

Weighted Tree Automata

Ambiguity and Logics

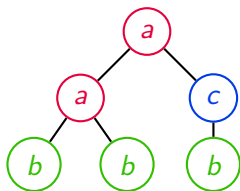
Erik Paul

November 10, 2015

Trees

(Γ, rk)

ranked alphabet



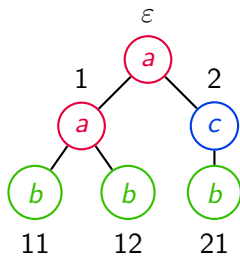
Trees

(Γ, rk)

ranked alphabet

$\text{pos}(t)$

positions



Trees

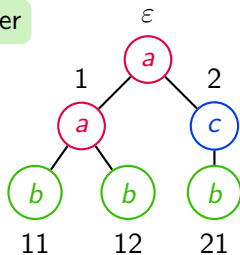
(Γ, rk)

ranked alphabet

$\text{pos}(t)$

positions

lexicographic order



Weighted Tree Automata

$(K, \oplus, \odot, 0, 1)$

commutative semiring

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$(K, \oplus, \odot, 0, 1)$

commutative semiring

$\mathcal{A} = (Q, \Gamma, \mu, \nu)$

automaton

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automaton

$\mu: (p_1, \dots, p_m, a, p) \mapsto \kappa$

transition weights

Weighted Tree Automata

$(K, \oplus, \odot, 0, 1)$

commutative semiring

$\mathcal{A} = (Q, \Gamma, \mu, \nu)$

automaton

$\mu: (p_1, \dots, p_m, a, p) \mapsto \kappa$

transition weights

$\nu: p \mapsto \kappa$

final weights

Runs

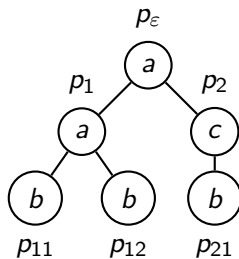
$r: \text{pos}(t) \rightarrow Q$

run

Runs

$r: \text{pos}(t) \rightarrow Q$

run



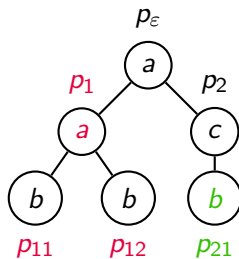
Runs

$r: \text{pos}(t) \rightarrow Q$

run

(p_{11}, p_{12}, a, p_1)

(b, p_{21})



Runs

$r: \text{pos}(t) \rightarrow Q$

run

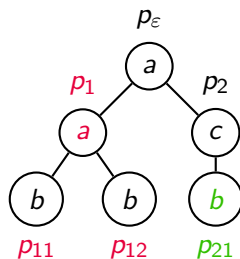
$\text{Run}(t)$

$\mu(\text{ transitions }) \neq 0$

$\nu(\text{ root }) \neq 0$

(p_{11}, p_{12}, a, p_1)

(b, p_{21})



Runs

$r: \text{pos}(t) \rightarrow Q$

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$\text{Run}(t)$

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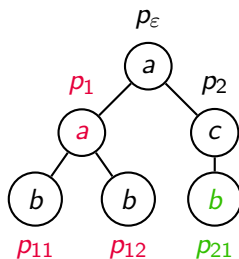
$\nu(\text{ root }) \neq 0$

$\text{wt}(r)$

$\nu(\text{ root }) \odot \prod \mu(\text{ transitions })$

(p_{11}, p_{12}, a, p_1)

(b, p_{21})



Runs

$r: \text{pos}(t) \rightarrow Q$

run

$\text{Run}(t)$

$\mu(\text{ transitions }) \neq 0$

$\nu(\text{ root }) \neq 0$

$wt(r)$

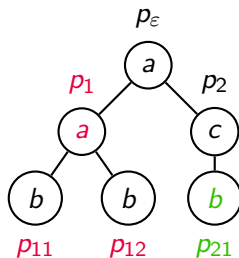
$\nu(\text{ root }) \odot \prod \mu(\text{ transitions })$

$\llbracket \mathcal{A} \rrbracket(t) = \sum_r wt(r)$

tree series

(p_{11}, p_{12}, a, p_1)

(b, p_{21})



MSO

$\varphi ::= \text{label}_a(x) \mid \text{edge}_i(x, y) \mid x \in X \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \exists X.\varphi$

MSO

$\varphi ::= \text{label}_a(x) \mid \text{edge}_i(x, y) \mid x \in X \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \exists X.\varphi$

QMSO

$\theta ::= \varphi \mid \kappa \mid \theta \oplus \theta \mid \theta \odot \theta \mid \Sigma x.\theta \mid \Sigma X.\theta \mid \Pi x.\theta$

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$$\llbracket \varphi \rrbracket = \mathbb{1}_{\mathcal{L}(\varphi)}$$

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$\llbracket \varphi \rrbracket = \mathbb{1}_{\mathcal{L}(\varphi)}$

$\llbracket \Sigma x.\text{label}_a(x) \rrbracket: t \mapsto |t|_a$

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QMSO

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$\llbracket \Sigma x.\text{label}_a(x) \rrbracket: t \mapsto |t|_a$

Theorem (Droste/Gastin/Vogler)

Weighted Tree Automata = restricted QMSO

Ambiguity

Ambiguity

unamb

$$|\text{Run}(t)| \leq 1$$

Ambiguity

Ambiguity

unamb

$$|\text{Run}(t)| \leq 1$$

fin-amb

$$|\text{Run}(t)| \leq C$$

Ambiguity

Ambiguity

unamb

$$|\text{Run}(t)| \leq 1$$

fin-amb

$$|\text{Run}(t)| \leq C$$

poly-amb

$$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$$

Ambiguity

Ambiguity

unamb	$ \text{Run}(t) \leq 1$
-------	--------------------------

fin-amb	$ \text{Run}(t) \leq C$
---------	--------------------------

poly-amb	$ \text{Run}(t) \leq P(\text{pos}(t))$
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Theorem

unamb	=	$\Pi y.\theta$
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Ambiguity

Ambiguity

$$\text{unamb} \qquad | \text{Run}(t) | \leq 1$$

$$\text{fin-amb} \qquad | \text{Run}(t) | \leq C$$

$$\text{poly-amb} \qquad | \text{Run}(t) | \leq P(| \text{pos}(t) |)$$

Theorem

$$\text{unamb} \quad = \quad \prod y.\theta$$

$$\text{fin-amb} \quad = \quad \prod y.\theta_1 \quad \oplus \cdots \oplus \quad \prod y.\theta_n$$

Ambiguity

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Theorem

$$\text{unamb} \quad = \quad \prod y. \theta$$

$$\text{fin-amb} \quad = \quad \prod y. \theta_1 \quad \oplus \cdots \oplus \quad \prod y. \theta_n$$

$$\text{poly-amb} \quad = \quad \sum x_1 \dots \sum x_{k_1}. \prod y. \theta_1 \quad \oplus \cdots \oplus \quad \sum x_1 \dots \sum x_{k_n}. \prod y. \theta_n$$

Finite Ambiguity

Theorem

Finitely Ambiguous

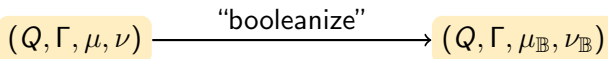
$$[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$$

Unambiguous

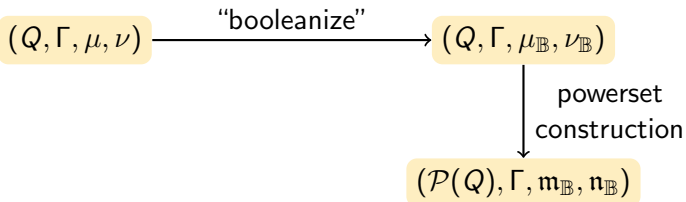
Finite Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

(Q, Γ, μ, ν)

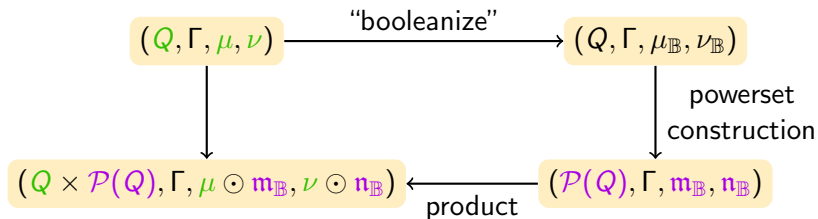
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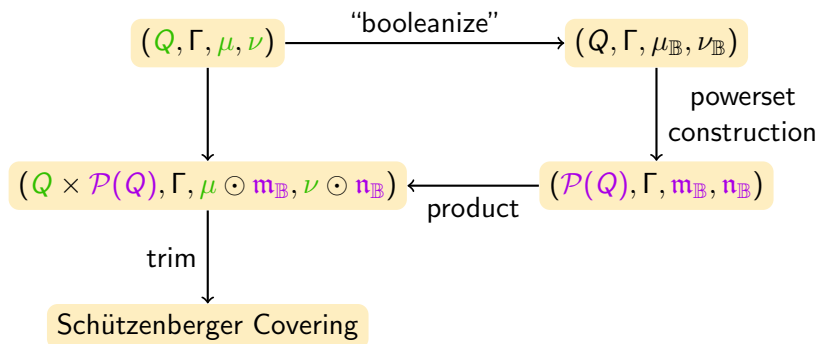
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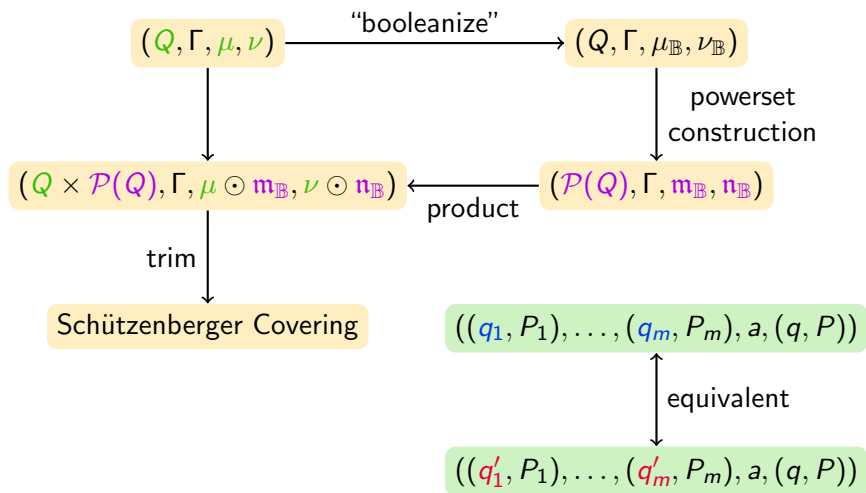
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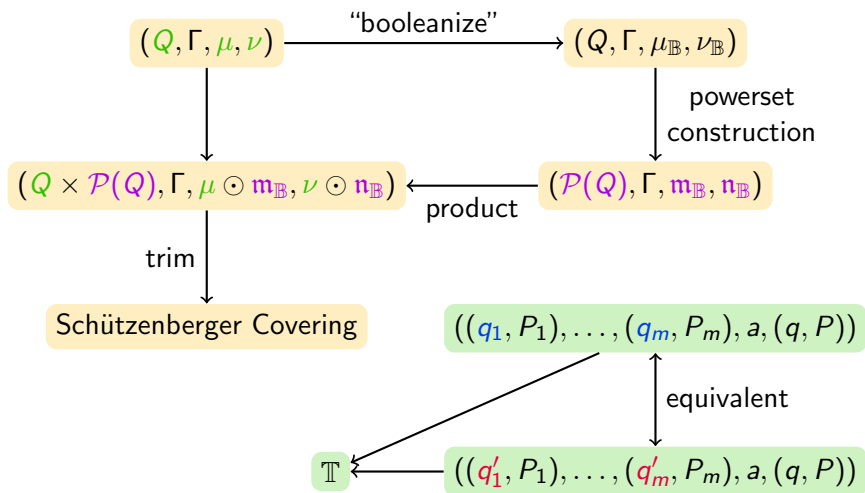
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Finite Ambiguity: $\llbracket \mathcal{A} \rrbracket = \llbracket \mathcal{A}_1 \rrbracket \oplus \dots \oplus \llbracket \mathcal{A}_n \rrbracket$



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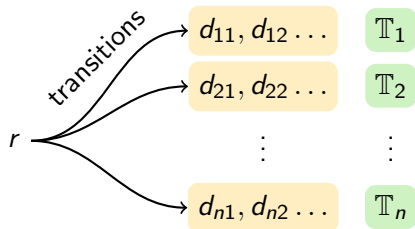
T_1

T_2

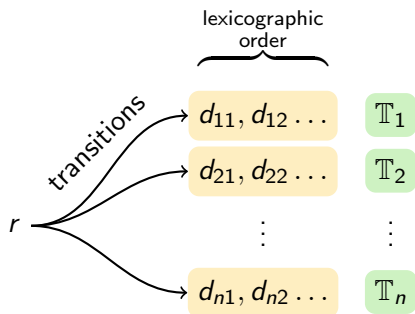
\vdots

T_n

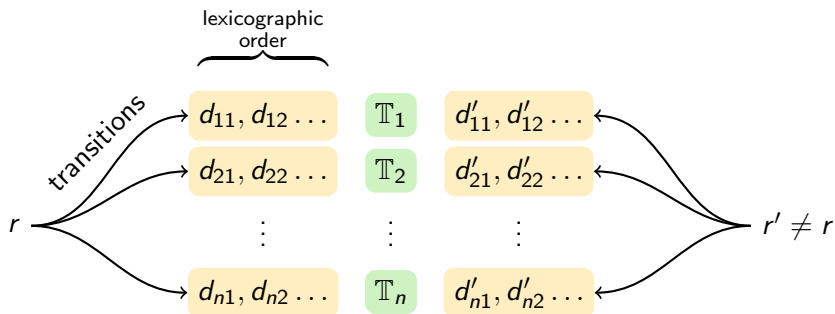
Finite Ambiguity: $[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$



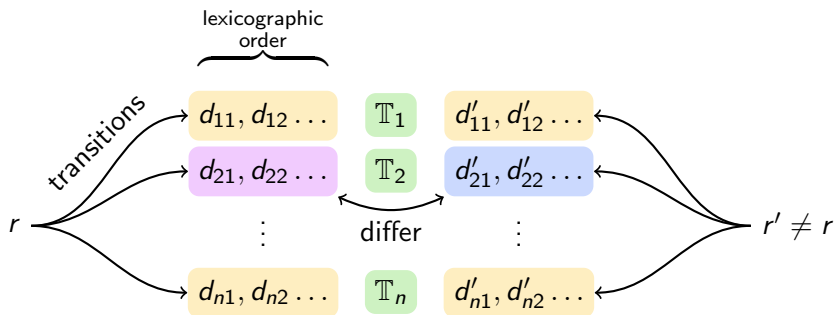
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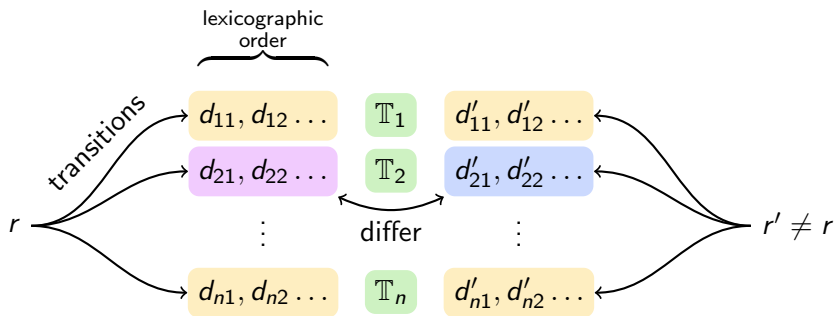
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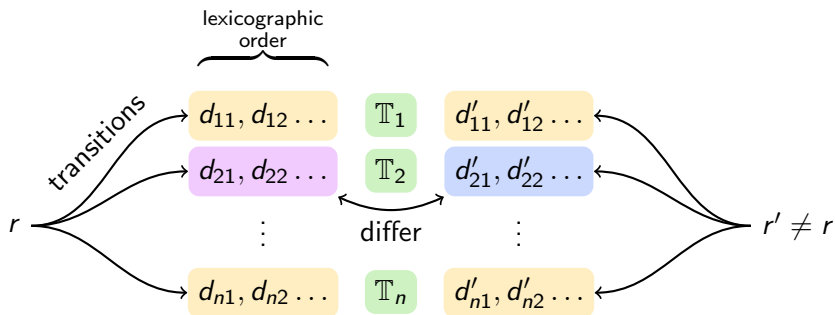


Finite Ambiguity: $[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$



$$\begin{pmatrix} (p, P) \\ d_{11}, d_{12} \dots \\ d_{21}, d_{22} \dots \\ \vdots \\ d_{n1}, d_{n2} \dots \end{pmatrix}$$

Finite Ambiguity: $[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$

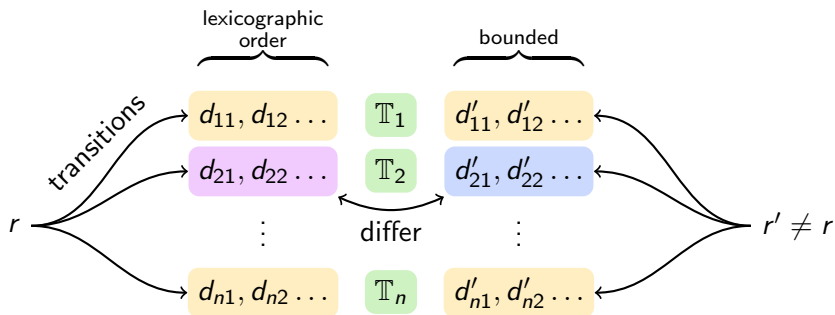


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\longrightarrow

separate final states

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\longrightarrow

separate final states

Polynomial Ambiguity

k -poly-amb

$$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$$

$$\deg(P) = k$$

Polynomial Ambiguity

k -poly-amb

$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$

$\deg(P) = k$

$\deg(\mathcal{A})$

min k

Polynomial Ambiguity

k -poly-amb

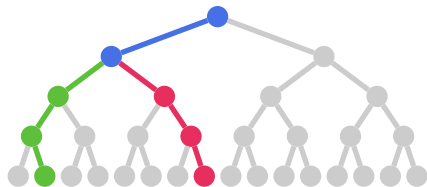
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$Q = \{p_0, p_1, p_2, q\}$



Polynomial Ambiguity

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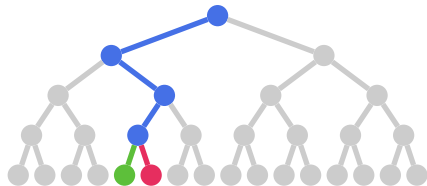
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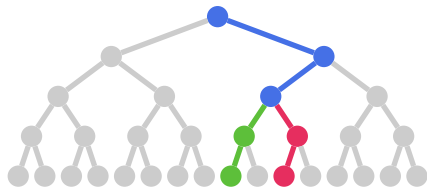
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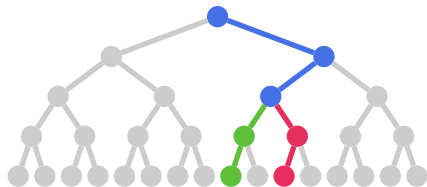
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$$P(x) = x^2$$

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Polynomial Ambiguity

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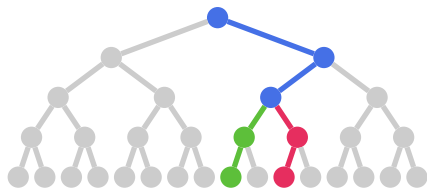
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$$|t_n| = 2n + 1$$

Polynomial Ambiguity

k -poly-amb

$$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$$

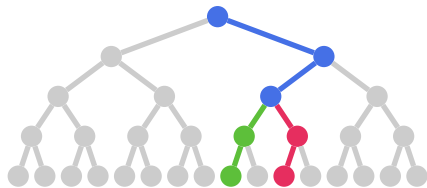
$$\deg(P) = k$$

$\deg(\mathcal{A})$

$\min k$

$$P(x) = x^2$$

$$Q = \{p_0, p_1, p_2, q\}$$



$$|t_n| = 2n + 1$$

$$|R(t_n)| \geq \frac{1}{2}n^2$$

Polynomial Ambiguity

k -poly-amb

$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$

$\deg(P) = k$

$\deg(\mathcal{A})$

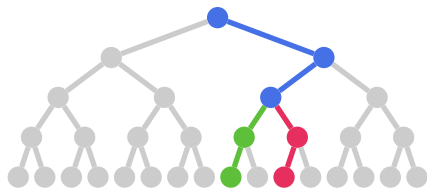
$\min k$

$\deg(p)$

$\deg(Q, \Gamma, \mu, \mathbb{1}_p)$

$$P(x) = x^2$$

$$Q = \{p_0, p_1, p_2, q\}$$



$$|t_n| = 2n + 1$$

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Polynomial Ambiguity

k -poly-amb

$$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$$

$$\deg(P) = k$$

$\deg(\mathcal{A})$

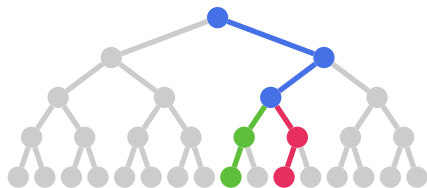
$\min k$

$\deg(p)$

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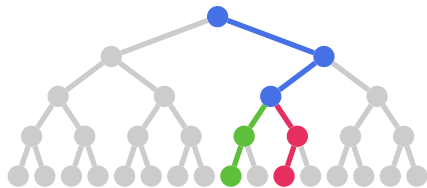
$$Q = \{p_0, p_1, p_2, q\}$$



$$|t_n| = 2n + 1$$

$$|R(t_n)| = n + 1$$

Polynomial Ambiguity

 $p \preceq q$ $\exists \text{path } p \searrow q$ $P(x) = x^2$ $Q = \{p_0, p_1, p_2, q\}$ 

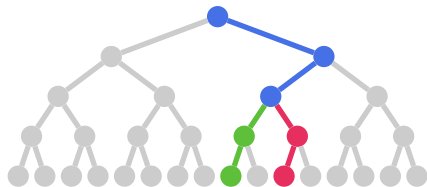
Polynomial Ambiguity

$$p \preceq q$$

$$\exists \text{path } p \searrow q$$

$$P(x) = x^2$$

$$Q = \{p_0, p_1, p_2, q\}$$



$$p_0 \preceq p_1 \preceq q$$

$$p_0 \preceq p_2 \preceq q$$

Polynomial Ambiguity

$$p \preceq q$$

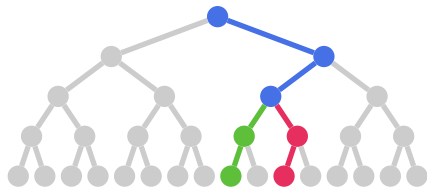
$$\exists \text{path } p \searrow q$$

$$\mathcal{C}(p)$$

strongly connected component

$$P(x) = x^2$$

$$Q = \{p_0, p_1, p_2, q\}$$



$$p_0 \preceq p_1 \preceq q$$

$$p_0 \preceq p_2 \preceq q$$

Polynomial Ambiguity

$$p \preceq q$$

$$\exists \text{path } p \searrow q$$

$\mathcal{C}(p)$

strongly connected component

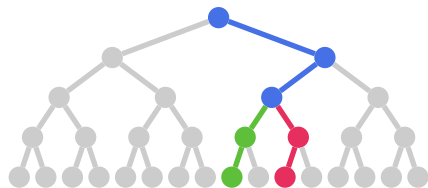
bridge out of $\mathcal{C}(p)$

$$(p_1, \dots, p_m, a, p_0)$$

$$p \approx p_0 \quad \forall i \ p_i \not\approx p_0$$

$$P(x) = x^2$$

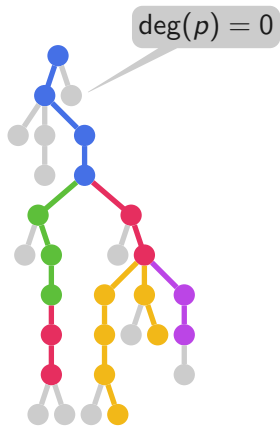
$$Q = \{p_0, p_1, p_2, q\}$$



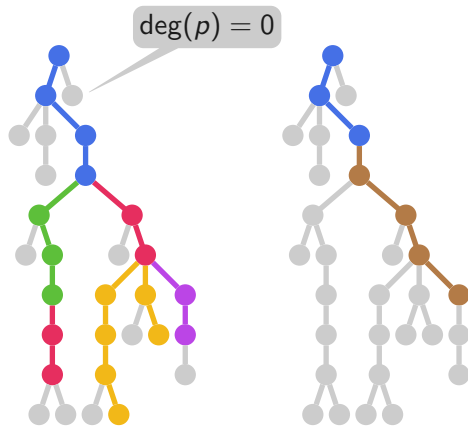
$$p_0 \preceq p_1 \preceq q$$

$$p_0 \preceq p_2 \preceq q$$

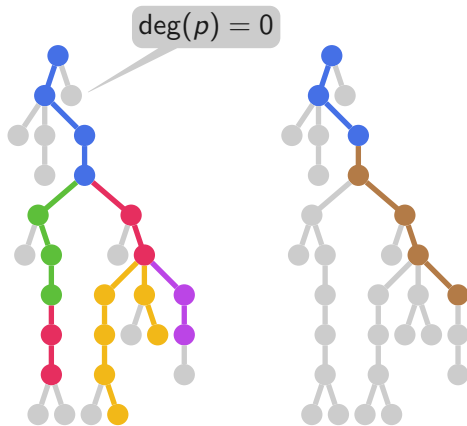
Polynomial Ambiguity



Polynomial Ambiguity



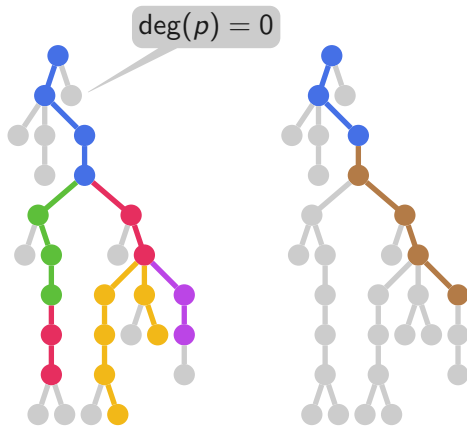
Polynomial Ambiguity



Standard Form

$\exists!$ final state

Polynomial Ambiguity



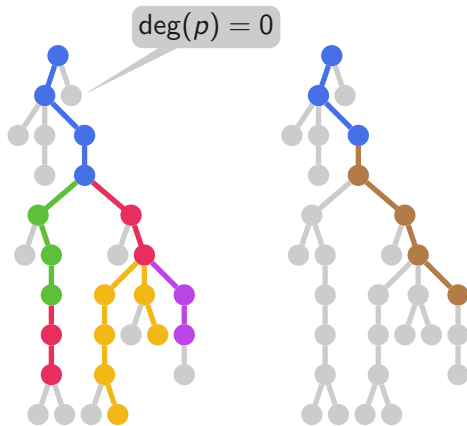
Standard Form

$\exists!$ final state

$\exists!$ bridge $d(p)$ out of $\mathcal{C}(p)$

\uparrow
 $\text{deg}(p) > 0$

Polynomial Ambiguity



Standard Form

$\exists!$ final state

$\exists!$ bridge $d(p)$ out of $\mathcal{C}(p)$

$\text{deg}(p) > 0$

$\forall r \exists! w : d(p) \text{ at } w$

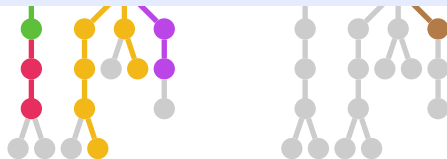
Polynomial Ambiguity

Theorem

polynomially ambiguous

$$[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$$

standardized



Standard Form

$\exists!$ final state

$\exists!$ bridge $d(p)$ out of $\mathcal{C}(p)$

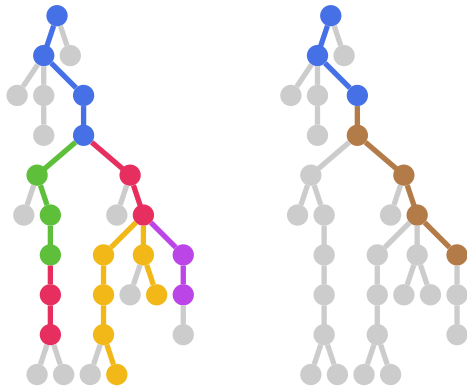
$\deg(p) > 0$

$\forall r \exists! w : d(p)$ at w

Polynomial Ambiguity

Theorem

$$[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$$



Standard Form

$\exists!$ final state

$\exists!$ bridge $d(p)$ out of $\mathcal{C}(p)$

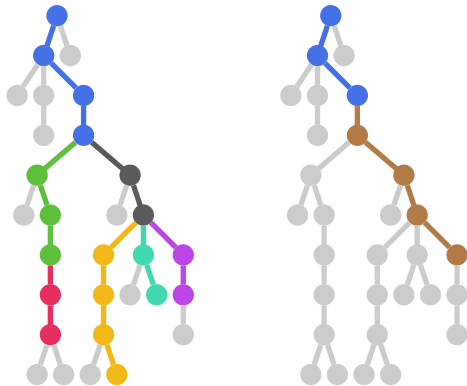
$\deg(p) > 0$

$\forall r \exists! w : d(p)$ at w

Polynomial Ambiguity

Theorem

$$[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$$



Standard Form

$\exists!$ final state

$\exists!$ bridge $d(p)$ out of $\mathcal{C}(p)$

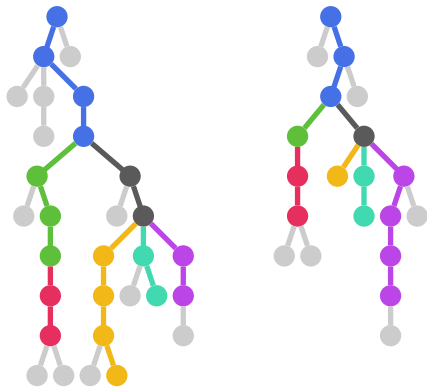
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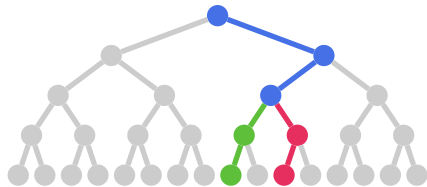
Polynomial Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

$\text{Run}(t; w_1 \dots w_k, d_1 \dots d_k)$

d_i at w_i

$$P(x) = x^2$$

$$Q = \{p_0, p_1, p_2, q\}$$



Polynomial Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

Theorem

\mathcal{A} std + $\deg(\mathcal{A}) = k$

■ \exists transitions $d_1 \dots d_k \exists C$

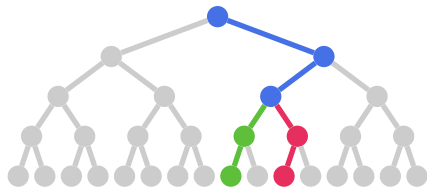
$\forall t \forall \vec{w} : |\text{Run}(t; \vec{w}, \vec{d})| \leq C \quad \wedge \quad \forall r \forall i : d_i \in r$

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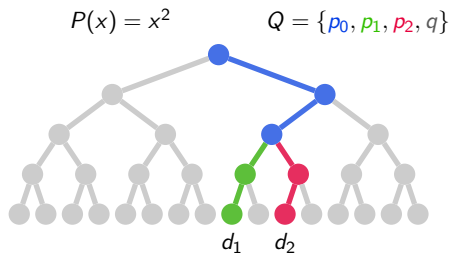
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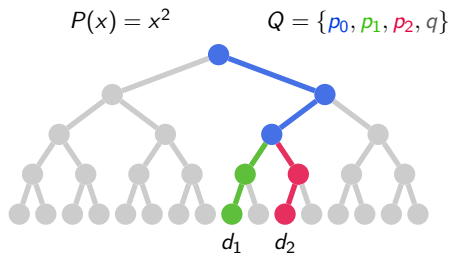
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$$|t_n| \leq Cn \quad \wedge \quad |\text{Run}(t_n)| \geq n^k$$



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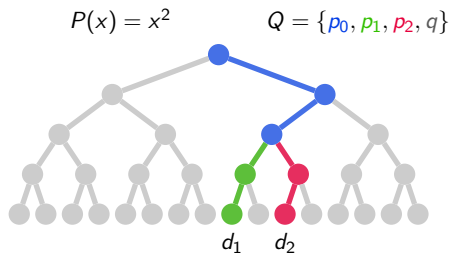
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$$|t_n| = 2n + 1$$

$$|R(t_n)| \geq \frac{1}{2}n^2$$

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Theorem

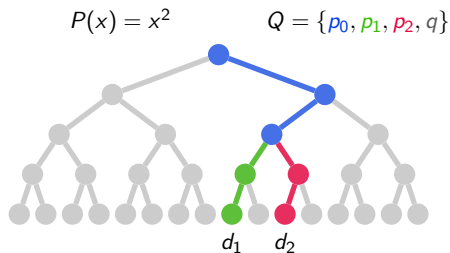
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