Finite Sequentiality of Finitely Ambiguous Max-Plus Tree Automata

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Leipzig University

QuantLA

Max-Plus Automata

$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{a} q_4 \xrightarrow{b} q_5 \xrightarrow{a} q_6$$

Max-Plus Automata

Weights in $\mathbb{R} \cup \{-\infty\}$

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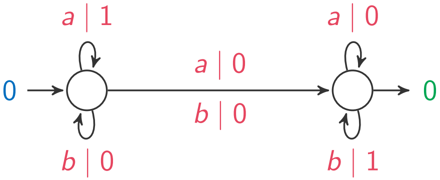

Max-Plus Automata: Ambiguity

one "initial state"

sequential / deterministic

no two valid $p \xrightarrow{a} q_1,\ p \xrightarrow{a} q_2$

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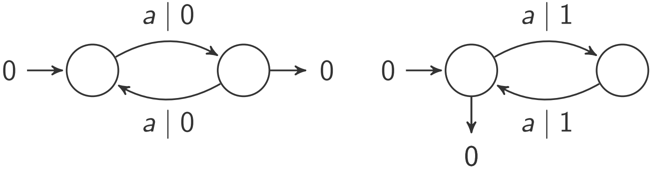

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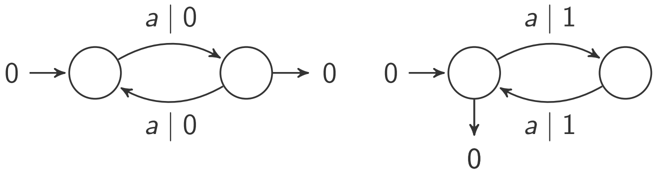
$\mathcal{A}$ max-plus automaton

p, q states



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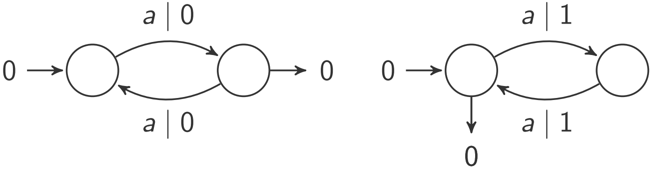

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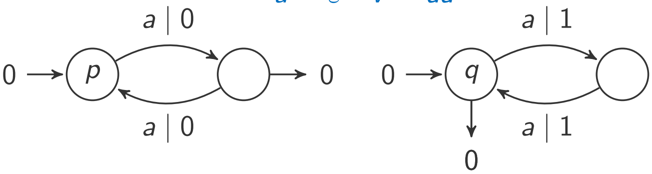

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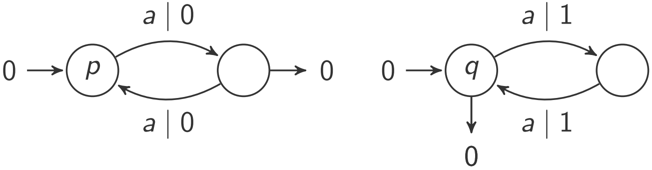

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THM [Mohri] $\mathcal{A}$ unamb $\Rightarrow$ $\qquad$ $\mathcal{A}$ determinizable $\leftrightarrow$ no rivals in $\mathcal{A}$

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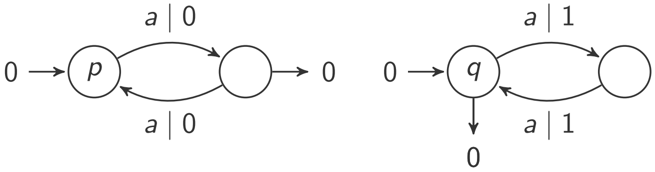

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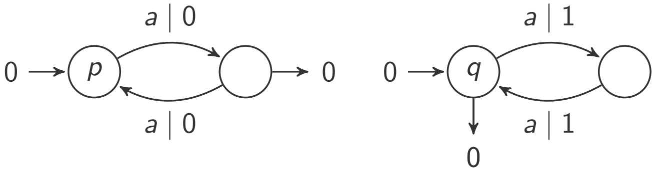

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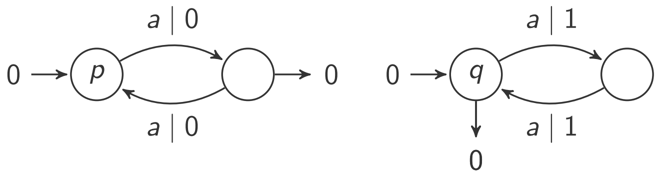

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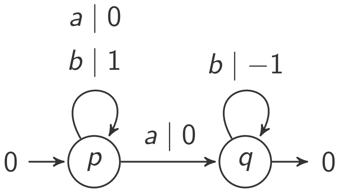

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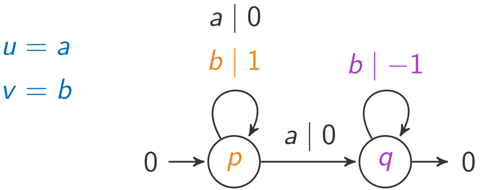

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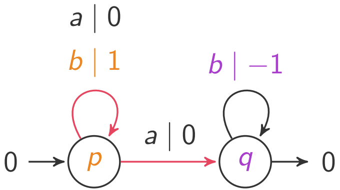

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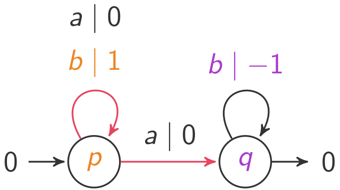

b's before last a

$\Updownarrow$

b's after last a

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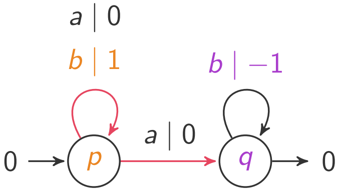

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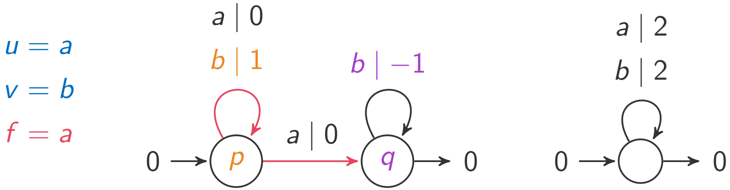

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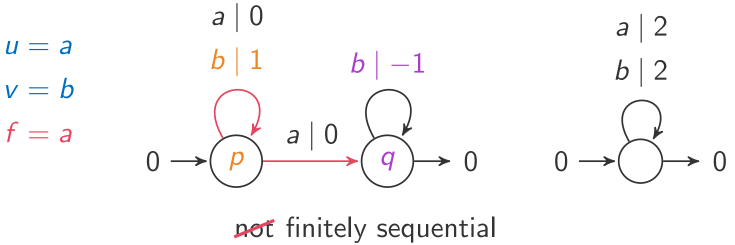

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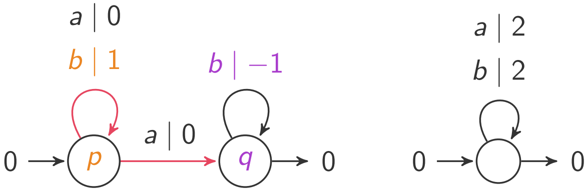

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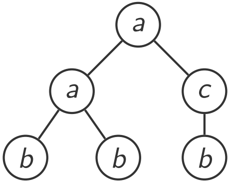
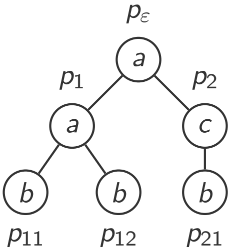
THM [BALA] $\mathcal{A}$ fin amb $\Rightarrow$ $[\![\mathcal{A}]\!]$ fin seq $\leftrightarrow$ $\exists C \forall w \exists i: |[\![\mathcal{A}]\!](w) - [\![\mathcal{U}_i]\!](w)| \leq C$

no fork $p \xrightarrow{f} q$ in run of $\mathcal{U}_i$ on w

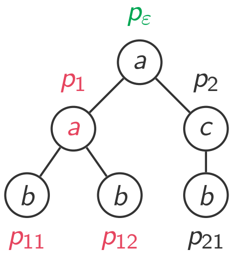

a
a
c
b
b
b


p_ε
a
p_1
p_2
a
c
b
b
b
p_{11}
p_{12}
p_{21}

weight of run =

transition weights + final weight

(p_{11}, p_{12}, a, p_1)



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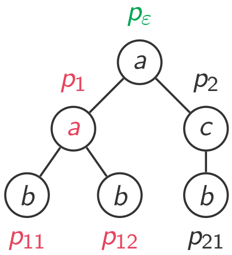

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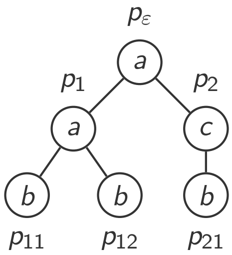

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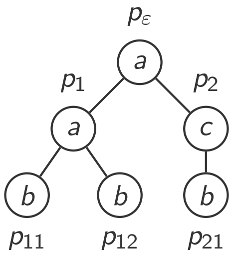

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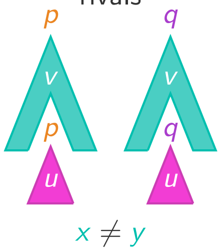

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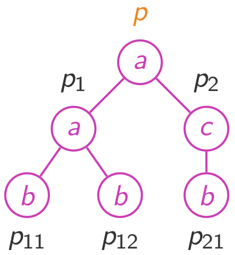

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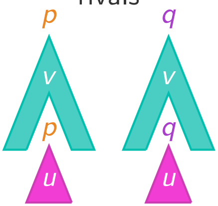

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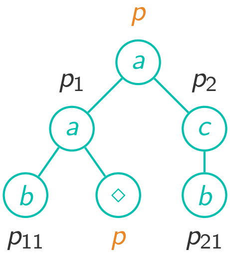

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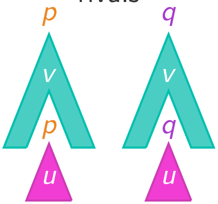

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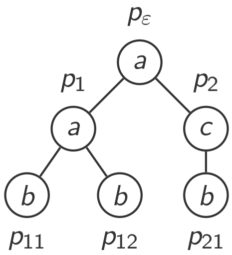

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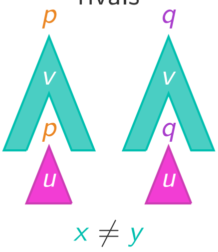
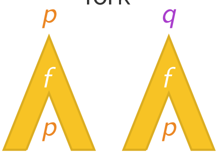


fork



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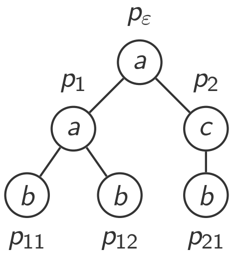

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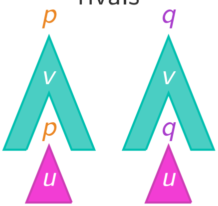

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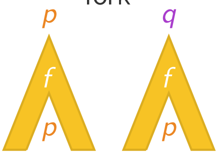
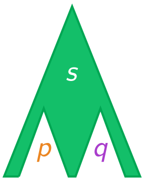


split



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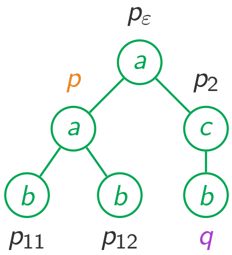

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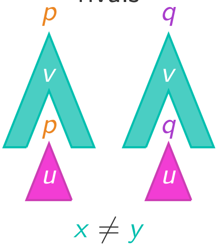
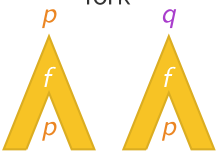
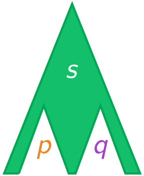


fork



split

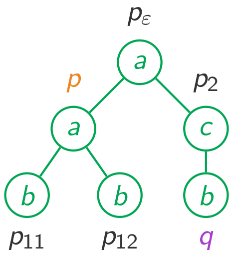


weight of run =

transition weights + final weight

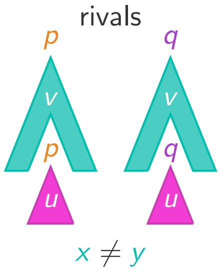
(p_{11}, p_{12}, a, p_1)

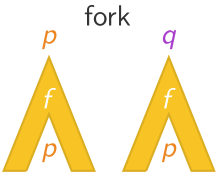
determinism: bottom-up

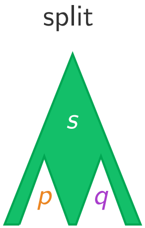


Finite Sequentiality:

$[\![\mathcal{A}]\!] = \max_{i=1}^{n} [\![\mathcal{A}_i]\!]$ for some determ $\mathcal{A}_i$?







$\mathcal{A}$ fin amb $\Rightarrow$ $[\![\mathcal{A}]\!] = \max_{i=1}^{n} [\![\mathcal{U}_i]\!]$ for unamb $\mathcal{U}_i$

weight of run =

transition weights + final weight

(p_{11}, p_{12}, a, p_1)

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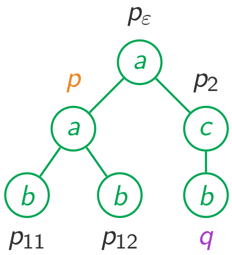
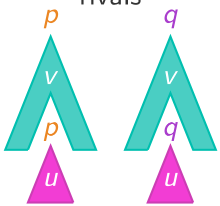


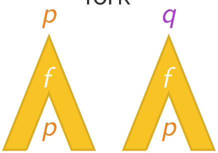
Finite Sequentiality: $[\![\mathcal{A}]\!] = \max_{i=1}^{n} [\![\mathcal{A}_i]\!]$ for some determ $\mathcal{A}_i$?

rivals

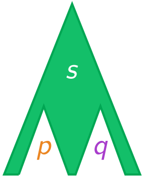


$x \neq y$

fork



split



NEW THM

$\mathcal{A}$ fin amb $\Rightarrow$ $[\![\mathcal{A}]\!]$ fin seq $\leftrightarrow$ $\exists C \forall t \exists i : |[\![\mathcal{A}]\!](t) - [\![\mathcal{U}_i]\!](t)| \leq C$

no fork, no split in run of $\mathcal{U}_i$ on t