On Ambiguity of Max-Plus Tree Automata

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Leipzig University

Max-Plus Automata

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Max-Plus Automata

Weights in $\mathbb{R} \cup \{-\infty\}$

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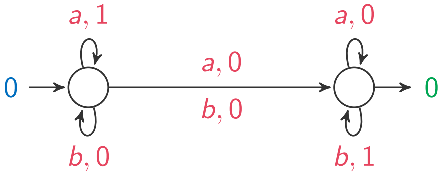

Max-Plus Automata: Ambiguity

one "initial state"

sequential / deterministic — no two valid $p \xrightarrow{a} q_1,\ p \xrightarrow{a} q_2$

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Given $\mathcal{A}_1, \mathcal{A}_2$ Is $[\![\mathcal{A}_1]\!](w) = [\![\mathcal{A}_2]\!](w)$ for all w?

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Finite Sequentiality problem

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Decidability for max-plus automata on words

	Equivalence	Unambiguity	Sequentiality	Fin Seq
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Krob

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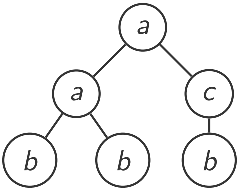

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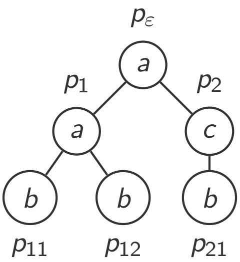

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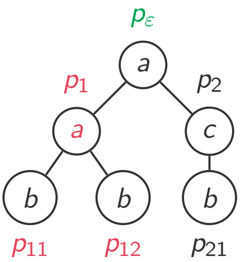

$J \subseteq \mathbb{N}_0^n$ called linear iff $\exists \bar{v} \in \mathbb{N}_0^n, V \in \mathbb{N}_0^{n \times k}$

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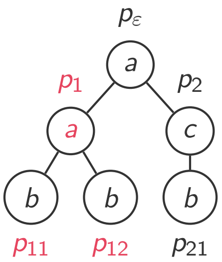

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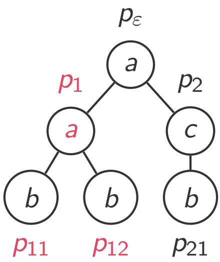

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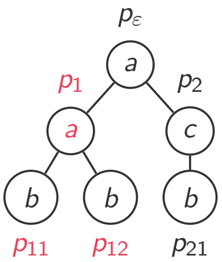

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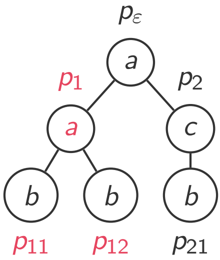

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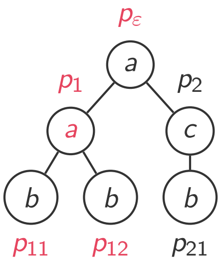

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$\bar{w} = (w_1, \ldots, w_{M+1})$ $\qquad W_1, \ldots, W_{M+1}$ rows of W

We show: $\mathcal{A}_1$ fin-amb $\implies$ $\mathcal{A}_1 \geq \mathcal{A}_2$ decidable

$\mathbb{O} = \{(\mathrm{wt}_1, \ldots, \mathrm{wt}_M, \mathrm{wt}_{M+1}) \mid t \text{ tree}\} \qquad = W \cdot \mathbb{P}(\mathcal{A})$

$$\mathbb{P}(\mathcal{A}) \text{ semilinear} \;\Rightarrow\; \mathbb{P}(\mathcal{A}) = \bigcup_{i=1}^{l} \bar{v}_i + V_i \cdot \mathbb{N}_0^k$$

$$\Rightarrow\; \mathbb{O} = \bigcup_{i=1}^{l} W\bar{v}_i + W V_i \cdot \mathbb{N}_0^k$$

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not $\mathcal{A}_1 \geq \mathcal{A}_2$ iff $v_i < v_{M+1}\ \forall i$ for some $\bar{v} \in \mathbb{O}$

$\rightsquigarrow$ decidable for set $\bar{w} + W \cdot \mathbb{N}_0^k$

$\bar{w} = (w_1, \ldots, w_{M+1})$ $\qquad W_1, \ldots, W_{M+1}$ rows of W

find solution $\bar{X} \in \mathbb{N}_0^k$

$$\begin{array}{ccccccc} w_1 & + & W_1 \cdot \bar{X} & < & w_{M+1} & + & W_{M+1} \cdot \bar{X} \\ & \vdots & & & & \vdots & \\ w_M & + & W_M \cdot \bar{X} & < & w_{M+1} & + & W_{M+1} \cdot \bar{X} \end{array}$$