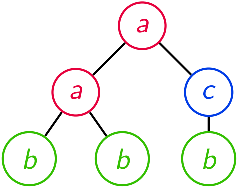
On Finite and Polynomial Ambiguity of Weighted Tree Automata

Erik Paul

April 29, 2016

Trees

(Γ, rk) ranked alphabet



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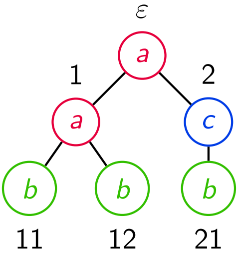
$\mathrm{pos}(t)$	positions



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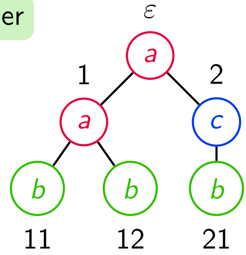
lexicographic order



Weighted Tree Automata

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$\mu\colon (p_1, \ldots, p_m, a, p) \mapsto \kappa$	transition weights
$\nu\colon p \mapsto \kappa$	final weights

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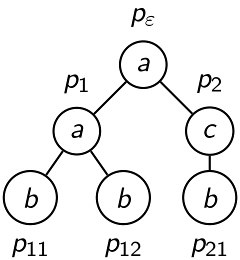

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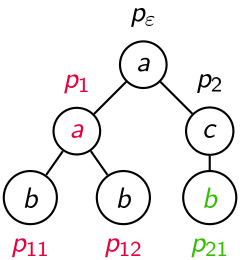
(p_{11}, p_{12}, a, p_1)

(b, p_{21})



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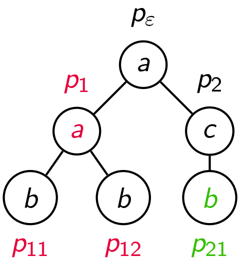

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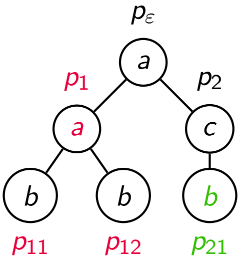

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$[\![\mathcal{A}]\!](t) = \sum_r wt(r)$	tree series

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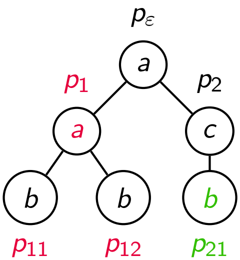

Logics

MSO

$\varphi ::= \mathrm{label}_a(x) \mid \mathrm{edge}_i(x, y) \mid x \in X \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \exists X.\varphi$

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Theorem (Droste/Gastin/Vogler)

Weighted Tree Automata = restricted QMSO

Ambiguity

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unamb	$|\mathrm{Run}(t)| \leq 1$

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unamb	$=$	$\Pi y.\theta$		
fin-amb	$=$	$\Pi y.\theta_1$	$\oplus \cdots \oplus$	$\Pi y.\theta_n$
poly-amb	$=$	$\Sigma x_1 \ldots \Sigma x_{k_1}.\Pi y.\theta_1$	$\oplus \cdots \oplus$	$\Sigma x_1 \ldots \Sigma x_{k_n}.\Pi y.\theta_n$

Finite Ambiguity

Theorem

Finitely Ambiguous

$$[\![\mathcal{A}]\!] = [\![\mathcal{A}_1]\!] \oplus \ldots \oplus [\![\mathcal{A}_n]\!]$$

Unambiguous

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(Q, Γ, μ, ν)

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$$(Q, \Gamma, \mu, \nu) \xrightarrow{\text{"booleanize"}} (Q, \Gamma, \mu_{\mathbb{B}}, \nu_{\mathbb{B}})$$

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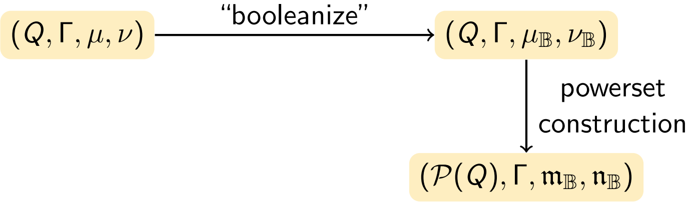

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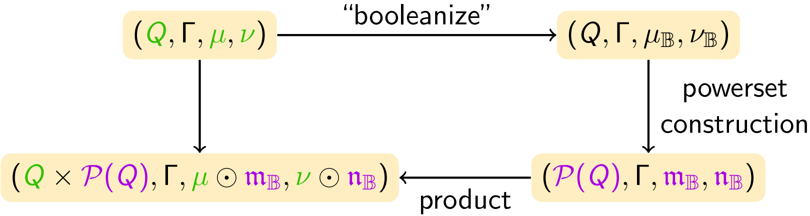

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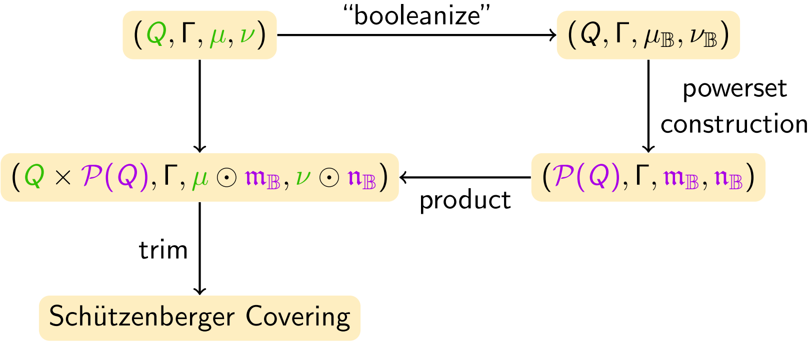

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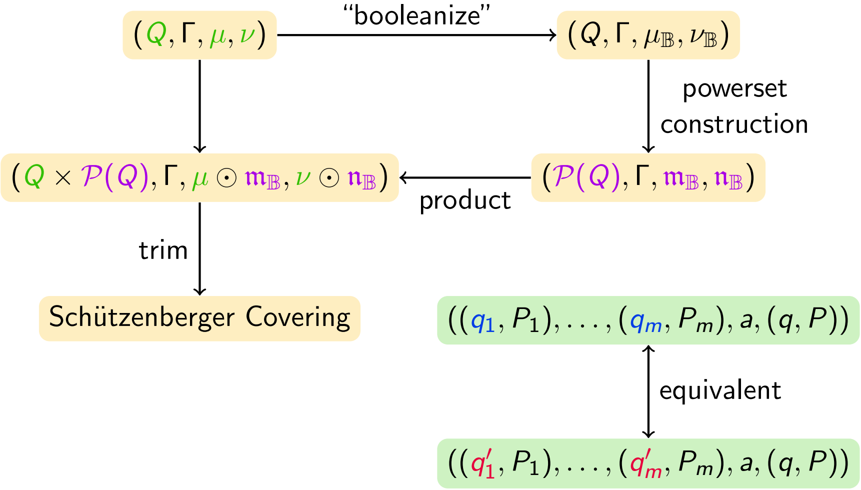

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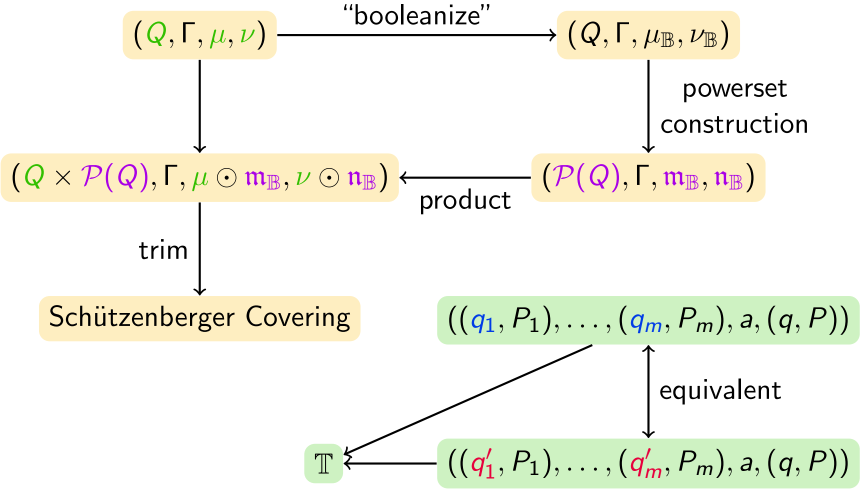

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$\mathbb{T}_1$

$\mathbb{T}_2$

$\vdots$

$\mathbb{T}_n$

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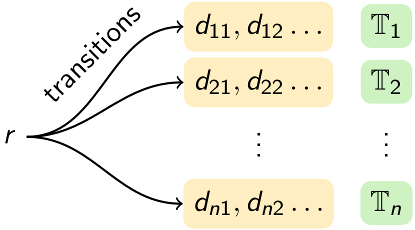

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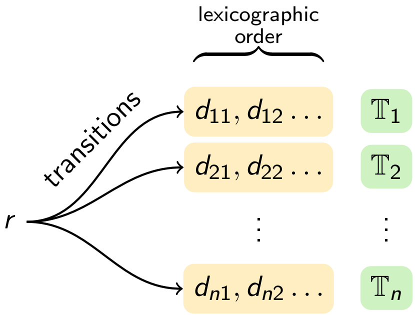

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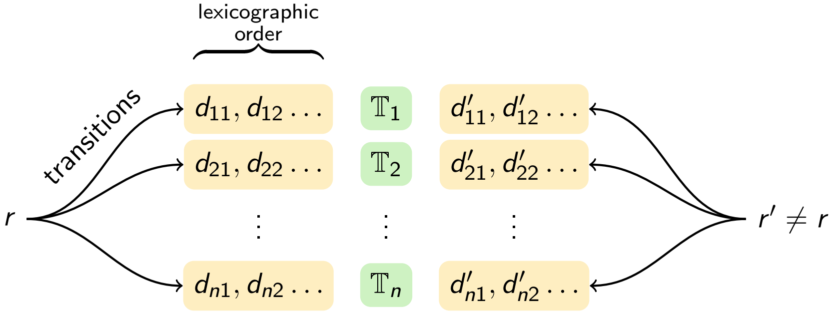

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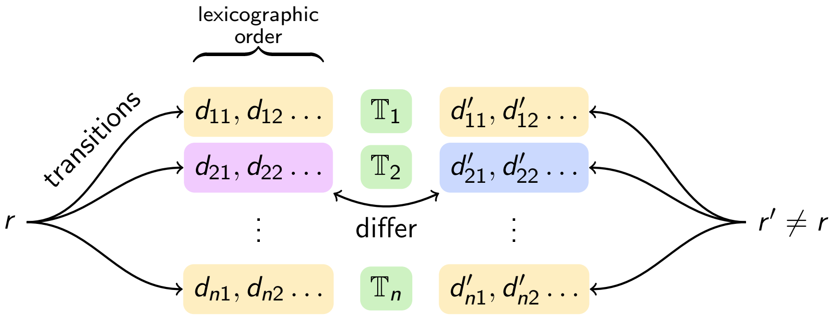

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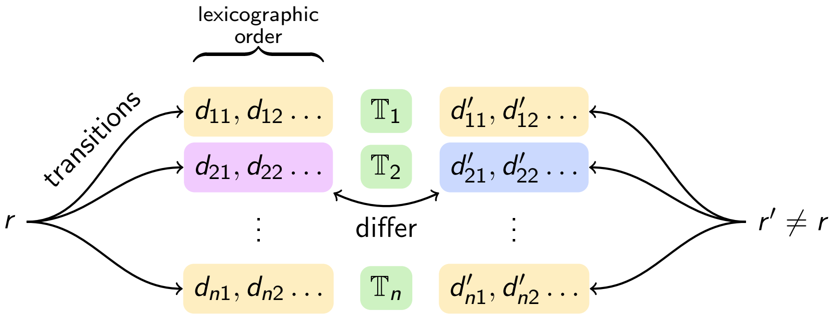

$$\begin{pmatrix} (p, P) \\ d_{11}, d_{12} \ldots \\ d_{21}, d_{22} \ldots \\ \vdots \\ d_{n1}, d_{n2} \ldots \end{pmatrix}$$

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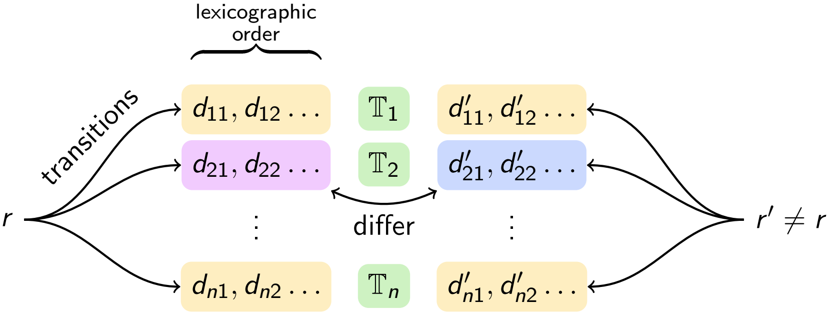

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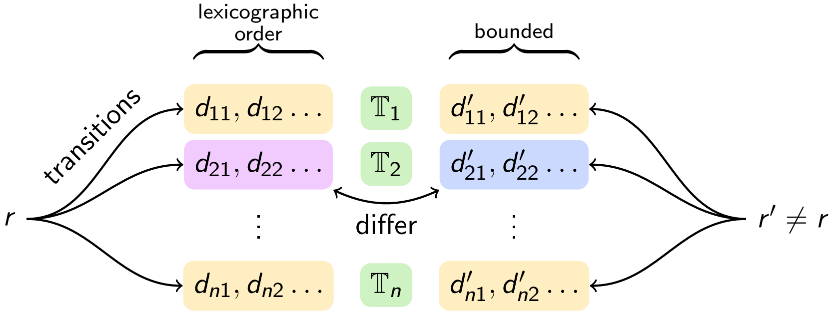

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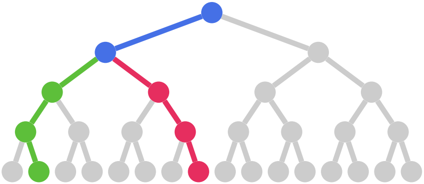

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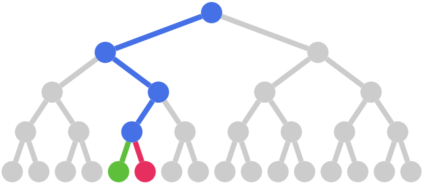

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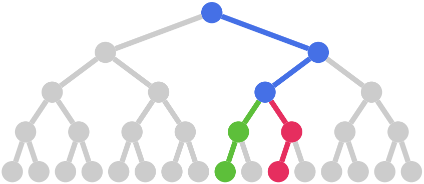

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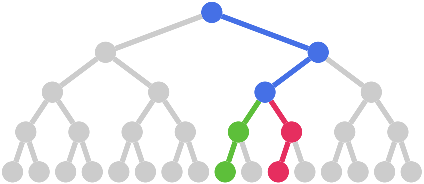

Polynomial Ambiguity

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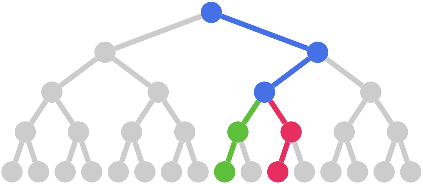

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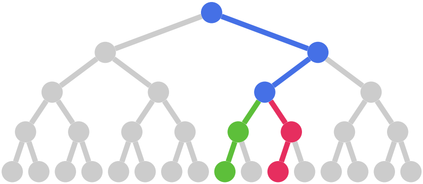

(q, q, a, p)

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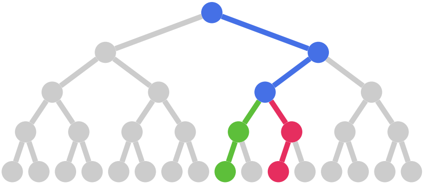

(q, q, a, p)

$\downarrow$

(q_1, q_2, a, p)

Polynomial Ambiguity: $[\![\mathcal{A}]\!] = [\![\mathcal{A}_1]\!] \oplus \ldots \oplus [\![\mathcal{A}_n]\!]$

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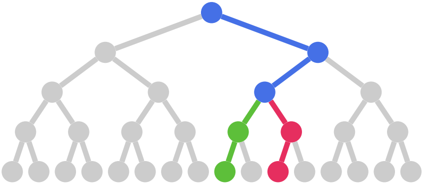

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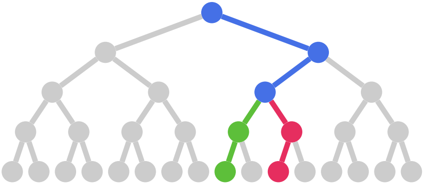

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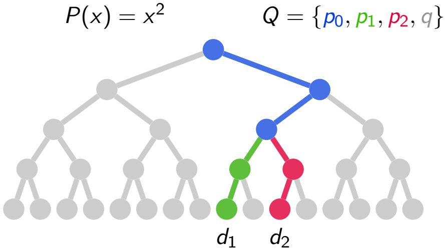

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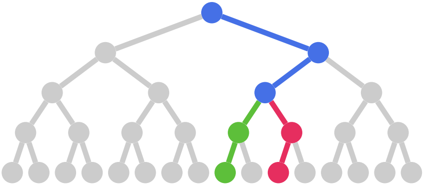

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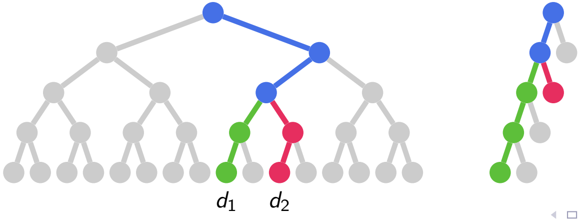


Polynomial Ambiguity: $[\![\mathcal{A}]\!] = [\![\mathcal{A}_1]\!] \oplus \ldots \oplus [\![\mathcal{A}_n]\!]$

Theorem

$\mathcal{A}$ std + $\deg(\mathcal{A}) = k$

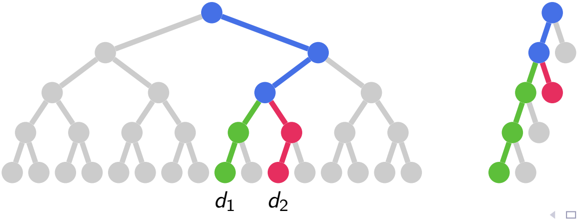
- $\exists$transitions $d_1 \ldots d_k$ $\exists C$

  $$\forall t\ \forall \vec{w} : |\mathrm{Run}(t; \vec{w}, \vec{d})| \leq C \quad \wedge \quad \forall r\ \forall i : d_i \in r$$

- $\exists t_n$ $\exists C$

  $$|t_n| \leq Cn \quad \wedge \quad |\mathrm{Run}(t_n)| \geq n^k$$

$P(x) = x^2$ $\qquad$ $Q = \{p_0, p_1, p_2, q\}$



$|t_n| = 2n + 1$

Polynomial Ambiguity: $[\![\mathcal{A}]\!] = [\![\mathcal{A}_1]\!] \oplus \ldots \oplus [\![\mathcal{A}_n]\!]$

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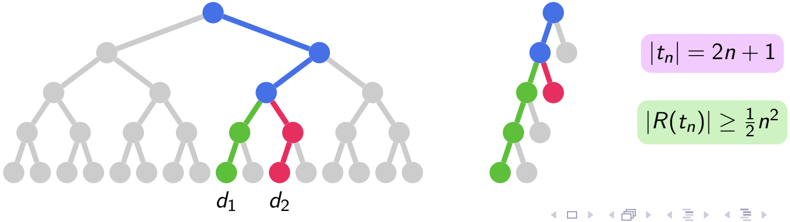

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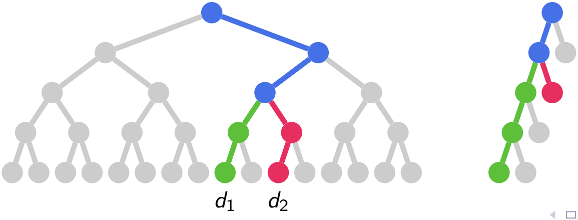

$|t_n| = 2n + 1$

$|R(t_n)| \geq \frac{1}{2} n^2$