notions of substitutions also defined for monoids. Note that $\overset{\mathrm{mod}}{\longleftarrow}$ refers to $\longleftarrow = \overset{\varepsilon}{\longleftarrow}$, if $\mathrm{mod} = \varepsilon$, and to $\overset{o}{\longleftarrow}$, if $\mathrm{mod} = o$.

Let $(A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid, $\varphi \in A[T_\Delta(X_n)]$, $\psi_1, \ldots, \psi_n \in A[T_\Delta(V)]$ be an n-tuple of monomials, and $\mathrm{mod} \in \{\varepsilon, o\}$ be a modifier. The mod-*substitution of* $(\psi_1, \ldots, \psi_n)$ *into* φ, denoted by $\varphi \overset{\mathrm{mod}}{\longleftarrow}_\star (\psi_1, \ldots, \psi_n)$, is defined for every $a, a_1, \ldots, a_n \in A \setminus \{\mathbf{0}\}$, $t \in T_\Delta(X_n)$, $i \in [n]$, and $t_1, \ldots, t_n \in T_\Delta(V)$ by the following axioms.

$$\varphi \overset{\mathrm{mod}}{\longleftarrow}_\star () = \varphi \tag{1}$$

$$\widetilde{\mathbf{0}} \overset{\mathrm{mod}}{\longleftarrow}_\star (\psi_1, \ldots, \psi_n) = \widetilde{\mathbf{0}} \tag{2}$$

$$\begin{aligned}
\varphi \overset{\text{mod}}{\longleftarrow}_{\star} (\psi_1, \dots, \psi_{i-1}, \widetilde{\mathbf{0}}, \psi_{i+1}, \dots, \psi_n) &= \widetilde{\mathbf{0}} && (3)\\
a\, t \longleftarrow_{\star} (a_1\, t_1, \dots, a_n\, t_n) &= \Big(a \odot \prod_{i \in [n]} a_i\Big)\, t[t_1, \dots, t_n] && (4)\\
a\, t \overset{o}{\longleftarrow}_{\star} (a_1\, t_1, \dots, a_n\, t_n) &= \Big(a \odot \prod_{i \in [n]} a_i^{|t|_{x_i}}\Big)\, t[t_1, \dots, t_n] && (5)
\end{aligned}$$

This way (1)–(4) characterize pure substitution on monomials, and (1)–(3) and (5) characterize o-substitution on monomials. It is easily seen using Proposition 3.4 of [21], that these are really the restrictions of the respective notions of substitution, which are defined for semirings $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, to their multiplicative monoid $(A, \odot, \mathbf{1}, \mathbf{0})$, i.e.,

$$\varphi \overset{\text{mod}}{\longleftarrow} (\psi_1, \dots, \psi_n) = \varphi \overset{\text{mod}}{\longleftarrow}_{\star} (\psi_1, \dots, \psi_n) \ .$$

Henceforth, we drop the star from the substitution on monomials.

Finally, we mention that [6, 31] introduce a notion of substitution based on the OI-substitution approach [16, 15], in which different trees may be substituted for different occurrences of one variable. There the number of occurrences of a certain formal variable is taken into account as well. In this paper, we only deal with the IO-substitution approach.

3. Deterministic bottom-up weighted tree transducers

In this section, we recall the notion of a deterministic bottom-up tree series transducer [14, 21]. Then we present another model called deterministic bottom-up weighted tree transducer (abbreviated deterministic bu-w-tt), and show that deterministic bu-w-tt over the multiplicative monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ of a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ are equivalent to deterministic bottom-up tree series transducers over $\mathcal{A}$. The main advantage of deterministic bu-w-tt is the fact that they are defined over a monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ only, and hence that we can deal with more general algebraic structures (cf. Proposition 2.2). We present the necessary definitions in a compact style and refer the reader to [14, 21] for an elaborated introduction into general tree series transducers and weighted tree transducers.

Before we proceed with the definition of deterministic bottom-up tree series transducers, we recall some basic notions concerning matrices. Let I and J be countable *index sets* and let S be a set of *entries*. An $(I \times J)$-*matrix over* S is a mapping $K : I \times J \longrightarrow S$. The set of all matrices over S with indices of $I \times J$ is denoted by $S^{I \times J}$. The element $K(i, j)$ is called the (i, j)-*entry* of the matrix K and also written as $K_{i,j}$. If it is understood that the matrix K is a *row-vector* or *column-vector* (i.e., I or J is a singleton set, respectively), then we generally omit the element of the singleton set when indexing elements of the matrix K. Accordingly, we write, for example, K^I instead of $K^{I \times \{1\}}$, whenever we do not want to stress that the matrix K is a column-vector.

Given a finite set Q of states, input and output ranked alphabets Σ and Δ, respectively, and a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, a *deterministic bottom-up tree representation (over Q, Σ, Δ, and $\mathcal{A}$)* is a family $(\mu_k)_{k \in \mathbb{N}}$ of mappings, where for every $k \in \mathbb{N}$ the mapping μ_k has type

$$\mu_k : \Sigma^{(k)} \longrightarrow A[T_\Delta(X_k)]^{Q \times Q^k} \ .$$

Moreover, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $w \in Q^k$ there exists at most one $q \in Q$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$. A *deterministic bottom-up tree series transducer (over Σ and Δ)* is defined as a six-tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, where

- Q and $F \subseteq Q$ are nonempty, finite sets of *states* and *final states*, respectively,
- Σ and Δ are the *input* and *output ranked alphabet*, respectively; both disjoint to Q;
- $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, and
- μ is a deterministic bottom-up tree representation over Q, Σ, Δ, and $\mathcal{A}$.

For every mod $\in \{\varepsilon, o\}$, $k \in \mathbb{N}$, and $\sigma \in \Sigma^{(k)}$ the deterministic bottom-up tree representation μ induces $\overline{\mu_k(\sigma)}^{\mathrm{mod}} : (A\langle T_\Delta \rangle^Q)^k \longrightarrow A\langle T_\Delta \rangle^Q$ defined componentwise for every $q \in Q$ and $R_1, \ldots, R_k \in A\langle T_\Delta \rangle^Q$ by

$$\overline{\mu_k(\sigma)}^{\mathrm{mod}}(R_1, \ldots, R_k)_q = \sum_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \overset{\mathrm{mod}}{\longleftarrow} ((R_1)_{q_1}, \ldots, (R_k)_{q_k}) \enspace .$$

Note that $(A\langle T_\Delta \rangle^Q, (\overline{\mu_k(\sigma)}^{\mathrm{mod}})_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}})$ defines a Σ-algebra, and T_Σ is the initial Σ-algebra. There exists a unique homomorphism $h_\mu^{\mathrm{mod}} : T_\Sigma \longrightarrow A\langle T_\Delta \rangle^Q$, which is defined for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$ by

$$h_\mu^{\mathrm{mod}}(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}^{\mathrm{mod}}(h_\mu^{\mathrm{mod}}(s_1), \ldots, h_\mu^{\mathrm{mod}}(s_k)) \enspace .$$

It can easily be proved by structural induction that $h_\mu^{\mathrm{mod}}(s) \in A[T_\Delta]^Q$ for every $s \in T_\Sigma$, hence we can replace $A\langle T_\Delta \rangle^Q$ by $A[T_\Delta]^Q$ in the types of $\overline{\mu_k(\sigma)}^{\mathrm{mod}}$ and h_μ^{mod}. Finally, the mod-*tree-to-tree-series transformation*, for short: mod-t-ts transformation, computed by M is $\tau_M^{\mathrm{mod}} : T_\Sigma \longrightarrow A[T_\Delta]$ specified for every $s \in T_\Sigma$ by $\tau_M^{\mathrm{mod}}(s) = \sum_{q \in F} h_\mu^{\mathrm{mod}}(s)_q$.

Definition 3.1. A *deterministic bottom-up weighted tree transducer* (over $\mathcal{A}$), abbreviated deterministic bu-w-tt, is defined as $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ where

- Q and $F \subseteq Q$ are finite and nonempty sets of *states* and *final states*, respectively,
- Σ and Δ are the *input* and *output ranked alphabet*, respectively; both disjoint to Q;
- $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ is a monoid with an absorbing element $\mathbf{0}$,
- $\delta = (\, \delta_\sigma^k : Q^k \longrightarrow Q \,)_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}}$ is a family of *state transition mappings*, and
- $\mu = (\, \mu_\sigma^k : Q^k \longrightarrow A[T_\Delta(X_k)] \,)_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}}$ is a family of *output mappings*.

The deterministic bu-w-tt M is *boolean*, if for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$ every monomial in the range of μ_σ^k is boolean. We also make use of the following syntactic restrictions of deterministic bu-w-tt. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ be a deterministic bu-w-tt; we say that M is

- *nondeleting* (respectively, *linear*), if for every $k \in \mathbb{N}$, $q_1, \ldots, q_k \in Q$, and $\sigma \in \Sigma^{(k)}$ every variable $x \in X_k$ appears at least once, i.e., $|t|_x \geq 1$, (respectively, at most once, i.e., $|t|_x \leq 1$) in any $t \in \mathrm{supp}(\mu_\sigma^k(q_1, \ldots, q_k))$,
- *total*, if $F = Q$ and $\mu_\sigma^k(q_1, \ldots, q_k) \neq \widetilde{\mathbf{0}}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q$, and
- a *homomorphism*, if M is total and Q is a singleton.

In case M is a deterministic homomorphism bu-w-tt, we just say that M is a homomorphism bu-w-tt. Finally, we should assign a formal semantics to deterministic bu-w-tt. In fact, we define two different semantics, namely the tree-to-tree-series transformation, abbreviated t-ts transformation, and the o-tree-to-tree-series transformation, abbreviated o-t-ts transformation. Both are defined in the very same manner except for the type of substitution being used.

Definition 3.2. Let mod $\in \{\varepsilon, o\}$ and $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ be a deterministic bu-w-tt over $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$. For every $s \in T_\Sigma$ we define $\widehat{\delta} : T_\Sigma \longrightarrow Q$ and $\widehat{\mu}_{\text{mod}} : T_\Sigma \longrightarrow A[T_\Delta]$ by structural recursion as follows. For every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$ we let $\widehat{\delta}(\sigma(s_1, \ldots, s_k)) = \delta^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k))$ and

$$\widehat{\mu}_{\text{mod}}(\sigma(s_1, \ldots, s_k)) = \mu^k_\sigma(\widehat{\delta}(s_1), \ldots, \widehat{\delta}(s_k)) \overset{\text{mod}}{\longleftarrow} (\widehat{\mu}_{\text{mod}}(s_1), \ldots, \widehat{\mu}_{\text{mod}}(s_k)) \enspace .$$

The mod-*tree-to-tree-series transformation computed by M* is the mapping $\tau^{\text{mod}}_M : T_\Sigma \longrightarrow A[T_\Delta]$ specified for every $s \in T_\Sigma$ by

$$\tau^{\text{mod}}_M(s) = \begin{cases} \widehat{\mu}_{\text{mod}}(s) & \text{, if } \widehat{\delta}(s) \in F \\ \widetilde{\mathbf{0}} & \text{, otherwise} \enspace . \end{cases}$$

Example 3.3. *The deterministic bu-w-tt $M_{\text{size}} = (\{\star\}, \Sigma, \Sigma, \mathbb{Z}_\infty, \{\star\}, \delta, \mu)$ with input and output ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, state transition mappings $\delta = (\delta^2_\sigma, \delta^0_\alpha)$, and output mappings $\mu = (\mu^2_\sigma, \mu^0_\alpha)$ is defined by*

$$\delta^2_\sigma(\star, \star) = \delta^0_\alpha() = \star \quad , \quad \mu^2_\sigma(\star, \star) = 1\ \sigma(x_1, x_2) \quad \text{, and} \quad \mu^0_\alpha() = 1\ \alpha \enspace .$$

We observe that for every $s \in T_\Sigma$ we have $\tau_{M_{\text{size}}}(s) = \tau^o_{M_{\text{size}}}(s) = \text{size}(s)\ s$. Moreover, M_{size} is a linear and nondeleting homomorphism bu-w-tt, which is not boolean.

In the sequel, we investigate the computational power of various subclasses of deterministic bu-w-tt and compare their computational power by means of set inclusion. The next definition establishes shorthands for such classes of mod-t-ts transformations also taking the two different notions of substitution into account.

Definition 3.4. Let mod $\in \{\varepsilon, o\}$ and $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid. Further, let Pref $= \{\text{n}, \text{l}, \text{t}, \text{h}\}$ be a set of abbreviations standing for nondeleting, linear, total, and homomorphism, respectively. Moreover, let $r \subseteq$ Pref. The class dr–BOT$^{\text{mod}}(\mathcal{A})$ denotes the class of all mod-t-ts transformations $\tau : T_\Sigma \longrightarrow A[T_\Delta]$ such that there exists a deterministic bu-w-tt

$$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$$

with $\tau^{\text{mod}}_M = \tau$, and M obeys all the restrictions abbreviated in r. Henceforth, we omit the set braces and the separating commata and just list the letters in r. We say that r is a prefix.

We generally omit the d and the prefix t (standing for deterministic and total) in case the prefix h (standing for homomorphism) is present, because homomorphism tree transducers are deterministic and total by definition. We define the set $\Pi = \{\text{d}, \text{dn}, \text{dl}, \text{dt}, \text{h}, \text{dnl}, \text{dnt}, \text{hn}, \text{dlt}, \text{hl}, \text{dnlt}, \text{hnl}\}$ of sensible combinations and the restrictions $\Pi_r = \{\, \pi \in \Pi \mid r \in \pi \,\}$ for every $r \in$ Pref.

We note that all the restrictions and classes have been defined for deterministic bottom-up tree series transducers [14, 21] as well. Next, we establish relations between deterministic bu-w-tt, deterministic bottom-up tree series transducers, and deterministic bottom-up tree transducers.

Let us start by showing that deterministic bu-w-tt over multiplicative monoids of semirings compute the same class of mod-t-ts transformations as deterministic bottom-up tree series transducers. Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring, $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \mu_1)$ be a deterministic bottom-up tree series transducer, and $M_2 = (Q_2, \Sigma, \Delta, (A, \odot, \mathbf{1}, \mathbf{0}), F_2, \delta_2, \mu_2)$ be a deterministic bu-w-tt over the multiplicative monoid of $\mathcal{A}$. The device M_1 is *related to* M_2, if $Q_1 = Q_2$, $F_1 = F_2$, and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q, q_1, \ldots, q_k \in Q_1$ we have $(\mu_1)_k(\sigma)_{q,(q_1,\ldots,q_k)} \neq \mathbf{0}$ implies

$$(\delta_2)^k_\sigma(q_1, \ldots, q_k) = q \;\text{ and }\; (\mu_2)^k_\sigma(q_1, \ldots, q_k) = (\mu_1)_k(\sigma)_{q,(q_1,\ldots,q_k)} \;,$$

as well as $(\mu_1)_k(\sigma)_{(\delta_2)^k_\sigma(q_1,\ldots,q_k),(q_1,\ldots,q_k)} = (\mu_2)^k_\sigma(q_1, \ldots, q_k)$. A straightforward induction on the structure of $s \in T_\Sigma$ then shows for every mod $\in \{\varepsilon, o\}$ that

$$(\widehat{\mu_2})_{\mathrm{mod}}(s) = h^{\mathrm{mod}}_{\mu_1}(s)_{\widehat{\delta_2}(s)}$$

and thus $\tau^{\mathrm{mod}}_{M_1}(s) = \tau^{\mathrm{mod}}_{M_2}(s)$, whenever M_1 is related to M_2. Note that M_1 obeys the restrictions of $\pi \in \Pi$, if and only if M_2 obeys the restrictions of π.

Proposition 3.5. *Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring. Then for every $\pi \in \Pi$ and* mod $\in \{\varepsilon, o\}$ *we have*

$$\pi\text{–BOT}^{\mathrm{mod}}_{\text{t-ts}}(\mathcal{A}) = \pi\text{–BOT}^{\mathrm{mod}}((A, \odot, \mathbf{1}, \mathbf{0})) \;,$$

where π–BOT$^{\mathrm{mod}}_{\text{t-ts}}(\mathcal{A})$ denotes the class of all mod*-t-ts transformations computable by bottom-up tree series transducers obeying all the restrictions of π (cf. [14, 21]).*

Next, we transfer the obvious relationship between deterministic bottom-up tree transducers on the one hand and deterministic bottom-up tree series transducers over the Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ on the other hand (cf. Corollary 4.7 of [14] and Corollary 5.9 of [21]) to the corresponding relationship between deterministic bottom-up tree transducers and deterministic bu-w-tt over $\mathbb{Z}_2$. Let $S = \{ L \in \mathcal{P}_{\mathrm{f}}(T_\Delta) \mid \mathrm{card}(L) \leq 1 \}$ and $\sim\, \subseteq \mathbb{Z}_2[T_\Delta] \times S$ be the relation defined by $\varphi \sim L$, if and only if $L = \mathrm{supp}(\varphi)$. Indeed the relation $\sim$ is a bijection. Consequently, for every $\tau_1 : T_\Sigma \longrightarrow \mathbb{Z}_2[T_\Delta]$ and $\tau_2 : T_\Sigma \longrightarrow S$, let $\tau_1 \sim \tau_2$ if and only if for every $s \in T_\Sigma$ we have $\tau_1(s) \sim \tau_2(s)$. Moreover, let $\sim$ also be defined on classes of mappings in the obvious way.

Proposition 3.6. *For every $\pi \in \Pi$ and modifier* mod $\in \{\varepsilon, o\}$ *we have*

$$\pi\text{–BOT}^{\mathrm{mod}}(\mathbb{Z}_2) \sim \pi\text{–BOT}_{\mathrm{tt}} \;,$$

where π–BOT$_{\mathrm{tt}}$ denotes the class of all tree transformations computable by bottom-up tree transducers obeying all the restrictions of π (cf. [12]).

Proof. In the same spirit as $\sim$, a relation between deterministic bottom-up tree transducers and deterministic bu-w-tt over the group $\mathbb{Z}_2$ can be established (cf. Corollary 4.7 of [14]). More precisely, a deterministic bottom-up tree transducer $M_1 = (Q_1, \Sigma, \Delta, F_1, \delta_1, \mu_1)$ is *related to* a deterministic bu-w-tt $M_2 = (Q_2, \Sigma, \Delta, \mathbb{Z}_2, F_2, \delta_2, \mu_2)$, if $Q_1 = Q_2$, $F_1 = F_2$, $\delta_1 = \delta_2$, and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q_1$ the following condition holds.

$$(\mu_1)^k_\sigma(q_1, \ldots, q_k) = \mathrm{supp}((\mu_2)^k_\sigma(q_1, \ldots, q_k))$$

Note that for every combination $\pi \in \Pi$ we have that M_1 obeys the restrictions of π, if and only if M_2 obeys them. Moreover, if M_1 is related to M_2, then $\tau_{M_1} \sim \tau_{M_2}^{\mathrm{mod}}$ (cf. Corollary 4.7 of [14] and Corollary 5.9 of [21]). The proof of the last statement is straightforward and left to the reader. □

Thus, deterministic bottom-up tree transducers and deterministic bu-w-tt over the group $\mathbb{Z}_2$ are equally powerful, which allows us to treat deterministic bottom-up tree transducers as if they were deterministic bu-w-tt over the group $\mathbb{Z}_2$ in order to have a unique presentation.

Corollary 3.7. *For every combination $\pi \in \Pi$ we have*

$$\pi\text{–BOT}^o(\mathbb{Z}_2) = \pi\text{–BOT}(\mathbb{Z}_2) \ .$$

4. Hasse diagrams

In this section, we investigate the relation between classes of t-ts and o-t-ts transformations computed by deterministic bu-w-tt with respect to set inclusion. We derive several Hasse diagrams displaying the relationships given certain properties of the underlying monoid. As a starting point, we state the well-known Hasse diagram for deterministic bu-w-tt over the group $\mathbb{Z}_2$, i.e., for deterministic bottom-up tree transducers. Figure 1 displays the Hasse diagram for all classes of t-ts and o-t-ts transformations defined in Definition 3.4 (for $\mathcal{A} = \mathbb{Z}_2$). In order to present concise diagrams, we shorten the denotation of the classes from $\pi\text{–BOT}^{\mathrm{mod}}(\mathcal{A})$ to just π^{mod} for every combination $\pi \in \Pi$ and $\mathrm{mod} \in \{\varepsilon, o\}$. Moreover, we use $\pi^{=}$ to express that $\pi\text{–BOT}^o(\mathcal{A}) = \pi\text{–BOT}(\mathcal{A})$.

Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative monoid with at least three elements. In Section 4.1, we derive some statements which hold for every such monoid $\mathcal{A}$. In the sequel, we consider the case that $\mathcal{A}$ is non-periodic (cf. Section 4.2). Section 4.3 is dedicated to periodic, but non-regular monoids $\mathcal{A}$. Automatically, such a monoid $\mathcal{A}$ is non-idempotent and no group with an absorbing element by Proposition 2.1. The next case, which is handled in Section 4.4, additionally assumes that $\mathcal{A}$ is regular, but still not idempotent and no group with an absorbing element. Thereafter, we consider the case in which $\mathcal{A}$ is idempotent. This again excludes the case that $\mathcal{A}$ is actually a group with an absorbing element. The final case of groups (with an absorbing element) is taken care of in Section 4.6.

Theorem 4.1. *Figure 1 is the* Hasse *diagram of the displayed classes of t-ts and o-t-ts transformations over $\mathbb{Z}_2$ ordered by set inclusion.*

Proof. The equalities are concluded from Corollary 3.7 and all the inclusions hold by definition. Finally, the following four statements are sufficient to prove strictness and incomparability.

$$\text{dnlt–BOT}(\mathbb{Z}_2) \not\subseteq \text{h–BOT}(\mathbb{Z}_2) \tag{6}$$
$$\text{dnl–BOT}(\mathbb{Z}_2) \not\subseteq \text{dt–BOT}(\mathbb{Z}_2) \tag{7}$$
$$\text{hn–BOT}(\mathbb{Z}_2) \not\subseteq \text{dl–BOT}(\mathbb{Z}_2) \tag{8}$$
$$\text{hl–BOT}(\mathbb{Z}_2) \not\subseteq \text{dn–BOT}(\mathbb{Z}_2) \tag{9}$$

The inequalities (6) and (7) are trivial, and (8) and (9) are due to Theorem 3.3 of [19]. □

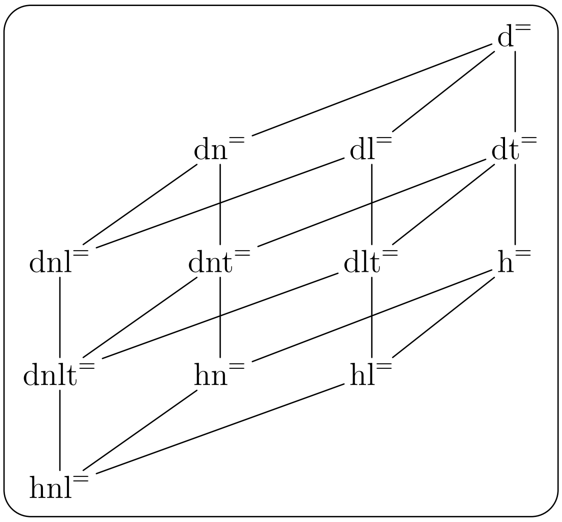

Figure 1: Hasse diagram for the group $\mathbb{Z}_2$.

4.1. Results for arbitrary monoids

In this section, we derive some statements which hold irrespective of the underlying monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$. We show how to use the results of the Hasse diagram in Figure 1 in order to obtain incomparability results for classes of t-ts and o-t-ts transformations over monoids $\mathcal{A}$ different from $\mathbb{Z}_2$. Roughly speaking, we show that all inequalities present in Figure 1 are preserved in the transition from $\mathbb{Z}_2$ to $\mathcal{A}$. This is mainly due to the fact that $\mathbb{Z}_2$ is a submonoid (with absorbing $\mathbf{0}$) of $\mathcal{A}$. Hence we take a counterexample in $\mathbb{Z}_2$, i.e., a mod_1-t-ts transformation τ which is in the class $\pi_1\text{–BOT}^{\text{mod}_1}(\mathbb{Z}_2)$, but not in the class $\pi_2\text{–BOT}^{\text{mod}_2}(\mathbb{Z}_2)$ for some modifiers $\text{mod}_1, \text{mod}_2 \in \{\varepsilon, o\}$ and $\pi_1, \pi_2 \in \Pi$. Then we prove that τ is also a counterexample for the inclusion $\pi_1\text{–BOT}^{\text{mod}_1}(\mathcal{A}) \subseteq \pi_2\text{–BOT}^{\text{mod}_2}(\mathcal{A})$, i.e., τ is trivially in $\pi_1\text{–BOT}^{\text{mod}_1}(\mathcal{A})$ because $\mathbb{Z}_2$ is a submonoid of $\mathcal{A}$, but still not in $\pi_2\text{–BOT}^{\text{mod}_2}(\mathcal{A})$.

For the purpose of the next lemma, we restrict the counterexample τ to be computed by a total deterministic bu-w-tt $M = (Q, \Sigma, \Delta, \mathbb{Z}_2, F, \delta, \mu)$. Now assume that $\tau \in \pi_2\text{–BOT}^{\text{mod}_2}(\mathcal{A})$, i.e., there exists a deterministic bu-w-tt $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \delta', \mu')$ such that $\tau_{M'}^{\text{mod}_2} = \tau$. It follows from the totality of M that for every $s \in T_\Sigma$ there exists a unique $t \in T_\Delta$ such that $\tau(s) = 1\ t$. Moreover, it follows that all reachable states of M' must be final and that for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and all reachable states $q_1, \ldots, q_k \in Q'$ of M' we have that $(\mu')^k_\sigma(q_1, \ldots, q_k)$ is boolean. Then we can easily drop the states which are not reachable from M' and obtain a boolean total deterministic bu-w-tt M'' with $\tau_{M''}^{\text{mod}_2} = \tau$. However, boolean deterministic bu-w-tt compute solely in $\mathbb{Z}_2$, and therefore, M'' can equivalently be specified as deterministic bu-w-tt over $\mathbb{Z}_2$, which is a contradiction to the assumption that $\tau \notin \pi_2\text{–BOT}^{\text{mod}_2}(\mathbb{Z}_2)$.

Lemma 4.2. *Let* $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ *be a monoid and* $\text{mod}_1, \text{mod}_2 \in \{\varepsilon, o\}$. *Furthermore, let* $\pi_1 \in \Pi_t$ *and* $\pi_2 \in \Pi$. *If* $\pi_1\text{–BOT}^{\text{mod}_1}(\mathbb{Z}_2) \nsubseteq \pi_2\text{–BOT}^{\text{mod}_2}(\mathbb{Z}_2)$, *then* $\pi_1\text{–BOT}^{\text{mod}_1}(\mathcal{A}) \nsubseteq \pi_2\text{–BOT}^{\text{mod}_2}(\mathcal{A})$.

Proof. Let $\tau \in \pi_1\text{–BOT}^{\text{mod}_1}(\mathbb{Z}_2) \setminus \pi_2\text{–BOT}^{\text{mod}_2}(\mathbb{Z}_2)$ be a mod_1-t-ts transformation, hence there exists a deterministic bu-w-tt M' obeying the restrictions π_1 such that $\tau = \tau_{M'}^{\text{mod}_1}$. Obviously,

π_1–$\mathrm{BOT}^{\mathrm{mod}_1}(\mathbb{Z}_2) \subseteq \pi_1$–$\mathrm{BOT}^{\mathrm{mod}_1}(\mathcal{A})$, because $\mathbb{Z}_2$ is a submonoid of $\mathcal{A}$. Thus there exists a total deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π_1 such that $\tau_{M_1}^{\mathrm{mod}_1} = \tau$. Note that $\widehat{\mu_1}_{\mathrm{mod}_1}(s) \neq \widetilde{\mathbf{0}}$ for every $s \in T_\Sigma$.

Now we prove by contradiction that $\tau \notin \pi_2$–$\mathrm{BOT}^{\mathrm{mod}_2}(\mathcal{A})$. Therefore, let $\tau \in \pi_2$–$\mathrm{BOT}^{\mathrm{mod}_2}(\mathcal{A})$, i.e., there exists a deterministic bu-w-tt $M_2 = (Q_2, \Sigma, \Delta, \mathcal{A}, F_2, \delta_2, \mu_2)$ obeying the restrictions of π_2 with $\tau_{M_2}^{\mathrm{mod}_2} = \tau$. The remaining proof first shows that there also exists a boolean deterministic bu-w-tt M'' obeying the restrictions of π_2 such that $\tau_{M''}^{\mathrm{mod}_2} = \tau$. The final step then shows that the existence of M'' would yield that $\tau \in \pi_2$–$\mathrm{BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$ contrary to the fact that $\tau \notin \pi_2$–$\mathrm{BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$. Hence $\tau \notin \pi_2$–$\mathrm{BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

We construct a boolean deterministic bu-w-tt $M'' = (Q_2, \Sigma, \Delta, \mathcal{A}, F_2, \delta_2, \mu'')$ obeying the restrictions π_2 and $\tau_{M''}^{\mathrm{mod}_2} = \tau_{M_2}^{\mathrm{mod}_2} = \tau$. Let $\mu'' = ((\mu'')_\sigma^k)_{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}}$ and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q_2$ let

$$(\mu'')_\sigma^k(q_1, \ldots, q_k) = \mathrm{char}(\mathrm{supp}((\mu_2)_\sigma^k(q_1, \ldots, q_k))) \ .$$

Obviously, M'' is boolean and obeys the restrictions of π_2. For our subgoal, it remains to show that $\tau_{M''}^{\mathrm{mod}_2} = \tau_{M_2}^{\mathrm{mod}_2}$. Therefore we obviously have to prove that $\widehat{\mu''}_{\mathrm{mod}_2}(s) = \widehat{\mu_2}_{\mathrm{mod}_2}(s)$ for every $s \in T_\Sigma$. We perform induction over the structure of s.

Let $s = \sigma(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$. We distinguish two separate cases.

(i) Let $i \in [k]$ be such that $\widehat{\mu_2}_{\mathrm{mod}_2}(s_i) = \widetilde{\mathbf{0}}$ or $(\mu_2)_\sigma^k(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) = \widetilde{\mathbf{0}}$. Then $\tau_{M_2}^{\mathrm{mod}_2}(s) = \widetilde{\mathbf{0}}$, but contrary $\tau_{M_2}^{\mathrm{mod}_2}(s) = \tau_{M_1}^{\mathrm{mod}_1}(s) \neq \widetilde{\mathbf{0}}$ because M_1 is boolean and total.

(ii) Assume that for every $i \in [k]$ we have $\widehat{\mu_2}_{\mathrm{mod}_2}(s_i) \neq \widetilde{\mathbf{0}}$ and

$$(\mu_2)_\sigma^k(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) = a\, t$$

for some $a \in A \setminus \{\mathbf{0}\}$ and $t \in T_\Delta(X_k)$. By induction hypothesis also $\widehat{\mu_2}_{\mathrm{mod}_2}(s_i) = \widehat{\mu''}_{\mathrm{mod}_2}(s_i)$ holds, and consequently, $\widehat{\mu_2}_{\mathrm{mod}_2}(s_i) = \mathbf{1}\, t_i$ for some $t_i \in T_\Delta$ because M'' is boolean. Then

$$\begin{aligned} &\widehat{\mu_2}_{\mathrm{mod}_2}(\sigma(s_1, \ldots, s_k)) \\ &= (\mu_2)_\sigma^k(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \overset{\mathrm{mod}_2}{\longleftarrow} (\widehat{\mu_2}_{\mathrm{mod}_2}(s_1), \ldots, \widehat{\mu_2}_{\mathrm{mod}_2}(s_k)) \\ &= a\, t \overset{\mathrm{mod}_2}{\longleftarrow} (\mathbf{1}\, t_1, \ldots, \mathbf{1}\, t_k) \\ &= a\, t[t_1, \ldots, t_k] \ . \end{aligned}$$

Since $\tau_{M_1}^{\mathrm{mod}_1}(s) \neq \widetilde{\mathbf{0}}$ we conclude that $\tau_{M_2}^{\mathrm{mod}_2}(s) = \widehat{\mu_2}_{\mathrm{mod}_2}(s)$. Further, M_1 is boolean, so also $\widehat{\mu_2}_{\mathrm{mod}_2}(s)$ is boolean, and we continue with

$$\begin{aligned} &\widehat{\mu_2}_{\mathrm{mod}_2}(\sigma(s_1, \ldots, s_k)) \\ &= a\, t[t_1, \ldots, t_k] \\ &= \mathbf{1}\, t[t_1, \ldots, t_k] \\ &= \mathbf{1}\, t \overset{\mathrm{mod}_2}{\longleftarrow} (\mathbf{1}\, t_1, \ldots, \mathbf{1}\, t_k) \\ &= (\mu'')_\sigma^k(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \overset{\mathrm{mod}_2}{\longleftarrow} (\widehat{\mu''}_{\mathrm{mod}_2}(s_1), \ldots, \widehat{\mu''}_{\mathrm{mod}_2}(s_k)) \\ &= \widehat{\mu''}_{\mathrm{mod}_2}(\sigma(s_1, \ldots, s_k)) \ . \end{aligned}$$

Hence there also exists a boolean deterministic bu-w-tt M'' obeying the restrictions of π_2 such that $\tau_{M''}^{\mathrm{mod}_2} = \tau$. Immediately, we obtain that

$$M = (Q_2, \Sigma, \Delta, \mathbb{Z}_2, F_2, \delta_2, \mu'')$$

is a deterministic bu-w-tt obeying all the restrictions of π_2 over $\mathbb{Z}_2$ such that $\tau_M^{\mathrm{mod}_2} = \tau$. However, this is contradictory to the assumption, because τ was chosen such that $\tau \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$, which finally proves the lemma. □

Thus we can derive inequality for classes of t-ts and o-t-ts transformations over the monoid $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ simply by observing inequality for the respective classes of t-ts and o-t-ts transformations over the group $\mathbb{Z}_2$. Roughly speaking, these latter inequalities are based solely on a deficiency in the tree output component of one class. For example, for any mod $\in \{\varepsilon, o\}$ the mod-t-ts transformation which maps each input tree to a fully balanced binary tree of the same height with whatever nonzero cost cannot be computed by a linear deterministic bu-w-tt. In order to generate the fully balanced binary trees, one definitely needs the copying of output trees. Another example is totality. The mod-t-ts transformation which maps every input tree to $\widetilde{\mathbf{0}}$ obviously cannot be computed by a total deterministic bu-w-tt.

The following lemma presents the conclusions of Figure 1 and Lemma 4.2. Moreover, it adds the missing case of totality, which is straightforward using the remark of the previous paragraph.

Lemma 4.3. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid and* $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$. *For every $\pi_1, \pi_2 \in \Pi$ such that there exists $r \in$* Pref *which occurs in π_2 but not in π_1, i.e., $r \in \pi_2 \setminus \pi_1$, we have*

$$\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \ .$$

Proof. We distinguish two cases.

(i) Let $r \neq \mathrm{t}$. Apparently, $r \notin \pi_1 \cup \{t\}$, so let $\pi_1' = \pi_1 \cup \{\mathrm{t}\}$. From Figure 1, we can check that $\pi_1'\text{–BOT}^{\mathrm{mod}_1}(\mathbb{Z}_2) \nsubseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathbb{Z}_2)$ and with the help of Lemma 4.2 we conclude $\pi_1'\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$. Trivially, $\pi_1'\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$, hence

$$\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \ .$$

(ii) Let $r = \mathrm{t}$. Moreover, let $\Sigma = \{\alpha^{(0)}\}$. We construct the linear and nondeleting deterministic bu-w-tt $M = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta, \mu)$ with transition mappings $\delta = (\delta_\alpha^0)$ and output mappings $\mu = (\mu_\alpha^0)$ specified by $\delta_\alpha^0() = \star$ and $\mu_\alpha^0() = \widetilde{\mathbf{0}}$. Apparently, $\tau_M^{\mathrm{mod}_1} \in \pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$ and $\tau_M^{\mathrm{mod}_1} \notin \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, because $\mathrm{t} \in \pi_2$. Hence

$$\pi_1\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \pi_2\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \ .$$

□

Due to the previous corollary, we can restrict our attention to the comparison of classes of t-ts transformations with the corresponding classes of o-t-ts transformations. As a first comparison we restate the equality of the classes of t-ts and o-t-ts transformations for all restrictions which contain both the nondeletion as well as the linearity restriction. This equality was shown for tree series transducers in [21], but can also be seen from the definition of pure and o-substitution, because both notions coincide whenever the participating tree series are nondeleting and linear.

Proposition 4.4. *Let* $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ *be a monoid. Then*

$$\pi\text{–BOT}^o(\mathcal{A}) = \pi\text{–BOT}(\mathcal{A})$$

for every $\pi \in \{\text{dnl}, \text{dnlt}, \text{hnl}\}$.

The final result of this section shows two inequality results. Essentially, we prove that the classes of t-ts transformations and o-t-ts transformations computed by linear homomorphism bu-w-tt are incomparable. Due to the Hasse diagram presented in Figure 1, we cannot prove this result for every monoid with absorbing element, but rather we require that the monoid $(A, \odot, \mathbf{1}, \mathbf{0})$ has at least three elements, i.e., $\mathbf{0} \neq \mathbf{1}$, and it is not isomorphic to $\mathbb{Z}_2$.

Since we often deal with homomorphism bu-w-tt, of which the state behaviour is completely determined, in the sequel, we do not explicitly specify the state transition mappings δ, but assume that they are specified in the only possible way. The result hl–BOT$(\mathcal{A}) \nsubseteq$ h–BOT$^o(\mathcal{A})$ is proved essentially by exploiting the property that pure substitution can distinguish two output trees with different weights, although it deletes them. On the other hand, this distinction vanishes in o-substitution, and we cannot use the state to signal the difference, because we consider homomorphism bu-w-tt. The same properties are used to prove hl–BOT$^o(\mathcal{A}) \nsubseteq$ h–BOT$(\mathcal{A})$.

Lemma 4.5. *Let* $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ *be a monoid and* $A \neq \{\mathbf{0}, \mathbf{1}\}$. *Then*

$$\text{hl–BOT}(\mathcal{A}) \nsubseteq \text{h–BOT}^o(\mathcal{A}) \quad \textit{and} \quad \text{hl–BOT}^o(\mathcal{A}) \nsubseteq \text{h–BOT}(\mathcal{A}) \ .$$

Proof. Let us prove the former statement. We choose $a \in A \setminus \{\mathbf{0}, \mathbf{1}\}$ arbitrarily. Suppose that $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$ and $M_1 = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ is the linear homomorphism bu-w-tt with $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha, (\mu_1)^0_\beta)$ specified by

$$(\mu_1)^1_\gamma(\star) = \mathbf{1}\,\alpha \quad , \quad (\mu_1)^0_\alpha() = a\,\alpha \quad , \quad (\mu_1)^0_\beta() = \mathbf{1}\,\beta \ .$$

Let $\tau = \tau_{M_1}$. Clearly, $\tau \in$ hl–BOT$(\mathcal{A})$, and moreover, $\tau(\gamma(\alpha)) = a\,\alpha$ and $\tau(\gamma(\beta)) = \mathbf{1}\,\alpha$.

Now let us prove that $\tau \notin$ h–BOT$^o(\mathcal{A})$. We prove this statement by contradiction, so assume that there exists a homomorphism bu-w-tt

$$M_2 = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$$

such that $\tau^o_{M_2} = \tau$. Trivially, $\delta_2 = \delta_1$ and $\mu_2 = ((\mu_2)^1_\gamma, (\mu_2)^0_\alpha, (\mu_2)^0_\beta)$ with

$$(\mu_2)^1_\gamma(\star) = c\,t \quad , \quad (\mu_2)^0_\alpha() = a\,\alpha \quad , \quad (\mu_2)^0_\beta() = \mathbf{1}\,\beta$$

for some $c \in A$ and $t \in T_\Sigma(X_1)$. Moreover, we readily observe $t = \alpha$, otherwise we have $\text{supp}(\tau^o_{M_2}(\gamma(\beta))) \neq \{\alpha\}$. Consequently, $\tau^o_{M_2}(\gamma(\alpha)) = \tau^o_{M_2}(\gamma(\beta)) = c\,\alpha$. Thus we obtain the contradiction $a = \mathbf{1}$ and conclude that $\tau \notin$ h–BOT$^o(\mathcal{A})$.

To show the latter statement, i.e., hl–BOT$^o(\mathcal{A}) \nsubseteq$ h–BOT$(\mathcal{A})$, let $\tau^o = \tau^o_{M_1}$. Obviously, $\tau^o \in$ hl–BOT$^o(\mathcal{A})$, and moreover, $\tau^o(\gamma(\alpha)) = \tau^o(\gamma(\beta)) = \mathbf{1}\,\alpha$. Let us prove that $\tau^o \notin$ h–BOT$(\mathcal{A})$. We prove this statement by contradiction, so suppose that there exists a homomorphism bu-w-tt

$$M_3 = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, \{\star\}, \delta_3, \mu_3)$$

such that $\tau_{M_3} = \tau^o$. Trivially, we see that $\delta_3 = \delta_1$ and $\mu_3 = ((\mu_3)^1_\gamma, (\mu_3)^0_\alpha, (\mu_3)^0_\beta)$ with

$$(\mu_3)^1_\gamma(\star) = c\,t \quad , \quad (\mu_3)^0_\alpha() = a\,\alpha \quad , \quad (\mu_3)^0_\beta() = \mathbf{1}\,\beta$$

for some $c \in A$ and $t \in T_\Sigma(X_1)$. Moreover, we again readily observe $t = \alpha$, else we have $\mathrm{supp}(\tau_{M_3}(\gamma(\beta))) \neq \{\alpha\}$. Consequently,

$$\tau_{M_3}(\gamma(\alpha)) = (c \odot a)\,\alpha = \mathbf{1}\,\alpha = c\,\alpha = \tau_{M_3}(\gamma(\beta)) \ ,$$

which yields $c = \mathbf{1}$ and hence also $a = \mathbf{1}$. This is contrary to the assumption that $a \in A \setminus \{\mathbf{0}, \mathbf{1}\}$. Thus we conclude that $\tau^o \notin$ h–BOT($\mathcal{A}$). □

In particular, the former lemma also proves that the classes of t-ts and o-t-ts transformations computed by homomorphism bu-w-tt are incomparable for all monoids different from $\mathbb{Z}_2$. In fact, it can be seen from the proof of the previous lemma that there is a single homomorphism bu-w-tt M such that $\tau_M \notin$ h–BOT^o($\mathcal{A}$) and $\tau_M^o \notin$ h–BOT($\mathcal{A}$).

Corollary 4.6. *We have* $\mathcal{A} = \mathbb{Z}_2$, *if and only if the equality* π–BOT^o($\mathcal{A}$) = π–BOT($\mathcal{A}$) *holds for every* $\pi \in \Pi$.

Proof. The equality in $\mathbb{Z}_2$ is shown in Theorem 4.1, and Lemma 4.5 proves the incomparability of hl–BOT^o($\mathcal{A}$) and hl–BOT($\mathcal{A}$) in all other monoids. □

However, without additional information about the monoid we are unable to prove further comparability or incomparability results. Hence we consider monoids with certain properties in subsequent sections. The properties are chosen such that we obtain a Hasse diagram for every commutative monoid.

4.2. Non-periodic monoids

In this section, we show that for non-periodic monoids almost all classes of t-ts and o-t-ts transformations (except the ones containing both the nondeletion and linearity restriction) computed by restricted deterministic bu-w-tt are incomparable with respect to set inclusion. An example of a non-periodic monoid is the multiplicative monoid of $\mathbb{N}$. To be precise, we even show that

$$\pi\text{–BOT}(\mathcal{A}) \nsubseteq \text{d–BOT}^o(\mathcal{A}) \quad \text{and} \quad \pi\text{–BOT}^o(\mathcal{A}) \nsubseteq \text{d–BOT}(\mathcal{A})$$

for every $\pi \in \{\mathrm{hn}, \mathrm{hl}\}$ and non-periodic monoid $\mathcal{A}$.

The general idea of the proof is the following. Let $a \in A$ be such that $a^i \neq a^j$, whenever $i \neq j$ where $i, j \in \mathbb{N}$. We construct a homomorphism bu-w-tt M_1, which computes a t-ts transformation τ in which arbitrarily large powers of a occur as weights in the range. Let us first consider the result hl–BOT$^{\mathrm{mod}_1}$($\mathcal{A}$) $\nsubseteq$ d–BOT$^{\mathrm{mod}_2}$($\mathcal{A}$) where mod_1 and mod_2 are different. Our input ranked alphabet will have two unary symbols; encountering γ_1 in the input we stack another a to the weight computed so far and output a prolonged output tree, and encountering γ_2 we delete the computed output tree at no cost. Since every deterministic bu-w-tt $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$, which also computes τ but as a mod_2-t-ts transformation, has only finitely many states, it must permit at least one final state q which accepts infinitely many input trees. In particular, the transition from q to some state reading γ_2 is interesting. In the case of $\mathrm{mod}_2 = o$, the weight of the outputted tree is reset to the weight present in the monomial $\mu^1_{\gamma_2}(q)$, which is to be defined. On the other hand, pure substitution stacks another a to the weight of the output tree computed. It can be shown that among those infinitely many input trees which q accepts, there are two for which the weights a^{n_1} and a^{n_2} of their corresponding output trees is different (this is mainly due to the fact that arbitrarily large powers of a can occur). Since all the powers of a are different, there is no consistent way to define $\mu^1_{\gamma_2}(q)$. Similarly, when $\mathrm{mod}_2 = \varepsilon$ one encounters the problem that o-substitution resets the weight to $\mathbf{1}$, whenever a γ_2 is read in the input. The above remarks about the weights a^{n_1} and a^{n_2} apply as well and in order to define $\mu^1_{\gamma_2}(q)$ in this case there should be an element $b \in A$ such that $a^{n_1} \odot b = \mathbf{1} = a^{n_2} \odot b$ which is shown to be contradictory. Summing up, with pure substitution one can remember the number of γ_2 encountered in the whole input tree even if a part of the transformation of the input tree was deleted. On the other hand, using o-substitution when deleting a computed output tree, we can easily reset the weight to a determined value irrespective of the weight of the output tree computed so far.

The arguments required for the result on nondeleting homomorphism bu-w-tt are similar, but use copying instead of deletion. In principle, pure substitution has the problem that it is supposed to square the weight of the computed output tree. However, those output trees may have infinitely many different weights, so that this information cannot be stored in the states and there is no element $b \in A$ which squares a^{n_1} and a^{n_2}, i.e., $a^{2n_1} = a^{n_1} \odot b$ and $a^{2n_2} = a^{n_2} \odot b$, for suitable $n_1, n_2 \in \mathbb{N}$. Conversely, o-substitution squares the weight of the computed output tree and therefore needs an element which when multiplied to a^{2n_1} and a^{2n_2} computes their square roots. It is shown that for selected $n_1, n_2 \in \mathbb{N}$ such an element cannot exist.

Lemma 4.7. *Let $\mathcal{A}$ be a non-periodic monoid. For every $\pi \in \{\mathrm{hn}, \mathrm{hl}\}$ and $\{\mathrm{mod}_1, \mathrm{mod}_2\} = \{\varepsilon, o\}$ we have*

$$\pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \enspace .$$

Proof. Since $\mathcal{A}$ is non-periodic, there exists an $a \in A$ such that for every $i, j \in \mathbb{N}$ we have $a^i = a^j$, if and only if $i = j$. Further let $\Delta = \{\gamma^{(1)}, \alpha^{(0)}\}$. Let us prove the statement by case analysis on π. Case (1) considers the case where $\pi = \mathrm{hl}$ and Case (2) supposes $\pi = \mathrm{hn}$.

1. Let $\Gamma = \{\gamma_1^{(1)}, \gamma_2^{(1)}, \alpha^{(0)}\}$. We construct the linear homomorphism bu-w-tt

$$M_1 = (\{\star\}, \Gamma, \Delta, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$$

with $\mu_1 = ((\mu_1)^1_{\gamma_1}, (\mu_1)^1_{\gamma_2}, (\mu_1)^0_{\alpha})$ specified by

$$(\mu_1)^1_{\gamma_1}(\star) = a\,\gamma(x_1) \quad , \quad (\mu_1)^1_{\gamma_2}(\star) = (\mu_1)^0_{\alpha}() = \mathbf{1}\,\alpha \enspace .$$

Moreover, we define $l_1 : T_\Gamma \longrightarrow \mathbb{N}$ recursively for every $t \in T_\Gamma$ as follows.

$$l_1(\gamma_1(t)) = l_1(t) + 1 \quad \text{and} \quad l_1(\gamma_2(t)) = l_1(\alpha) = 0 \enspace .$$

Note that M_1 computes the t-ts transformation $\tau_{M_1} : T_\Gamma \longrightarrow A[T_\Delta]$ mapping every $s \in T_\Gamma$ to the monomial $a^{|s|_{\gamma_1}}\, \gamma^{l_1(s)}(\alpha)$, and the o-t-ts transformation $\tau^o_{M_1} : T_\Gamma \longrightarrow A[T_\Delta]$ mapping s to the monomial $a^{l_1(s)}\, \gamma^{l_1(s)}(\alpha)$.
Next, we prove that $\tau^{\mathrm{mod}_1}_{M_1} \notin \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, which yields

$$\text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \nsubseteq \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \enspace .$$

Suppose there exists a deterministic bu-w-tt $M = (Q, \Gamma, \Delta, \mathcal{A}, F, \delta, \mu)$ with $\tau^{\mathrm{mod}_2}_M = \tau^{\mathrm{mod}_1}_{M_1}$. We observe that for every $s \in T_\Gamma$ we have that $\tau^{\mathrm{mod}_1}_{M_1}(s) \neq \widetilde{\mathbf{0}}$, and thus, $\tau^{\mathrm{mod}_2}_M(s) = \widehat{\mu}_{\mathrm{mod}_2}(s)$ as well as $\widehat{\delta}(s) \in F$. (Note that if $a^n = \mathbf{0}$ for some $n \in \mathbb{N}$, then $a^n = a^{n+1}$ which contradicts to our assumption.) Next we prove that there are $q \in F$ and $s_1, s_2 \in T_\Gamma$ such that $\widehat{\delta}(s_1) = q = \widehat{\delta}(s_2)$ and $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$ and $l_1(s_1) \neq l_1(s_2)$. Therefore, we let

$\Gamma' = \{\gamma_1^{(1)}, \alpha^{(0)}\} \subset \Gamma$, hence $T_{\Gamma'} \subseteq T_\Gamma$. We show that s_1 and s_2 can actually be chosen from $T_{\Gamma'}$. Clearly, there exist $q \in F$ and an infinite set $S \subseteq T_{\Gamma'}$ such that $q = \widehat{\delta}(s)$ for every $s \in S$, because Q is finite whereas $T_{\Gamma'}$ is infinite. For every $s \in S$ we have $\text{size}(s) = |s|_{\gamma_1} + 1 = l_1(s) + 1$, because $S \subseteq T_{\Gamma'}$. We observe that $[s]_{\equiv_{\text{size}}}$, where $s \equiv_{\text{size}} s'$ if and only if $\text{size}(s) = \text{size}(s')$, is finite for every $s \in S$, hence by the pigeon-hole principle there must exist $s_1, s_2 \in S$ such that $\text{size}(s_1) \neq \text{size}(s_2)$, i.e., $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$ and $l_1(s_1) \neq l_1(s_2)$.
Hence we can assume that there exist $q \in F$ and $s_1, s_2 \in T_\Gamma$ such that $\widehat{\delta}(s_1) = q = \widehat{\delta}(s_2)$ and $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$ and $l_1(s_1) \neq l_1(s_2)$. Since

$$\text{supp}(\tau_{M_1}^{\text{mod}_1}(\gamma_2(s_1))) = \text{supp}(\tau_{M_1}^{\text{mod}_1}(\gamma_2(s_2))) = \{\alpha\} ,$$

and

$$\tau_M^{\text{mod}_2}(\gamma_2(s_i)) = \widehat{\mu}_{\text{mod}_2}(\gamma_2(s_i)) = \mu^1_{\gamma_2}(q) \quad \overset{\text{mod}_2}{\longleftarrow} \quad (\widehat{\mu}_{\text{mod}_2}(s_i))$$

for every $i \in [2]$, we have $\mu^1_{\gamma_2}(q) \neq \widetilde{\mathbf{0}}$, and thereby, $\mu^1_{\gamma_2}(q) = a'\ t$ for some $a' \in A \setminus \{\mathbf{0}\}$ and $t \in T_\Delta(X_1)$. Next we prove that $t = \alpha$. Since $\tau_M^{\text{mod}_2} = \tau_{M_1}^{\text{mod}_1}$ we have that

$$\text{supp}(\tau_{M_1}^{\text{mod}_1}(s_i)) = \text{supp}(\widehat{\mu}_{\text{mod}_2}(s_i)) = \text{supp}(\tau_M^{\text{mod}_2}(s_i)) = \{\gamma^{l_1(s_i)}(\alpha)\} .$$

Then

$$\begin{aligned} \alpha &= \text{supp}(\tau_{M_1}^{\text{mod}_1}(\gamma_2(s_i))) = \text{supp}(\tau_M^{\text{mod}_2}(\gamma_2(s_i))) \\ &= \text{supp}(\mu^1_{\gamma_2}(q) \quad \overset{\text{mod}_2}{\longleftarrow} \quad (\widehat{\mu}_{\text{mod}_2}(s_i))) \\ &= t[\gamma^{l_1(s_i)}(\alpha)] . \end{aligned}$$

Now using $l_1(s_1) \neq l_1(s_2)$ we conclude $|t|_{x_1} = 0$, thus finally, $t = \alpha$.
We obtain for every $i \in [2]$

$$\begin{aligned} \tau_M^{\text{mod}_2}(\gamma_2(s_i)) &= a'\ \alpha \quad \overset{\text{mod}_2}{\longleftarrow} \quad (\tau_{M_1}^{\text{mod}_1}(s_i)) \\ &= \begin{cases} (a' \odot a^{l_1(s_i)})\ \alpha & \text{, if } \text{mod}_2 = \varepsilon \\ a'\ \alpha & \text{, if } \text{mod}_2 = o . \end{cases} \end{aligned}$$

Recall now that $\text{mod}_1 \neq \text{mod}_2$ and $\tau_{M_1}(\gamma_2(s_i)) = a^{|s_i|_{\gamma_1}}\ \alpha$ and

$$\tau^o_{M_1}(\gamma_2(s_i)) = a^{l_1(\gamma_2(s_i))}\ \alpha = \mathbf{1}\ \alpha .$$

Hence for every $i \in [2]$ we derive the equation

$$\begin{aligned} a' \odot a^{l_1(s_i)} = \mathbf{1} \quad &= (\tau^o_{M_1}(\gamma_2(s_i)), \alpha) \quad \text{, if } \text{mod}_2 = \varepsilon \\ a' = a^{|s_i|_{\gamma_1}} \quad &= (\tau_{M_1}(\gamma_2(s_i)), \alpha) \quad \text{, if } \text{mod}_2 = o . \end{aligned}$$

In case $\text{mod}_2 = o$ this yields a contradiction since $a' = a^{|s_1|_{\gamma_1}} = a^{|s_2|_{\gamma_1}}$, which apparently is contradictory due to $a^{|s_1|_{\gamma_1}} \neq a^{|s_2|_{\gamma_1}}$ by $|s_1|_{\gamma_1} \neq |s_2|_{\gamma_1}$. Finally, in the other case, i.e., $\text{mod}_2 = \varepsilon$, we effectively have

$$\mathbf{1} = a' \odot a^{l_1(s_1)} = a' \odot a^{l_1(s_2)} .$$

Now let $y_1 = \min(l_1(s_1), l_1(s_2))$, $y_2 = \max(l_1(s_1), l_1(s_2))$, and $d = y_2 - y_1$. Obviously, $y_1 \neq y_2$ and thereby $d \neq 0$ by $l_1(s_1) \neq l_1(s_2)$. We consider

$$\mathbf{1} = a' \odot a^{y_2} = a' \odot a^{y_1+d} = a' \odot a^{y_1} \odot a^d = \mathbf{1} \odot a^d = a^d \ ,$$

however $\mathbf{1} = a^0 = a^d$, if and only if $0 = d$, which is a contradiction. Irrespective of mod_2 we have thus proved that there is no deterministic bu-w-tt M having the property that $\tau_M^{\mathrm{mod}_2} = \tau_{M_1}^{\mathrm{mod}_1}$. Thus $\tau_{M_1}^{\mathrm{mod}_1} \notin \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

2. Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We define the nondeleting homomorphism bu-w-tt

$$M_2 = (\{\star\}, \Delta, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$$

with $\mu_2 = ((\mu_2)^1_\gamma, (\mu_2)^0_\alpha)$ given by

$$(\mu_2)^1_\gamma(\star) = a\, \sigma(x_1, x_1) \quad , \quad (\mu_2)^0_\alpha() = a\, \alpha \ .$$

For every $s \in T_\Delta$ let $t_s \in T_\Sigma$ be the fully balanced output tree with $\mathrm{height}(t_s) = \mathrm{height}(s)$. The t-ts transformation $\tau_{M_2} : T_\Delta \longrightarrow A[T_\Sigma]$ computed by M_2 maps s to $a^{\mathrm{size}(s)}\, t_s$, whereas the o-t-ts transformation $\tau^o_{M_2} : T_\Delta \longrightarrow A[T_\Sigma]$ computed by M_2 maps s to $a^{\mathrm{size}(t_s)}\, t_s$. Note that $\mathrm{size}(t_s) = 2^{\mathrm{size}(s)} - 1$.

Let us prove $\tau_{M_2}^{\mathrm{mod}_1} \notin \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A})$, thereby showing

$$\text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \ .$$

To derive a contradiction assume a deterministic bu-w-tt $M = (Q, \Delta, \Sigma, \mathcal{A}, F, \delta, \mu)$ such that $\tau_M^{\mathrm{mod}_2} = \tau_{M_2}^{\mathrm{mod}_1}$.

We again observe that for every $s \in T_\Delta$ we have $\tau_{M_2}^{\mathrm{mod}_1}(s) \neq \widetilde{\mathbf{0}}$, and thus, $\tau_M^{\mathrm{mod}_2}(s) = \widehat{\mu}_{\mathrm{mod}_2}(s)$ as well as $\widehat{\delta}(s) \in F$. Moreover, T_Δ is infinite. In contrast M has only a finite set of final states F; hence there must exist a final state $q \in F$ and $s_1, s_2 \in T_\Delta$ with $q = \widehat{\delta}(s_i)$ and $s_1 \neq s_2$ such that $t_{s_i} \in \mathrm{supp}(\widehat{\mu}_{\mathrm{mod}_2}(s_i))$ for $i \in [2]$. Since $s_1 \neq s_2$ we also have $\mathrm{size}(s_1) \neq \mathrm{size}(s_2)$ and $t_{s_1} \neq t_{s_2}$.

Apparently, $\widehat{\mu}_{\mathrm{mod}_2}(\gamma(s_i)) = \mu^1_\gamma(q) \overset{\mathrm{mod}_2}{\longleftarrow} (\tau_{M_2}^{\mathrm{mod}_1}(s_i))$, and furthermore, also $\tau_{M_2}^{\mathrm{mod}_1}(\gamma(s_i)) \neq \widetilde{\mathbf{0}}$, hence $\widehat{\delta}(\gamma(s_i)) \in F$ and $\mu^1_\gamma(q) \neq \widetilde{\mathbf{0}}$. Let $\mu^1_\gamma(q) = a'\, t$ for some $a' \in A \setminus \{\mathbf{0}\}$ and $t \in T_\Sigma(X_1)$.

Next, we observe that $t = \sigma(x_1, x_1)$, which can easily be proved by contradiction as follows. Assume that $t \neq \sigma(x_1, x_1)$. Then for some $j \in [2]$ the tree $t[t_{s_j}]$ is not fully balanced or its height is not $1 + \mathrm{height}(t_{s_j})$, because $t_{s_1} \neq t_{s_2}$. Hence we obtain for every $i \in [2]$

$$\begin{aligned}
\tau_M^{\mathrm{mod}_2}(\gamma(s_i)) &= a'\, \sigma(x_1, x_1) \overset{\mathrm{mod}_2}{\longleftarrow} (\tau_{M_2}^{\mathrm{mod}_1}(s_i)) \\
&= \begin{cases} (a' \odot a^{\mathrm{size}(t_{s_i})})\, \sigma(t_{s_i}, t_{s_i}) & \text{, if } \mathrm{mod}_2 = \varepsilon \\ (a' \odot a^{2 \cdot \mathrm{size}(s_i)})\, \sigma(t_{s_i}, t_{s_i}) & \text{, if } \mathrm{mod}_2 = o \ . \end{cases}
\end{aligned}$$

Recall that

$$\begin{aligned}
\tau_{M_2}(\gamma(s_i)) &= a^{\mathrm{size}(s_i)+1}\, \sigma(t_{s_i}, t_{s_i}) \\
\tau^o_{M_2}(\gamma(s_i)) &= a^{2 \cdot \mathrm{size}(t_{s_i})+1}\, \sigma(t_{s_i}, t_{s_i}) \ .
\end{aligned}$$

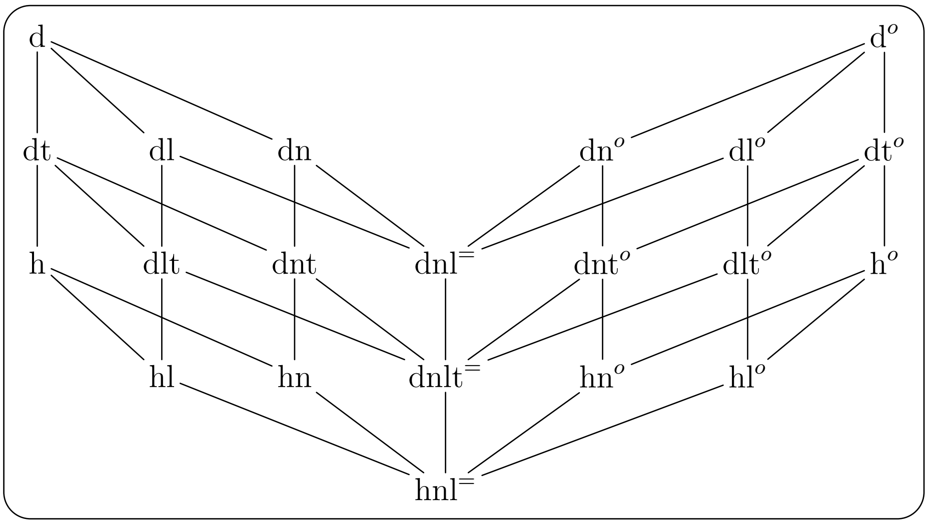

Figure 2: Hasse diagram for non-periodic monoids.

Hence for every $i \in [2]$ we derive the equation

$$\begin{array}{llll} a' \odot a^{\mathrm{size}(t_{s_i})} = a^{2\cdot\mathrm{size}(t_{s_i})+1} & = (\tau^o_{M_2}(\gamma(s_i)), \sigma(t_{s_i}, t_{s_i})) & , \text{if } \mathrm{mod}_2 = \varepsilon \\ a' \odot a^{2\cdot\mathrm{size}(s_i)} = a^{\mathrm{size}(s_i)+1} & = (\tau_{M_2}(\gamma(s_i)), \sigma(t_{s_i}, t_{s_i})) & , \text{if } \mathrm{mod}_2 = o \ . \end{array}$$

For every $i \in [2]$ we let $y_i = \mathrm{size}(t_{s_i})$, if $\mathrm{mod}_2 = \varepsilon$, whereas we let $y_i = \mathrm{size}(s_i)$ in case $\mathrm{mod}_2 = o$. Note that in both cases $y_1 \neq y_2$. We continue with the equations

$$\begin{array}{llll} a^{y_1+2\cdot y_2+1} = a' \odot a^{y_2} \odot a^{y_1} & = a^{2\cdot y_1+y_2+1} & , \text{if } \mathrm{mod}_2 = \varepsilon \\ a^{y_1+2\cdot y_2+1} = a' \odot a^{2\cdot y_1} \odot a^{2\cdot y_2} & = a^{2\cdot y_1+y_2+1} & , \text{if } \mathrm{mod}_2 = o \ . \end{array}$$

Thus in any case $a^{y_1+2\cdot y_2+1} = a^{2\cdot y_1+y_2+1}$. Since $a^i \neq a^j$ whenever $i \neq j$ for all $i, j \in \mathbb{N}$, we conclude $y_1 + 2 \cdot y_2 + 1 = 2 \cdot y_1 + y_2 + 1$ and thereby $y_1 = y_2$ which contradicts to $y_1 \neq y_2$. Consequently, irrespective of mod_2 we have proved that there is no deterministic bu-w-tt M having the property that $\tau^{\mathrm{mod}_2}_M = \tau^{\mathrm{mod}_1}_{M_2}$. Thus $\tau^{\mathrm{mod}_1}_{M_2} \notin \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A})$.

□

Together with the results of Section 4.1, we can already derive the Hasse diagram (cf. Figure 2) for non-periodic monoids. We observe that the classes of t-ts and o-t-ts transformations are incomparable, whenever inclusion is not trivial by definition or given as a result of Proposition 4.4.

Theorem 4.8. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a non-periodic monoid with an absorbing element $\mathbf{0}$. Figure 2 is the* Hasse *diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.*

Proof. All the inclusions are trivial and the equalities are due to Proposition 4.4. Then for every $\{\mathrm{mod}_1, \mathrm{mod}_2\} = \{\varepsilon, o\}$ the following six statements are sufficient to prove strictness and incomparability.

$$\begin{aligned} \text{dnlt–BOT}(\mathcal{A}) &\not\subseteq \text{h–BOT}^{\mathrm{mod}_1}(\mathcal{A}) && (10)\\ \text{dnl–BOT}(\mathcal{A}) &\not\subseteq \text{dt–BOT}^{\mathrm{mod}_1}(\mathcal{A}) && (11)\\ \text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) &\not\subseteq \text{dl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) && (12)\\ \text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) &\not\subseteq \text{dn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) && (13)\\ \text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) &\not\subseteq \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A}) && (14)\\ \text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) &\not\subseteq \text{d–BOT}^{\mathrm{mod}_2}(\mathcal{A}) && (15) \end{aligned}$$

The inequalities (10)–(13) are proved in Lemma 4.3, whereas inequalities (14) and (15) follow from Lemma 4.7. □

4.3. Periodic and commutative monoids

In this section, we consider monoids which are periodic and commutative. For example, the monoid $\mathbb{Z}_4$ is periodic and commutative (without being regular). It is easily seen that in commutative and periodic monoids $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ the carrier set $\langle A' \rangle_\odot$ of the least submonoid with the absorbing element $\mathbf{0}$ generated from a finite set $A' \subseteq A$ is again finite. This property is essential in the core construction of this section, because it allows us to keep track of the current weight in the states.

Proposition 4.9. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative and periodic monoid. For every finite $A' \subseteq A$ we have that $\langle A' \rangle_\odot$ is finite.*

Proof. We first observe that $\langle \emptyset \rangle_\odot = \{\mathbf{0}, \mathbf{1}\}$. Let $A' = \{a_1, \dots, a_k\} \subseteq A$ for some $k \in \mathbb{N}_+$. Then

$$\begin{aligned} \langle A' \rangle_\odot &= \{ a_1^{i_1} \odot \cdots \odot a_k^{i_k} \mid i_1, \dots, i_k \in \mathbb{N} \} \\ &= \{ a_1^{i_1} \odot \cdots \odot a_k^{i_k} \mid i_1 \in [0, n_1], \dots, i_k \in [0, n_k] \} \ , \end{aligned}$$

where for every $j \in [k]$ the integer $n_j \in \mathbb{N}$ is the smallest non-negative integer such that there exists $m_j \in \mathbb{N}$ with $m_j < n_j$ and $a_j^{n_j} = a_j^{m_j}$. Hence $\langle A' \rangle_\odot$ is a finite set. □

Given a deterministic bu-w-tt computing a t-ts transformation τ, we construct another deterministic bu-w-tt computing τ as o-t-ts transformation. Moreover, most of the restrictions defined for deterministic bu-w-tt (namely nondeletion, linearity, and totality) are preserved by this construction. However, a homomorphism bu-w-tt might yield a non-homomorphism bu-w-tt, because the construction increases the state-space compared to the given bu-w-tt.

The next definition abstracts the central feature required to model one type of substitution with the help of the other. We encapsulate this feature in a family of mappings in order to be able to invoke the construction later under different premises. More precisely, in subsequent lemmata we prove that such a family of mappings exists provided that the monoid has certain properties, e.g., is a group.

Definition 4.10. Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a monoid, $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \delta, \mu)$ be a deterministic bu-w-tt, and $\text{mod} \in \{\varepsilon, o\}$. Further, let $f_{M,\text{mod}} = (f^k_{M,\text{mod}})_{k\in\mathbb{N}}$ be a family of mappings where for every $k \in \mathbb{N}$ we have

$$f^k_{M,\text{mod}} : \Big(\bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q} \text{supp}(\mu^k_\sigma(q_1, \ldots, q_k))\Big) \times [k] \times A \longrightarrow A \ .$$

If f satisfies for every $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q} \text{supp}(\mu^k_\sigma(q_1, \ldots, q_k))$, $i \in [k]$, and $a \in A$ the statements

(i) $f^k_{M,\text{mod}}(t, i, a) = \mathbf{0}$, if $a = \mathbf{0}$,
(ii) $f^k_{M,\text{mod}}(t, i, a) \odot a^{|t|_{x_i}} = a$, if $\text{mod} = \varepsilon$, and
(iii) $f^k_{M,\text{mod}}(t, i, a) \odot a = a^{|t|_{x_i}}$, if $\text{mod} = o$,

then f is called a *family of* mod*-translation mappings* for M.

Let $\text{mod}_1, \text{mod}_2 \in \{\varepsilon, o\}$. For every deterministic bu-w-tt M_1, for which there exists a family of mod_1-translation mappings, we can construct another deterministic bu-w-tt M_2 computing the mod_2-t-ts transformation $\tau^{\text{mod}_2}_{M_2} = \tau^{\text{mod}_1}_{M_1}$. Due to the periodicity and commutativity of the monoid $\mathcal{A}$, the set of computable weights is finite (cf. Proposition 4.9). Let $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$. Given $s \in T_\Sigma$, we have already seen that $\widehat{\mu_1}_{\text{mod}_1}(s) = at$ for some $a \in A$ and $t \in T_\Delta$. Since the set of computable weights is finite, we can encode a into the state, i.e., we can construct a deterministic bu-w-tt

$$M'_1 = (Q'_1, \Sigma, \Delta, \mathcal{A}, F'_1, \delta'_1, \mu'_1)$$

such that $\tau^{\text{mod}_1}_{M_1} = \tau^{\text{mod}_1}_{M'_1}$ and $\widehat{\delta'_1}(s) = (\widehat{\delta_1}(s), a)$ and $\widehat{\mu'_1}_{\text{mod}_1}(s) = a\,t$.

Let us take a closer look at a family of translation mappings. Let $\text{mod}_1 = o$. Then, when substituting an output tree weighted a into a tree t for variable x_i, o-substitution accounts a exactly $|t|_{x_i}$-times, whereas pure substitution accounts a exactly once. In Item (iii) of Definition 4.10 we see that $f^k_{M'_1,o}(t, i, a)$ provides the factor which translates the pure substitution coefficient into the o-substitution coefficient, because $f^k_{M'_1,o}(t, i, a) \odot a = a^{|t|_{x_i}}$. So we need to multiply $f^k_{M'_1,o}(t, i, a)$ to the weight of the considered transition. This is possible, because a is encoded in the state, in which the bu-w-tt M'_1 processed the i-th direct input subtree of s. In this way we can define the weight of the transitions using the weight of the subcomputations.

Lemma 4.11. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative monoid and $\text{mod}_1, \text{mod}_2 \in \{\varepsilon, o\}$. Moreover, let $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ be a deterministic bu-w-tt obeying all the restrictions of $\pi \in \Pi \setminus \Pi_h$. If there exists a family of mod_1-translation mappings $f_{M_1,\text{mod}_1} = (f^k_{M_1,\text{mod}_1})_{k\in\mathbb{N}}$, there also exists a deterministic bu-w-tt $M_2 = (Q_2, \Sigma, \Delta, \mathcal{A}, F_2, \delta_2, \mu_2)$ obeying the restrictions of π such that $\tau^{\text{mod}_1}_{M_1} = \tau^{\text{mod}_2}_{M_2}$.*

Proof. If $\text{mod}_1 = \text{mod}_2$, then the statement becomes trivial. So it remains to prove the property for distinct mod_1 and mod_2. Let

$$C = \{\mathbf{0}\} \cup \left\{ ((\mu_1)^k_\sigma(q_1, \ldots, q_k), t) \,\middle|\, \begin{array}{l} k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1, \\ t \in \text{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k)) \end{array} \right\}$$

be the finite set of monoid elements occurring in the monomials in the range of μ_1. Since $\mathcal{A}$ is periodic and commutative, we conclude that $\langle C \rangle_\odot$ is finite. We construct the bu-w-tt M_2 by setting the set Q_2 of states to $Q_2 = Q_1 \times \langle C \rangle_\odot$ and the set F_2 of final states to $F_2 = F_1 \times \langle C \rangle_\odot$. Moreover, let $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $q_1, \ldots, q_k \in Q_1$, and $a_1, \ldots, a_k \in \langle C \rangle_\odot$. Now we define a and the monomial m as follows. If $(\mu_1)^k_\sigma(q_1, \ldots, q_k) = \widetilde{\mathbf{0}}$ or for some $i \in [k]$ we have $a_i = \mathbf{0}$, then let $a = \mathbf{0}$ and $m = \widetilde{\mathbf{0}}$. Otherwise suppose that $(\mu_1)^k_\sigma(q_1, \ldots, q_k) = a_0\, t$ for $a_0 \in C \setminus \{\mathbf{0}\}$ and $t \in T_\Delta(X_k)$ and let

$$a = \begin{cases} a_0 \odot a_1 \odot \cdots \odot a_k & \text{, if } \mathrm{mod}_1 = \varepsilon \\ a_0 \odot a_1^{|t|_{x_1}} \odot \cdots \odot a_k^{|t|_{x_k}} & \text{, if } \mathrm{mod}_1 = o \end{cases}$$

and $m = (f^k_{M_1,\mathrm{mod}_1}(t, 1, a_1) \odot \cdots \odot f^k_{M_1,\mathrm{mod}_1}(t, k, a_k) \odot a_0)\, t$. Clearly, $a \in \langle C \rangle_\odot$, so we let

$$\begin{aligned} (\delta_2)^k_\sigma((q_1, a_1), \ldots, (q_k, a_k)) &= ((\delta_1)^k_\sigma(q_1, \ldots, q_k), a) \\ (\mu_2)^k_\sigma((q_1, a_1), \ldots, (q_k, a_k)) &= m \ . \end{aligned}$$

Obviously, M_2 is nondeleting (respectively, linear and total), if M_1 is nondeleting (respectively, linear and total). Let $s \in T_\Sigma$. Finally, suppose that $\widehat{\mu_1}_{\mathrm{mod}_1}(s) = a\, t$ for some $a \in \langle C \rangle_\odot$ and $t \in T_\Delta$. We show that the following equalities hold.

$$\widehat{\mu_2}_{\mathrm{mod}_2}(s) = \widehat{\mu_1}_{\mathrm{mod}_1}(s) \quad \text{and} \quad \widehat{\delta_2}(s) = (\widehat{\delta_1}(s), a) \ .$$

1. Let $s = \alpha$ with $\alpha \in \Sigma^{(0)}$. Then

$$\widehat{\mu_2}_{\mathrm{mod}_2}(s) = (\mu_2)^0_\alpha() = (\mu_1)^0_\alpha() = \widehat{\mu_1}_{\mathrm{mod}_1}(s) \ .$$

   Moreover, $\widehat{\delta_2}(s) = (\delta_2)^0_\alpha() = ((\delta_1)^0_\alpha(), a') = (\widehat{\delta_1}(s), a')$ where

$$\begin{aligned} a' &= \begin{cases} \mathbf{0} & \text{, if } \mathrm{supp}((\mu_1)^0_\alpha()) = \emptyset \\ ((\mu_1)^0_\alpha, t') & \text{, if } \mathrm{supp}((\mu_1)^0_\alpha()) = \{t'\} \end{cases} \\ &= \begin{cases} \mathbf{0} & \text{, if } \mathrm{supp}(\widehat{\mu_1}_{\mathrm{mod}_1}(\alpha)) = \emptyset \\ (\widehat{\mu_1}_{\mathrm{mod}_1}, t') & \text{, if } \mathrm{supp}(\widehat{\mu_1}_{\mathrm{mod}_1}(\alpha)) = \{t'\} \end{cases} \\ &= a \ . \end{aligned}$$

2. Let $s = \sigma(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$. Then we have

$$\begin{aligned} &\widehat{\mu_2}_{\mathrm{mod}_2}(s) \\ &= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_2}_{\mathrm{mod}_2}(s_1), \ldots, \widehat{\mu_2}_{\mathrm{mod}_2}(s_k)) \\ &= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \ . \end{aligned}$$

   For every $i \in [k]$ let $\widehat{\mu_1}_{\mathrm{mod}_1}(s_i) = a_i\, t_i$ for some $a_i \in \langle C \rangle_\odot$ and $t_i \in T_\Delta$. By induction hypothesis we have further that $\widehat{\delta_2}(s_i) = (\widehat{\delta_1}(s_i), a_i)$.

   (a) In the first case, let $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = \widetilde{\mathbf{0}}$ or for some $i \in [k]$ let $a_i = \mathbf{0}$. Then by construction we obtain $(\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) = \widetilde{\mathbf{0}}$. Hence

$$\widehat{\mu_1}_{\mathrm{mod}_1}(s) = \widetilde{\mathbf{0}} = \widehat{\mu_2}_{\mathrm{mod}_2}(s) \ .$$

   (b) Let $a_0 \in C \setminus \{\mathbf{0}\}$ and $t' \in T_\Delta(X_k)$ be such that

$$(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = a_0\, t' \ .$$

We deduce

$$
\begin{aligned}
&\widehat{\mu_2}_{\mathrm{mod}_2}(s) \\
&= (\mu_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= (\mu_2)^k_\sigma((\widehat{\delta_1}(s_1), a_1), \ldots, (\widehat{\delta_1}(s_k), a_k)) \xleftarrow{\mathrm{mod}_2} \\
&\qquad \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= \Big(\prod_{i \in [k]} f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_0\Big)\, t' \xleftarrow{\mathrm{mod}_2} \\
&\qquad \xleftarrow{\mathrm{mod}_2} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= \Big(\prod_{i \in [k]} f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_0 \odot a_1^{m_1} \odot \cdots \odot a_k^{m_k}\Big)\, t'[t_1, \ldots, t_k] \\
&= \Big(\prod_{i \in [k]} (f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_i^{m_i}) \odot a_0\Big)\, t'[t_1, \ldots, t_k]
\end{aligned}
$$

where for every $i \in [k]$ we let

$$
m_i = \begin{cases} 1 & \text{, if } \mathrm{mod}_2 = \varepsilon \\ |t'|_{x_i} & \text{, if } \mathrm{mod}_2 = o \end{cases} .
$$

Recall that our general assumption was $\mathrm{mod}_1 \neq \mathrm{mod}_2$, so we now distinguish two cases, in each of which we take a closer look at the product $f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_i^{m_i}$ for every $i \in [k]$. Firstly, let $\mathrm{mod}_1 = \varepsilon$. Then $f^k_{M_1,\varepsilon}(t', i, a_i) \odot a_i^{|t'|_{x_i}} = a_i$ by Definition 4.10(ii). On the other hand, let $\mathrm{mod}_1 = o$. Immediately we obtain

$$
f^k_{M_1,o}(t', i, a_i) \odot a_i = a_i^{|t'|_{x_i}}
$$

by Definition 4.10(iii). Hence we continue with

$$
\begin{aligned}
&\widehat{\mu_2}_{\mathrm{mod}_2}(s) \\
&= \Big(\prod_{i \in [k]} (f^k_{M_1,\mathrm{mod}_1}(t', i, a_i) \odot a_i^{m_i}) \odot a_0\Big)\, t'[t_1, \ldots, t_k] \\
&= a_0\, t' \xleftarrow{\mathrm{mod}_1} (a_1\, t_1, \ldots, a_k\, t_k) \\
&= (\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) \xleftarrow{\mathrm{mod}_1} (\widehat{\mu_1}_{\mathrm{mod}_1}(s_1), \ldots, \widehat{\mu_1}_{\mathrm{mod}_1}(s_k)) \\
&= \widehat{\mu_1}_{\mathrm{mod}_1}(s) .
\end{aligned}
$$

This concludes the proof of the first property.

Let $\widehat{\mu_1}_{\mathrm{mod}_1}(s) = a\ t$ for some $a \in \langle C \rangle_\odot$ and $t \in T_\Delta$. Thus it remains to show that $\widehat{\delta_2}(s) = (\widehat{\delta_1}(s), a)$. In a straightforward manner we derive

$$
\begin{aligned}
\widehat{\delta_2}(s) &= (\delta_2)^k_\sigma(\widehat{\delta_2}(s_1), \ldots, \widehat{\delta_2}(s_k)) \\
&= (\delta_2)^k_\sigma((\widehat{\delta_1}(s_1), a_1), \ldots, (\widehat{\delta_1}(s_k), a_k)) \\
&= ((\delta_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)), a')
\end{aligned}
$$

$$= \quad (\widehat{\delta_1}(s), a') \ ,$$

where $a' = \mathbf{0}$, if $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = \widetilde{\mathbf{0}}$ or for some $i \in [k]$ we have $a_i = \mathbf{0}$. Hence $a' = a$. Otherwise let $(\mu_1)^k_\sigma(\widehat{\delta_1}(s_1), \ldots, \widehat{\delta_1}(s_k)) = a_0\, t'$ for some $a_0 \in C \setminus \{\mathbf{0}\}$ and $t' \in T_\Delta(X_k)$. Consequently,

$$a' = \begin{cases} a_0 \odot a_1 \odot \cdots \odot a_k & , \text{if } \mathrm{mod}_1 = \varepsilon \\ a_0 \odot a_1^{|t'|_{x_1}} \odot \cdots \odot a_k^{|t'|_{x_k}} & , \text{if } \mathrm{mod}_1 = o \ . \end{cases}$$

Hence $\widehat{\mu_2}_{\mathrm{mod}_2}(s) = \widehat{\mu_1}_{\mathrm{mod}_1}(s) = a'\, t'[t_1, \ldots, t_k]$ and $a = a'$, which concludes the proof of the statement.

□

The next lemma shows that in case we have a nondeleting (respectively, linear) deterministic bu-w-tt, then we can specify a family of mod-translation mappings with mod $= o$ (respectively, mod $= \varepsilon$) and then apply the previous lemma to obtain an inclusion result.

Lemma 4.12. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative monoid and* $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$. *We have π–$\mathrm{BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}^{\mathrm{mod}_2}(\mathcal{A})$ for every $\pi \in P$ where*

$$P = \begin{cases} \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}} & , \textit{if } \mathrm{mod}_1 = o \\ \Pi_{\mathrm{l}} \setminus \Pi_{\mathrm{h}} & , \textit{if } \mathrm{mod}_1 = \varepsilon \ . \end{cases}$$

Proof. Trivially the statement holds, if $\mathrm{mod}_1 = \mathrm{mod}_2$. Thus assume that mod_1 and mod_2 are distinct.

1. Let $\mathrm{mod}_1 = o$ and $\tau^o \in \pi$–$\mathrm{BOT}^o(\mathcal{A})$ for some $\pi \in \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a nondeleting deterministic bu-w-tt

$$M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$$

obeying the restrictions of π such that $\tau^o_{M_1} = \tau^o$. Let $f_{M_1,o} = (f^k_{M_1,o})_{k\in\mathbb{N}}$ be the family of mappings

$$f^k_{M_1,o} : \Big(\bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k)) \Big) \times [k] \times A \longrightarrow A$$

defined for every $k \in \mathbb{N}$, $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,o}(t, i, a) = \begin{cases} \mathbf{0} & , \text{if } a = \mathbf{0} \\ a^{|t|_{x_i} - 1} & , \text{otherwise} \ . \end{cases}$$

Each $f^k_{M_1,o}(t, i, a)$ is well-defined, because by the nondeletion restriction we have $|t|_{x_i} \geq 1$ for every $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$ and $i \in [k]$. Consequently, the exponent is non-negative in the definition of $f^k_{M_1,o}(t, i, a)$. Moreover, $f_{M_1,o}$ is trivially a family of o-translation mappings. Thus, due to Lemma 4.11, there exists a nondeleting deterministic bu-w-tt M_2 obeying the restrictions of π such that $\tau_{M_2} = \tau^o$. Hence π–$\mathrm{BOT}^o(\mathcal{A}) \subseteq \pi$–$\mathrm{BOT}(\mathcal{A})$ for every $\pi \in \Pi_{\mathrm{n}} \setminus \Pi_{\mathrm{h}}$.

2. Secondly, let $\mathrm{mod}_1 = \varepsilon$ and $\tau \in \pi$–BOT($\mathcal{A}$) for some $\pi \in \Pi_l \setminus \Pi_h$. Consequently, there exists a linear deterministic bu-w-tt

$$M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$$

obeying the restrictions of π such that $\tau_{M_1} = \tau$. Let $f_{M_1,\varepsilon} = (f^k_{M_1,\varepsilon})_{k\in\mathbb{N}}$ be the family of mappings

$$f^k_{M_1,\varepsilon} : \Big(\bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1,\ldots,q_k))\Big) \times [k] \times A \longrightarrow A$$

defined for every $k \in \mathbb{N}$, $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1,\ldots,q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,\varepsilon}(t,i,a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a^{1-|t|_{x_i}} & \text{, otherwise .} \end{cases}$$

Each $f^k_{M_1,\varepsilon}(t,i,a)$ is well-defined, because by the linearity restriction we obtain $|t|_{x_i} \le 1$ for every $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1,\ldots,q_k))$ and $i \in [k]$. Consequently, the exponent is non-negative in the definition of $f^k_{M_1,\varepsilon}(t,i,a)$. Moreover, $f_{M_1,\varepsilon}$ is obviously a family of translation mappings. Thus there exists a linear deterministic bu-w-tt M_2 obeying the restrictions of π such that $\tau^o_{M_2} = \tau$ due to Lemma 4.11. Hence π–BOT($\mathcal{A}$) $\subseteq$ π–BOT^o($\mathcal{A}$) for every $\pi \in \Pi_l \setminus \Pi_h$.

□

These are all the non-trivial inclusion results we are able to prove without requiring further properties of the monoid. So it remains to show incomparability results similar to Lemma 4.7. We start by showing that as long as the monoid is not regular, there exists a nondeleting homomorphism bu-w-tt computing a t-ts transformation, which cannot be computed by a deterministic bu-w-tt as o-t-ts transformation. We finally note that periodicity is not even required for the proof, which is similar to the proof of the corresponding statement in non-periodic semirings (cf. Lemma 4.7).

Lemma 4.13. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative and non-regular monoid.*

$$\text{hn–BOT}(\mathcal{A}) \nsubseteq \text{d–BOT}^o(\mathcal{A})$$

Proof. Since the monoid $\mathcal{A}$ is not regular, there exists an $a \in A$ such that there is no $b \in A$ with $b \odot a^2 = a$. Let $M_1 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the homomorphism bu-w-tt specified by the input ranked alphabet $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}\}$, output ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, and $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha)$.

$$(\mu_1)^1_\gamma(\star) = \mathbf{1}\, \sigma(x_1, x_1) \quad , \quad (\mu_1)^0_\alpha() = a\, \alpha \ .$$

Clearly, M_1 is a nondeleting homomorphism bu-w-tt, so $\tau_{M_1} \in$ hn–BOT($\mathcal{A}$). Let $\tau = \tau_{M_1}$. For every $s \in T_\Gamma$ let $t_s \in T_\Sigma$ be the fully balanced output tree such that the heights of the trees s and t_s are equal. An easy calculation yields that for every $s \in T_\Gamma$ the equality $\tau(s) = a\, t_s$ holds.

Next we prove that $\tau \notin$ d–BOT^o($\mathcal{A}$). In order to derive a contradiction, assume that there is a deterministic bu-w-tt $M_2 = (Q_2, \Gamma, \Sigma, \mathcal{A}, F_2, \delta_2, \mu_2)$ such that $\tau^o_{M_2} = \tau$. Since for every $s \in T_\Gamma$ it holds that $\tau(s) \neq \widetilde{\mathbf{0}}$ and M_2 has only a finite set Q_2 of states, there must exist a final state $q \in F_2$ such that for two distinct $s_1, s_2 \in T_\Gamma$, i.e., $s_1 \neq s_2$, we have $\widehat{\delta_2}(s_1) = q = \widehat{\delta_2}(s_2)$. Consequently, $\tau^o_{M_2}(s_i) = \widehat{\mu_2}_o(s_i)$ for every $i \in [2]$. Moreover, also $\widehat{\delta_2}(\gamma(s_i)) \in F$, hence

$$\tau^o_{M_2}(\gamma(s_i)) = \widehat{\mu_2}_o(\gamma(s_i)) = (\mu_2)^1_\gamma(\widehat{\delta_2}(s_i)) \overset{o}{\longleftarrow} (\widehat{\mu_2}_o(s_i)) = (\mu_2)^1_\gamma(q) \overset{o}{\longleftarrow} (\tau(s_i)) \ .$$

Trivially, $(\mu_2)^1_\gamma(q) \neq \widetilde{\mathbf{0}}$, otherwise $\tau^o_{M_2}(\gamma(s_i)) = \widetilde{\mathbf{0}}$. Let $(\mu_2)^1_\gamma(q) = b\ t$ for some $b \in A$ and $t \in T_\Sigma(X_1)$. Moreover, recall that $\tau(s_i) = a\ t_{s_i}$. We can readily conclude that $t = \sigma(x_1, x_1)$, else either $t[t_{s_1}]$ or $t[t_{s_2}]$ is not fully balanced or $\text{height}(t[t_{s_i}]) \neq \text{height}(s_i) + 1$ for some $i \in [2]$. We continue with

$$\tau^o_{M_2}(\gamma(s_i)) = (\mu_2)^1_\gamma(q) \overset{o}{\longleftarrow} (\tau(s_i)) = b\ \sigma(x_1, x_1) \overset{o}{\longleftarrow} (a\ t_{s_i}) = (b \odot a^2)\ \sigma(t_{s_i}, t_{s_i}) \ .$$

According to $\tau^o_{M_2} = \tau$, we also derive

$$\tau^o_{M_2}(\gamma(s_i)) = (b \odot a^2)\ \sigma(t_{s_i}, t_{s_i}) = a\ \sigma(t_{s_i}, t_{s_i}) = \tau(\gamma(s_i)) \ .$$

Consequently, we should have $b \odot a^2 = a$, but a was chosen such that this is impossible. Thus we arrived at a contradiction which yields $\tau \notin \text{d–BOT}^o(\mathcal{A})$. □

Next we show that there exists an o-t-ts transformation τ computed by a linear homomorphism bu-w-tt such that there exists no deterministic bu-w-tt computing τ as t-ts transformation, unless $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ is actually a group with an absorbing element $\mathbf{0}$.

Lemma 4.14. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative monoid which is no group.*

$$\text{hl–BOT}^o(\mathcal{A}) \nsubseteq \text{d–BOT}(\mathcal{A})$$

Proof. The monoid $\mathcal{A}$ is no group, hence there exists an $a \in A \setminus \{\mathbf{0}\}$, which cannot be inverted, i.e., there is no $b \in A$ such that $b \odot a = \mathbf{1}$. Let $M_1 = (\{\star\}, \Gamma, \Gamma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the homomorphism bu-w-tt specified by the ranked alphabet $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and output mappings $\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha)$.

$$(\mu_1)^1_\gamma(\star) = \mathbf{1}\ \alpha \quad , \quad (\mu_1)^0_\alpha() = a\ \alpha \ .$$

Clearly, M_1 is a linear homomorphism bu-w-tt, thus $\tau^o = \tau^o_{M_1} \in \text{hl–BOT}^o(\mathcal{A})$. Straightforward calculation yields $\tau^o(\alpha) = a\ \alpha$ and for every other $s \in T_\Gamma \setminus \{\alpha\}$ the equality $\tau^o(s) = \mathbf{1}\ \alpha$ holds.

Next we prove that $\tau^o \notin \text{d–BOT}(\mathcal{A})$. In order to derive a contradiction, assume that there exists a deterministic bu-w-tt $M_2 = (Q_2, \Gamma, \Gamma, \mathcal{A}, F_2, \delta_2, \mu_2)$ such that $\tau_{M_2} = \tau^o$. Obviously,

$$a\ \alpha = \tau^o(\alpha) = \tau_{M_2}(\alpha) = \widehat{\mu_2}(\alpha) = (\mu_2)^0_\alpha() \ .$$

Since we also have $\tau^o(\gamma(\alpha)) = \mathbf{1}\ \alpha$ we immediately obtain

$$\begin{aligned} &\tau_{M_2}(\gamma(\alpha)) \\ &\quad = \widehat{\mu_2}(\gamma(\alpha)) \\ &\quad = (\mu_2)^1_\gamma(\widehat{\delta_2}(\alpha)) \longleftarrow (\widehat{\mu_2}(\alpha)) \\ &\quad = (\mu_2)^1_\gamma((\delta_2)^0_\alpha()) \longleftarrow (a\ \alpha) \\ &\quad = b\ t \longleftarrow (a\ \alpha) \end{aligned}$$

$$= \quad (b \odot a)\ t[\alpha]$$

for some $b \in A$ and $t \in T_\Gamma(X_1)$. Moreover, we have that $(b \odot a)\ t[\alpha] = \mathbf{1}\ \alpha$, so $b \odot a = \mathbf{1}$. Contrary, a was chosen such that such an element b does not exist. Thus we derived the desired contradiction and conclude $\tau^o \notin$ d–BOT($\mathcal{A}$). □

We have already seen in Lemma 4.13 that the class of all t-ts transformations computed by nondeleting homomorphism bu-w-tt is not contained in the class of all o-t-ts transformations computed by deterministic bu-w-tt as long as the monoid $\mathcal{A}$ is not regular, i.e.,

$$\text{hn–BOT}(\mathcal{A}) \nsubseteq \text{d–BOT}^o(\mathcal{A})\ .$$

It is furthermore clear that the class of all o-t-ts transformations computed by nondeleting homomorphism bu-w-tt is properly contained in the class of all t-ts transformations computed by deterministic bu-w-tt due to Lemma 4.12 (on periodic and commutative monoids), i.e.,

$$\text{hn–BOT}^o(\mathcal{A}) \subseteq \text{d–BOT}(\mathcal{A})\ .$$

However, the relation between the class of o-t-ts transformations computed by nondeleting homomorphism bu-w-tt and the class of t-ts transformations computed by nondeleting homomorphism bu-w-tt is yet unsettled. The next lemma solves this question for all non-idempotent monoids.

Lemma 4.15. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a non-idempotent monoid.*

$$\text{hn–BOT}^o(\mathcal{A}) \nsubseteq \text{h–BOT}(\mathcal{A})$$

Proof. Let $a \in A \setminus \{\mathbf{0}, \mathbf{1}\}$ be such that $a \odot a \neq a$. Such an element exists due to the assumption that $\mathcal{A}$ is non-idempotent. Further, let $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $M_1 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the nondeleting homomorphism bu-w-tt with

$$\mu_1 = ((\mu_1)^1_\gamma, (\mu_1)^0_\alpha, (\mu_1)^0_\beta)$$

specified by

$$(\mu_1)^1_\gamma(\star) = \mathbf{1}\ \sigma(x_1, x_1) \quad , \quad (\mu_1)^0_\alpha() = a\ \alpha \quad , \quad (\mu_1)^0_\beta() = \mathbf{1}\ \alpha\ .$$

Let $\tau^o = \tau^o_{M_1}$. Clearly, $\tau^o \in$ hn–BOT$^o(\mathcal{A})$, and moreover, $\tau^o(\gamma(\alpha)) = a^2\ \sigma(\alpha, \alpha)$ as well as $\tau^o(\gamma(\beta)) = \mathbf{1}\ \sigma(\alpha, \alpha)$.

Now let us prove that $\tau^o \notin$ h–BOT($\mathcal{A}$). We prove this statement by contradiction, so assume that there exists a homomorphism bu-w-tt

$$M_2 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$$

such that $\tau_{M_2} = \tau^o$. Trivially, $\delta_2 = \delta_1$ and $\mu_2 = ((\mu_2)^1_\gamma, (\mu_2)^0_\alpha, (\mu_2)^0_\beta)$ with

$$(\mu_2)^1_\gamma(\star) = c\ t \quad , \quad (\mu_2)^0_\alpha() = a\ \alpha \quad , \quad (\mu_2)^0_\beta() = \mathbf{1}\ \alpha$$

for some $c \in A$ and $t \in T_\Sigma(X_1)$. Moreover, we readily observe $t = \sigma(x_1, x_1)$. Consequently, $\tau_{M_2}(\gamma(\alpha)) = (c \odot a)\ \sigma(\alpha, \alpha)$ and $\tau_{M_2}(\gamma(\beta)) = c\ \sigma(\alpha, \alpha)$. Thus we obtain the equalities $c = \mathbf{1}$ and $c \odot a = a^2$, which yield $a = a^2$. Contrary, a was chosen such that $a \neq a^2$. Thus we derived the desired contradiction and conclude that $\tau^o \notin$ h–BOT($\mathcal{A}$). □

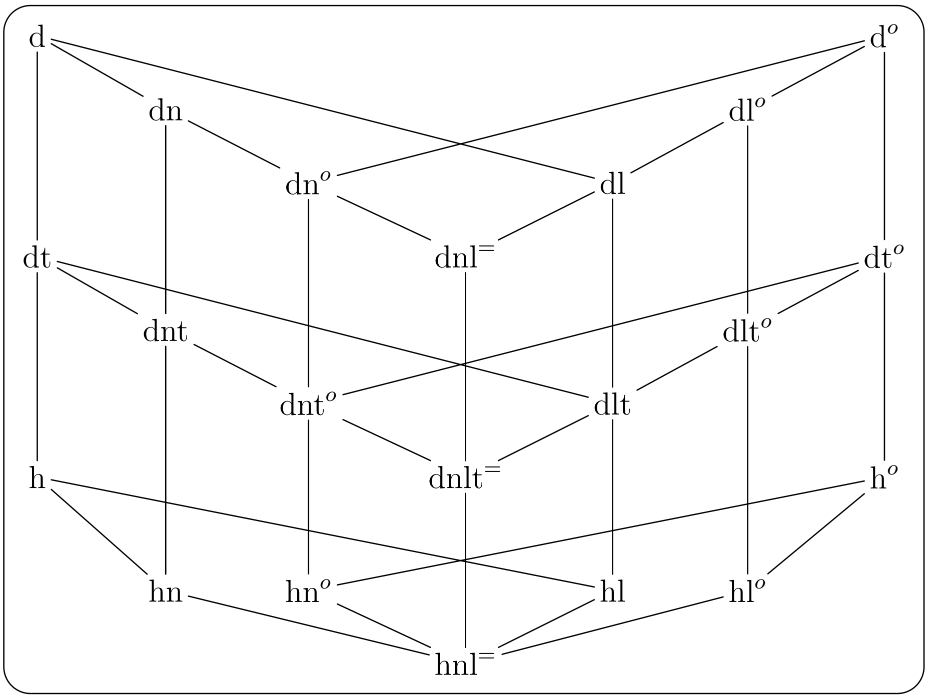

Figure 3: Hasse diagram for periodic, commutative, and non-regular monoids.

Finally, we are able to present the Hasse diagram for periodic and commutative monoids $\mathcal{A}$, which are not regular. The latter restriction assures that $\mathcal{A}$ is also neither idempotent nor a group. Those cases are handled in subsequent sections.

Theorem 4.16. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative, and non-regular monoid with an absorbing element $\mathbf{0}$. Figure 3 is the* Hasse *diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.*

Proof. All the inclusions are either trivial or follow from Lemma 4.12, whereas the equalities are due to Proposition 4.4. The following eight statements are sufficient to prove strictness and incomparability with mod $\in \{\varepsilon, o\}$.

$$\text{dnlt–BOT}(\mathcal{A}) \nsubseteq \text{h–BOT}^{\text{mod}}(\mathcal{A}) \tag{16}$$
$$\text{dnl–BOT}(\mathcal{A}) \nsubseteq \text{dt–BOT}^{\text{mod}}(\mathcal{A}) \tag{17}$$
$$\text{hn–BOT}^{o}(\mathcal{A}) \nsubseteq \text{dl–BOT}^{o}(\mathcal{A}) \tag{18}$$
$$\text{hl–BOT}(\mathcal{A}) \nsubseteq \text{dn–BOT}(\mathcal{A}) \tag{19}$$
$$\text{hn–BOT}(\mathcal{A}) \nsubseteq \text{d–BOT}^{o}(\mathcal{A}) \tag{20}$$
$$\text{hl–BOT}^{o}(\mathcal{A}) \nsubseteq \text{d–BOT}(\mathcal{A}) \tag{21}$$
$$\text{hn–BOT}^{o}(\mathcal{A}) \nsubseteq \text{h–BOT}(\mathcal{A}) \tag{22}$$
$$\text{hl–BOT}(\mathcal{A}) \nsubseteq \text{h–BOT}^{o}(\mathcal{A}) \tag{23}$$

The inequalities (16)–(19) are proved in Lemma 4.3, whereas we obtain (20) from Lemma 4.13, (21) from Lemma 4.14, (22) from Lemma 4.15, and (23) from Lemma 4.5. □

4.4. Periodic, commutative, and regular monoids

In this section we consider monoids $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ which are periodic, commutative, and regular. An example of a periodic, commutative, and regular monoid, which is neither idempotent nor a group, is $\mathbb{Z}_6$. Specifically the regularity allows us to derive more inclusion results. The next corollary states this formally. Roughly speaking, the classes of t-ts transformations become subsets of the corresponding classes of o-t-ts transformations, except for the classes bearing the homomorphism restriction.

Lemma 4.17. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative, and regular monoid. Then for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$ we have π–BOT($\mathcal{A}$) $\subseteq$ π–BOT^o($\mathcal{A}$).*

Proof. Let $\tau \in \pi$–BOT($\mathcal{A}$) for some $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π such that $\tau_{M_1} = \tau$. Moreover, let $f_{M_1,\varepsilon} = (f^k_{M_1,\varepsilon})_{k\in\mathbb{N}}$ be the family of mappings

$$f^k_{M_1,\varepsilon} : \Big(\bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))\Big) \times [k] \times A \longrightarrow A$$

defined for every $k \in \mathbb{N}$, $t \in \bigcup_{\sigma\in\Sigma^{(k)}, q_1,\ldots,q_k\in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,\varepsilon}(t, i, a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a & \text{, if } a \neq \mathbf{0}, |t|_{x_i} = 0 \\ b^{|t|_{x_i}-1} & \text{, otherwise} \end{cases} ,$$

where $b \in A$ is such that $a^2 \odot b = a$. Such $b \in A$ exists for every $a \in A$ due to regularity.

Each $f^k_{M_1,\varepsilon}(t, i, a)$ is well-defined, because in the case distinction every exponent is nonnegative in the definition of $f^k_{M_1,\varepsilon}(t, i, a)$. Moreover, it is straightforward to prove that $f_{M_1,\varepsilon}$ is a family of translation mappings for M_1. Thus, due to Lemma 4.11, there exists a deterministic bu-w-tt M_2 obeying the restrictions π such that $\tau^o_{M_2} = \tau$. Hence π–BOT($\mathcal{A}$) $\subseteq$ π–BOT^o($\mathcal{A}$) for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$.

Since we cannot apply Lemma 4.13 to show that the classes of t-ts and o-t-ts transformations computed by nondeleting homomorphism bu-w-tt are incomparable, but Lemma 4.15 already delivers one half, we establish the remaining half in the next lemma.

Lemma 4.18. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative and regular, but non-idempotent monoid.*

$$\text{hn–BOT}(\mathcal{A}) \nsubseteq \text{h–BOT}^o(\mathcal{A})$$

Proof. Since $\mathcal{A}$ is not idempotent, but regular, there exist $a, b \in A \setminus \{\mathbf{0}, \mathbf{1}\}$ such that $a \neq a^2$ and $a^2 \odot b = a$. Let $\Gamma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $M_1 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_1, \mu_1)$ be the nondeleting homomorphism bu-w-tt specified by

$$(\mu_1)^1_\gamma(\star) = a\, \sigma(x_1, x_1) \quad , \quad (\mu_1)^0_\alpha() = b\, \alpha \ .$$

Let $\tau = \tau_{M_1}$. Clearly, $\tau \in \text{hn–BOT}(\mathcal{A})$, and moreover,

$$\begin{aligned}
\tau(\gamma(\alpha)) &= (a \odot b)\, \sigma(\alpha, \alpha) \\
\tau(\gamma^2(\alpha)) &= a\, \sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha)) \\
\tau(\gamma^3(\alpha)) &= a^2\, \sigma(\sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha)), \sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha))) \ .
\end{aligned}$$

Now let us prove that $\tau \notin \text{h–BOT}^o(\mathcal{A})$. We prove this statement by contradiction, so assume that there exists a homomorphism bu-w-tt

$$M_2 = (\{\star\}, \Gamma, \Sigma, \mathcal{A}, \{\star\}, \delta_2, \mu_2)$$

such that $\tau^o_{M_2} = \tau$. Trivially, $\delta_2 = \delta_1$, $(\mu_2)^1_\gamma(\star) = c\ t$, and $(\mu_2)^0_\alpha() = b\ \alpha$ for some $c \in A$ and $t \in T_\Sigma(X_1)$. Moreover, we readily observe $t = \sigma(x_1, x_1)$, otherwise $\text{supp}(\tau^o_{M_2}(\gamma(\alpha))) \neq \{\sigma(\alpha, \alpha)\}$ or $\text{supp}(\tau^o_{M_2}(\gamma^2(\alpha))) \neq \{\sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha))\}$. Hence

$$\begin{aligned}
\tau^o_{M_2}(\gamma(\alpha)) &= (b^2 \odot c)\, \sigma(\alpha, \alpha) \\
\tau^o_{M_2}(\gamma^2(\alpha)) &= (b^4 \odot c^3)\, \sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha)) \\
\tau^o_{M_2}(\gamma^3(\alpha)) &= (b^8 \odot c^7)\, \sigma(\sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha)), \sigma(\sigma(\alpha, \alpha), \sigma(\alpha, \alpha))) \ .
\end{aligned}$$

Thus we obtain the equalities

$$b^2 \odot c = a \odot b \quad , \quad b^4 \odot c^3 = a \quad , \quad b^8 \odot c^7 = a^2 \ .$$

Now we compute as follows

$$\begin{aligned}
a &= b^4 \odot c^3 = (b^2 \odot c) \odot (b^2 \odot c) \odot c \\
&= (a \odot b) \odot (a \odot b) \odot c = (a^2 \odot b) \odot b \odot c = a \odot b \odot c
\end{aligned}$$

and $a^2 = b^8 \odot c^7 = (b^4 \odot c^3) \odot (b^4 \odot c^3) \odot c = a^2 \odot c$. Next we multiply the former equation with a, which gives $a^2 = a^2 \odot b \odot c = a \odot c$, and the latter equation with b, which yields $a = a^2 \odot b = a^2 \odot b \odot c = a \odot c$. Hence $a = a^2$, which is a contradiction, because a was chosen such that $a \neq a^2$. Thus we conclude that $\tau \notin \text{h–BOT}^o(\mathcal{A})$. □

At this point we have all the results necessary to derive the Hasse diagram for periodic, commutative, and regular monoids, which are neither idempotent nor groups.

Theorem 4.19. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic, commutative, and regular monoid, which is neither idempotent nor a group with an absorbing element $\mathbf{0}$. Figure 4 is the* Hasse *diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.*

Proof. All the inclusions are either trivial or follow from Lemma 4.12 and Lemma 4.17. The used equalities are due to Proposition 4.4, Lemma 4.12, and Lemma 4.17. The following seven statements are sufficient to prove strictness and incomparability. For every $\{\text{mod}_1, \text{mod}_2\} = \{\varepsilon, o\}$

$$\text{dnlt–BOT}(\mathcal{A}) \nsubseteq \text{h–BOT}^{\text{mod}_1}(\mathcal{A}) \tag{24}$$

$$\text{dnl–BOT}(\mathcal{A}) \nsubseteq \text{dt–BOT}^o(\mathcal{A}) \tag{25}$$

$$\text{hn–BOT}^{\text{mod}_1}(\mathcal{A}) \nsubseteq \text{dl–BOT}^o(\mathcal{A}) \tag{26}$$

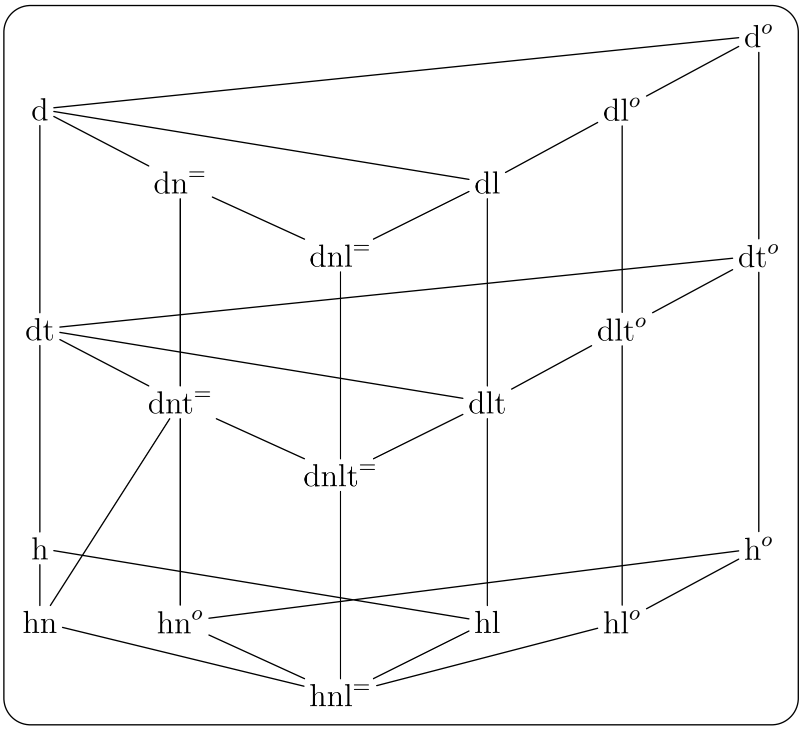

Figure 4: Hasse diagram for periodic, commutative, and regular monoids, which are neither idempotent nor a group.

$$\text{hl–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{dn–BOT}(\mathcal{A}) \tag{27}$$
$$\text{hl–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{h–BOT}^o(\mathcal{A}) \tag{28}$$
$$\text{hl–BOT}^o(\mathcal{A}) \quad \nsubseteq \quad \text{d–BOT}(\mathcal{A}) \tag{29}$$
$$\text{hl–BOT}^{\text{mod}_1}(\mathcal{A}) \quad \nsubseteq \quad \text{d–BOT}^{\text{mod}_2}(\mathcal{A}) \ . \tag{30}$$

The inequalities (24)–(27) are proved in Lemma 4.3, whereas (28) follows from Lemma 4.5, (29) follows from Lemma 4.14, and (30) follows from Lemmata 4.15 and 4.18. □

4.5. Commutative and idempotent monoids

This section is devoted to the study of commutative and idempotent monoids. The monoid $\mathbb{R}_{\max}$ is an example of such a monoid. Clearly, $a^n = a$ for every $n \in \mathbb{N}_+$ and $a \in A$ of such a monoid. Hence we easily derive the following observation.

Proposition 4.20. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be an idempotent monoid, $k \in \mathbb{N}$, and Δ be a ranked alphabet. For every nondeleting (in X_k) $t \in T_\Delta(X_k)$, $a \in A$, and monomials $m_1, \ldots, m_k \in A[T_\Delta]$ we have that*

$$a\, t \longleftarrow (m_1, \ldots, m_k) = a\, t \overset{o}{\longleftarrow} (m_1, \ldots, m_k) \ .$$

Corollary 4.21. *Let $\mathcal{A}$ be an idempotent monoid. For every $\pi \in \Pi_n$ we have*

$$\pi\text{–BOT}^o(\mathcal{A}) = \pi\text{–BOT}(\mathcal{A}) \ .$$

Corollary 4.22. *For every monoid $\mathcal{A}$, we have* hn–BOT$^o(\mathcal{A})$ = hn–BOT$(\mathcal{A})$ *if and only if $\mathcal{A}$ is idempotent.*

Proof. The equality in idempotent monoids is proved in Corollary 4.21 and Lemma 4.15 proves the inequality in all non-idempotent monoids. □

These are indeed all the new results necessary to prove the Hasse diagram. Note that idempotent monoids are trivially regular and periodic, so we apply some of the results derived in Section 4.4.

Theorem 4.23. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a commutative, idempotent monoid such that $A \neq \{\mathbf{0}, \mathbf{1}\}$. Figure 5 is the* Hasse *diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.*

Proof. All the inclusions are either trivial or follow from Lemma 4.17. The equalities are due to Proposition 4.4 and Corollary 4.21. Then the following six statements are sufficient to prove strictness and incomparability. For every mod $\in \{\varepsilon, o\}$

$$\text{dnlt–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{h–BOT}^{\text{mod}}(\mathcal{A}) \tag{31}$$
$$\text{dnl–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{dt–BOT}^o(\mathcal{A}) \tag{32}$$
$$\text{hn–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{dl–BOT}^o(\mathcal{A}) \tag{33}$$
$$\text{hl–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{dn–BOT}(\mathcal{A}) \tag{34}$$
$$\text{hl–BOT}(\mathcal{A}) \quad \nsubseteq \quad \text{h–BOT}^o(\mathcal{A}) \tag{35}$$
$$\text{hl–BOT}^o(\mathcal{A}) \quad \nsubseteq \quad \text{d–BOT}(\mathcal{A}) \ . \tag{36}$$

The inequalities (31)–(34) are proved in Lemma 4.3, whereas (35) follows from Lemma 4.5 and (36) follows from Lemma 4.14. □

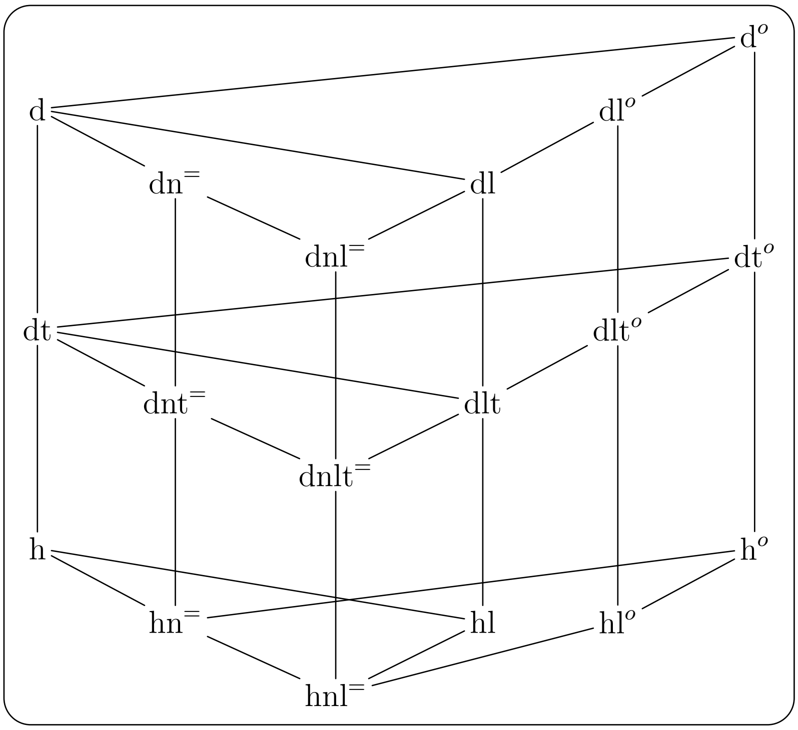

Figure 5: Hasse diagram for commutative and idempotent monoids with at least three elements.

4.6. *Periodic and commutative groups*

Finally, in this last section we consider periodic and commutative groups with an absorbing element $\mathbf{0}$. For example, the monoid $\mathbb{Z}_3$ fulfils all those restrictions. Note that all such monoids (except $\mathbb{Z}_2$) are non-idempotent. Due to the existence of inverses we can now easily derive a final lemma which follows from Lemma 4.11.

Lemma 4.24. *Let* $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ *be a periodic, commutative group and* $\mathrm{mod}_1, \mathrm{mod}_2 \in \{\varepsilon, o\}$. *For every* $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$

$$\pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \ .$$

Proof. The statement is trivial, if $\mathrm{mod}_1 = \mathrm{mod}_2$. Henceforth let mod_1 and mod_2 be distinct. Let $\tau \in \pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A})$ for some $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. Consequently, there exists a deterministic bu-w-tt $M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \delta_1, \mu_1)$ obeying the restrictions of π such that $\tau_{M_1}^{\mathrm{mod}_1} = \tau$. Moreover, let $f_{M_1,\mathrm{mod}_1} = (f^k_{M_1,\mathrm{mod}_1})_{k \in \mathbb{N}}$ be the family of mappings

$$f^k_{M_1,\mathrm{mod}_1} : \Big(\bigcup_{\sigma \in \Sigma^{(k)}, q_1, \ldots, q_k \in Q_1} \mathrm{supp}((\mu_1)^k_\sigma(q_1, \ldots, q_k)) \Big) \times [k] \times A \longrightarrow A$$

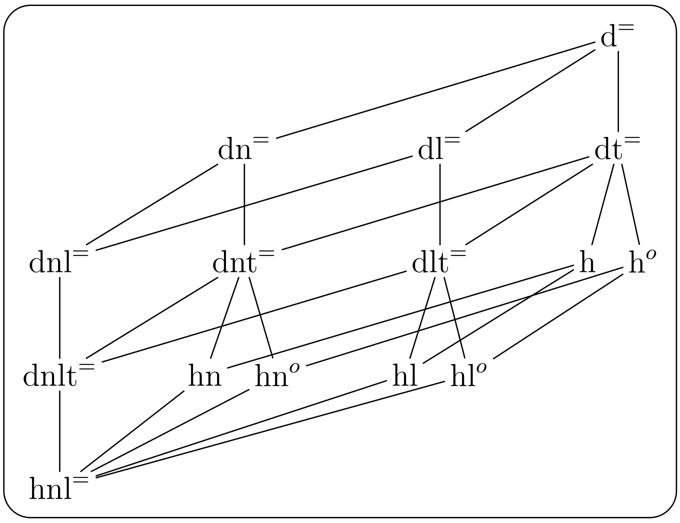

Figure 6: HASSE diagram for periodic and commutative groups with an absorbing element **0** and at least three elements.

defined for every $k \in \mathbb{N}$, $t \in \bigcup_{\sigma\in\Sigma^{(k)},q_1,\ldots,q_k\in Q_1} \operatorname{supp}((\mu_1)^k_\sigma(q_1,\ldots,q_k))$, $i \in [k]$, and $a \in A$ by

$$f^k_{M_1,\mathrm{mod}_1}(t,i,a) = \begin{cases} \mathbf{0} & \text{, if } a = \mathbf{0} \\ a^{1-|t|_{x_i}} & \text{, if } a \neq \mathbf{0}, \mathrm{mod}_1 = \varepsilon \\ a^{|t|_{x_i}-1} & \text{, if } a \neq \mathbf{0}, \mathrm{mod}_1 = o \ . \end{cases}$$

Each $f^k_{M_1,\mathrm{mod}_1}(t,i,a)$ is trivially well-defined due to the existence of inverses. Moreover, it is straightforward to prove that f_{M_1,mod_1} is a family of mod_1-translation mappings. Thus there exists a deterministic bu-w-tt M_2 obeying the restrictions π such that $\tau^{\mathrm{mod}_2}_{M_2} = \tau$ due to Lemma 4.11. Hence we can conclude $\pi\text{–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \subseteq \pi\text{–BOT}^{\mathrm{mod}_2}(\mathcal{A})$ for every $\pi \in \Pi \setminus \Pi_{\mathrm{h}}$. □

Since we demand that we have at least three elements, our group is non-idempotent, which allows us to reuse some the results of earlier sections. Finally, we present the last HASSE diagram.

Theorem 4.25. *Let $\mathcal{A} = (A, \odot, \mathbf{1}, \mathbf{0})$ be a periodic and commutative group with an absorbing element $\mathbf{0}$ such that $A \neq \{\mathbf{0}, \mathbf{1}\}$. Figure 6 is the* HASSE *diagram of the displayed classes of t-ts and o-t-ts transformations ordered by set inclusion.*

PROOF. All the inclusions are either trivial or follow from Lemma 4.24. The equalities are due to Proposition 4.4 and Lemma 4.24. Then the following six statements are sufficient to prove strictness and incomparability. For every $\{\mathrm{mod}_1, \mathrm{mod}_2\} = \{\varepsilon, o\}$

$$\text{dnlt–BOT}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \tag{37}$$
$$\text{dnl–BOT}(\mathcal{A}) \not\subseteq \text{dt–BOT}(\mathcal{A}) \tag{38}$$
$$\text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{dl–BOT}(\mathcal{A}) \tag{39}$$
$$\text{hl–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{dn–BOT}(\mathcal{A}) \tag{40}$$
$$\text{hn–BOT}^{\mathrm{mod}_1}(\mathcal{A}) \not\subseteq \text{h–BOT}^{\mathrm{mod}_2}(\mathcal{A}) \tag{41}$$

$$\text{hl–BOT}^{\text{mod}_1}(\mathcal{A}) \quad \nsubseteq \quad \text{h–BOT}^{\text{mod}_2}(\mathcal{A}) \ . \tag{42}$$

The inequalities (37)–(40) are proved in Lemma 4.3, whereas inequality (41) follows from Lemmata 4.15 and 4.18 and (42) follows from Lemma 4.5. □

5. Conclusions

We have investigated the power of deterministic bu-w-tt using pure and o-substitution. We presented Hasse diagrams conveying the relation between classes of t-ts and o-t-ts transformations for all sensible combinations of the common restrictions and all commutative monoids. It turned out that pure and o-substitution not only differ conceptually, but the induced classes of t-ts and o-t-ts transformations are also different for most monoids.

In principle, we observe that o-substitution is more appropriate, if the weight is related to the output tree, whereas pure substitution handles weights related to the input tree better. Concerning applications, deterministic bu-w-tt can be used to compute, for example, the topmost leftmost instance of a pattern in an input tree weighted by the size of the instance. For this purpose we would use o-substitution. Deterministic bu-w-tt using pure substitution can be applied to compute the same instance weighted by the size of the input tree.

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