

On Finite and Polynomial Ambiguity of Weighted Tree Automata

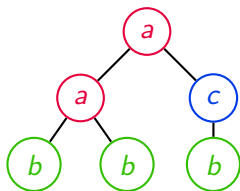
Erik Paul

July 28, 2016

Trees

(Γ, rk)

ranked alphabet



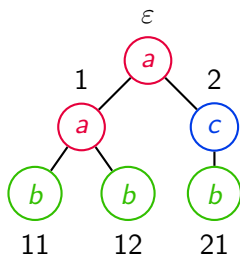
Trees

(Γ, rk)

ranked alphabet

$\text{pos}(t)$

positions



Trees

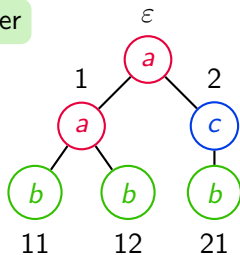
(Γ, rk)

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$\text{pos}(t)$

positions

lexicographic order



Weighted Tree Automata

$(K, \oplus, \odot, 0, 1)$

commutative semiring

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commutative semiring

$\mathcal{A} = (Q, \Gamma, \mu, \nu)$

automaton

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$\mu: (p_1, \dots, p_m, a, p) \mapsto \kappa$

transition weights

Weighted Tree Automata

$(K, \oplus, \odot, 0, 1)$

commutative semiring

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automaton

$\mu: (p_1, \dots, p_m, a, p) \mapsto \kappa$

transition weights

$\nu: p \mapsto \kappa$

final weights

Runs

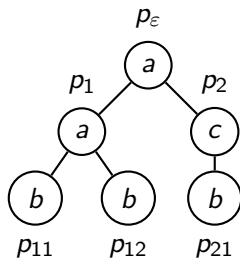
$r: \text{pos}(t) \rightarrow Q$

run

Runs

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run



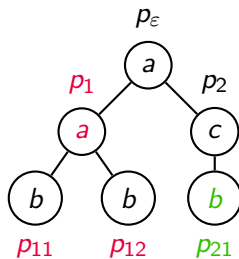
Runs

$r: \text{pos}(t) \rightarrow Q$

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(p_{11}, p_{12}, a, p_1)

(b, p_{21})



Runs

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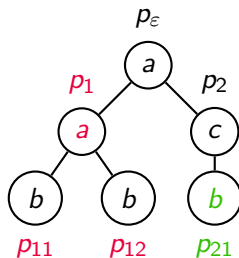
$\text{Run}(t)$

$\mu(\text{ transitions }) \neq 0$

$\nu(\text{ root }) \neq 0$

(p_{11}, p_{12}, a, p_1)

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Runs

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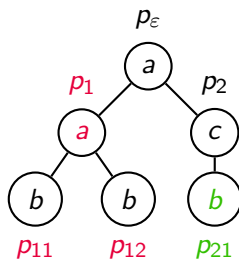
$\nu(\text{ root }) \neq 0$

$\text{wt}(r)$

$\nu(\text{ root }) \odot \prod \mu(\text{ transitions })$

(p_{11}, p_{12}, a, p_1)

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Runs

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$wt(r)$

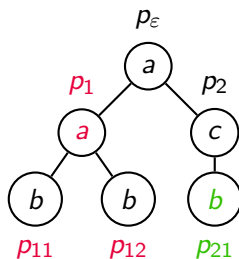
$\nu(\text{ root }) \odot \prod \mu(\text{ transitions })$

$\llbracket \mathcal{A} \rrbracket(t) = \sum_r wt(r)$

tree series

(p_{11}, p_{12}, a, p_1)

(b, p_{21})



MSO

$\varphi ::= \text{label}_a(x) \mid \text{edge}_i(x, y) \mid x \in X \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \exists X.\varphi$

MSO

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wMSO

$\theta ::= \varphi \mid \kappa \mid \theta \oplus \theta \mid \theta \odot \theta \mid \Sigma x.\theta \mid \Sigma X.\theta \mid \Pi x.\theta$

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$$\llbracket \varphi \rrbracket = \mathbb{1}_{\mathcal{L}(\varphi)}$$

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$$\llbracket \varphi \rrbracket = \mathbb{1}_{\mathcal{L}(\varphi)}$$

$$\llbracket \Sigma x.\text{label}_a(x) \rrbracket : t \mapsto |t|_a$$

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Theorem (Droste/Gastin/Vogler)

Weighted Tree Automata = restricted wMSO

Ambiguity

Ambiguity

unamb

$$|\text{Run}(t)| \leq 1$$

Ambiguity

Ambiguity

unamb

$$|\text{Run}(t)| \leq 1$$

fin-amb

$$|\text{Run}(t)| \leq C$$

Ambiguity

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unamb

$$|\text{Run}(t)| \leq 1$$

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$$|\text{Run}(t)| \leq C$$

poly-amb

$$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$$

Ambiguity

Ambiguity

unamb	$ \text{Run}(t) \leq 1$
-------	--------------------------

fin-amb	$ \text{Run}(t) \leq C$
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poly-amb	$ \text{Run}(t) \leq P(\text{pos}(t))$
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Theorem

unamb	=	$\Pi y.\theta$
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Ambiguity

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$$\text{unamb} \qquad | \text{Run}(t) | \leq 1$$

$$\text{fin-amb} \qquad | \text{Run}(t) | \leq C$$

$$\text{poly-amb} \qquad | \text{Run}(t) | \leq P(| \text{pos}(t) |)$$

Theorem

$$\text{unamb} \quad = \quad \prod y.\theta$$

$$\text{fin-amb} \quad = \quad \prod y.\theta_1 \quad \oplus \cdots \oplus \quad \prod y.\theta_n$$

Ambiguity

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Theorem

$$\text{unamb} \qquad = \qquad \prod y. \theta$$

$$\text{fin-amb} \qquad = \qquad \prod y. \theta_1 \oplus \dots \oplus \prod y. \theta_n$$

$$\text{poly-amb} \qquad = \qquad \sum x_1 \dots \sum x_{k_1}. \prod y. \theta_1 \oplus \dots \oplus \sum x_1 \dots \sum x_{k_n}. \prod y. \theta_n$$

Finite Ambiguity

Theorem

Finitely Ambiguous

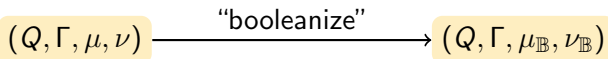
$$[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$$

Unambiguous

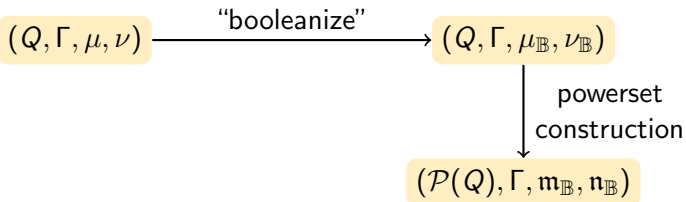
Finite Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

(Q, Γ, μ, ν)

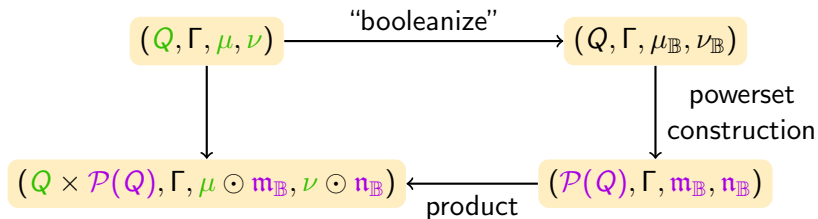
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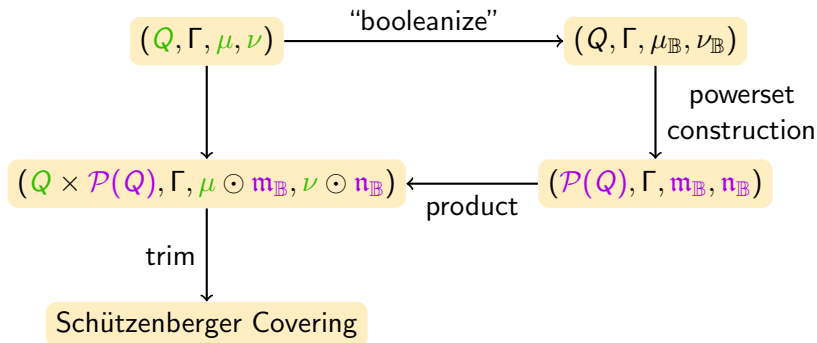
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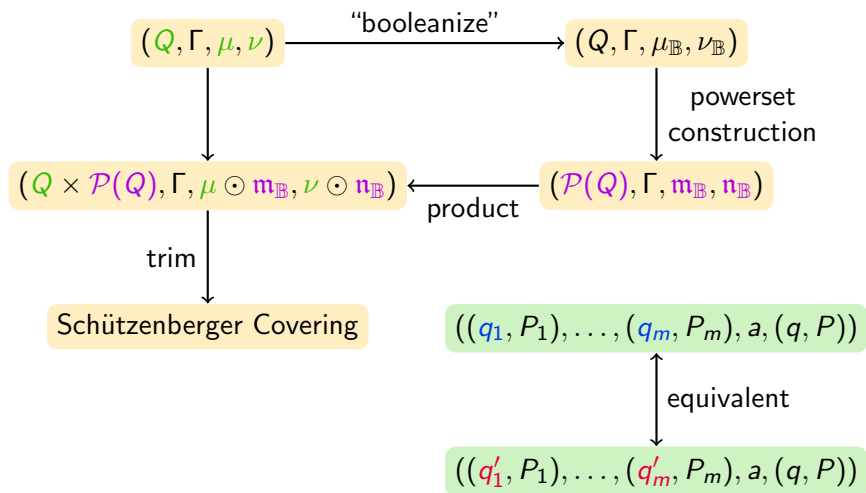
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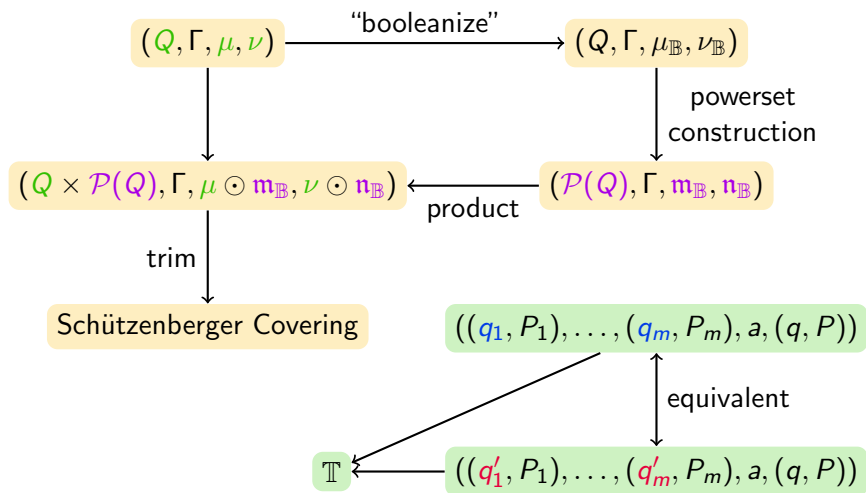
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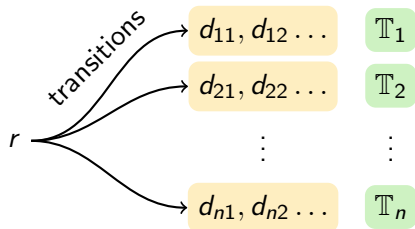
T_1

T_2

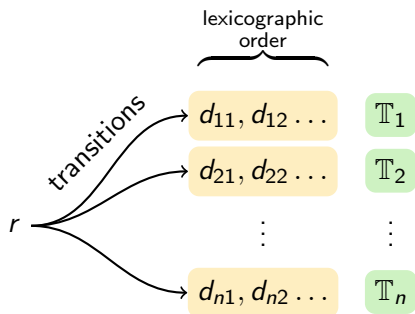
\vdots

T_n

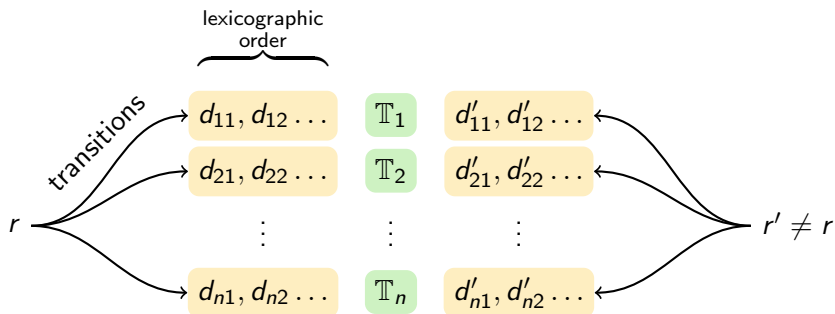
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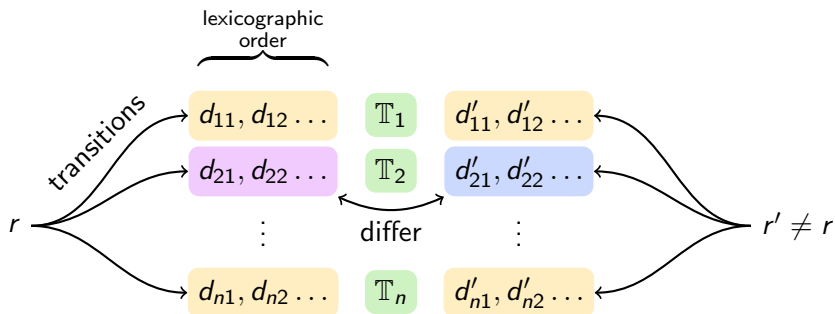
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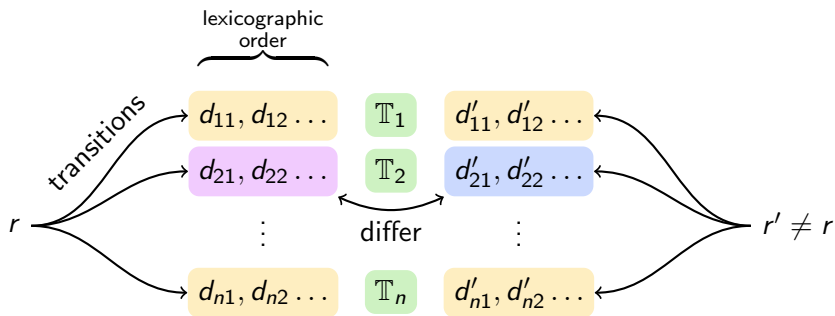
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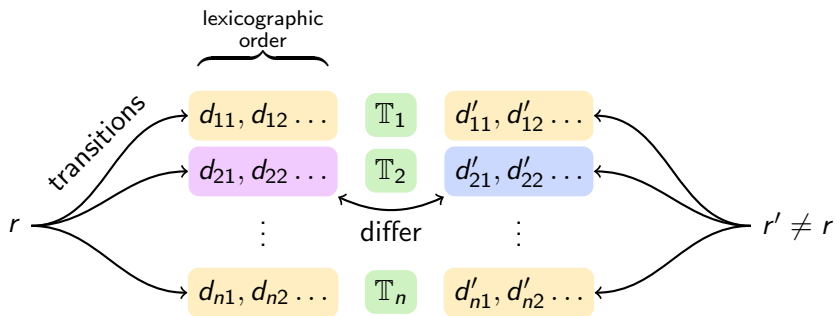


Finite Ambiguity: $[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$



$$\begin{pmatrix} (p, P) \\ d_{11}, d_{12} \dots \\ d_{21}, d_{22} \dots \\ \vdots \\ d_{n1}, d_{n2} \dots \end{pmatrix}$$

Finite Ambiguity: $[\mathcal{A}] = [\mathcal{A}_1] \oplus \dots \oplus [\mathcal{A}_n]$

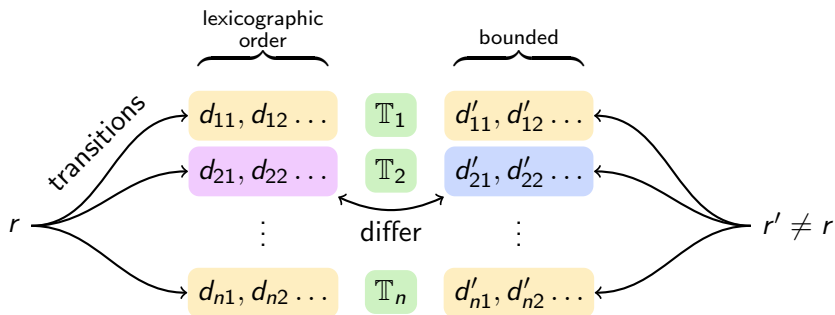


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→

separate final states

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\longrightarrow

separate final states

Polynomial Ambiguity

k -poly-amb

$$|\text{Run}(t)| \leq P(|\text{pos}(t)|)$$

$$\deg(P) = k$$

Polynomial Ambiguity

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$\deg(P) = k$

$\deg(\mathcal{A})$

$\min k$

Polynomial Ambiguity

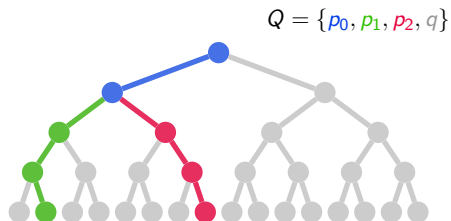
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Polynomial Ambiguity

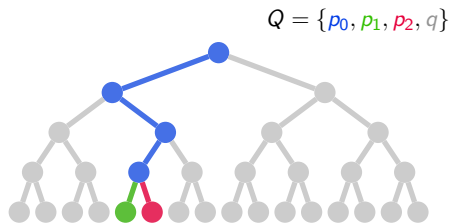
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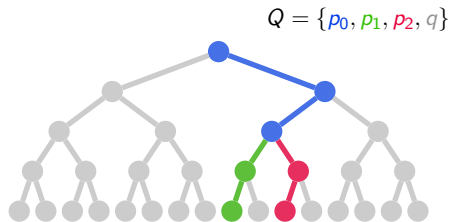
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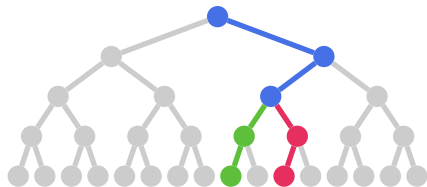
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$$P(x) = x^2$$

$$Q = \{p_0, p_1, p_2, q\}$$



Polynomial Ambiguity

Theorem

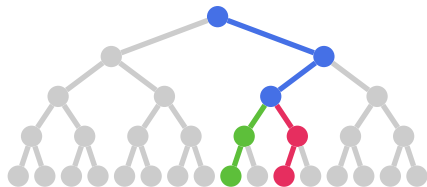
Polynomially Ambiguous

$$[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$$

↑
standardized

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Polynomial Ambiguity

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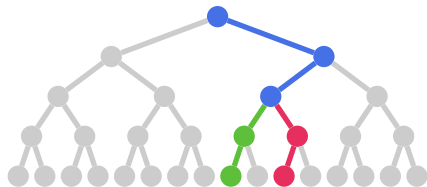
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(q, q, a, p)

Polynomial Ambiguity

Theorem

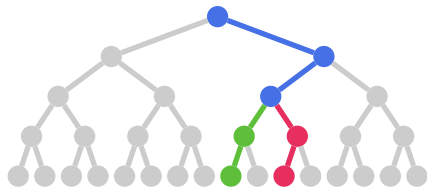
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$$(q, q, a, p)$$



$$(q_1, q_2, a, p)$$

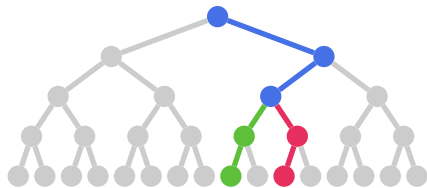
Polynomial Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

$\text{Run}(t; w_1 \dots w_k, d_1 \dots d_k)$

d_i at w_i

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Polynomial Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

Theorem

\mathcal{A} std + $\deg(\mathcal{A}) = k$

■ \exists transitions $d_1 \dots d_k \exists C$

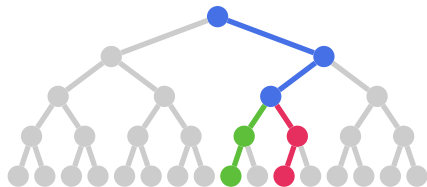
$$\forall t \forall \vec{w} : |\text{Run}(t; \vec{w}, \vec{d})| \leq C \quad \wedge \quad \forall r \forall i : d_i \in r$$

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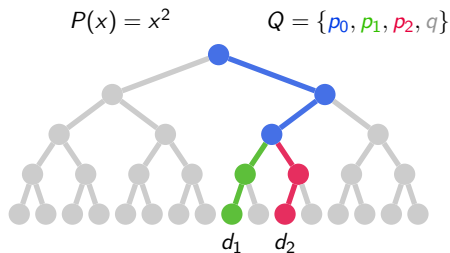
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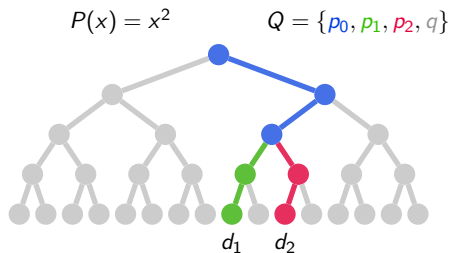
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■ $\exists t_n \exists C$

$$|t_n| \leq Cn \quad \wedge \quad |\text{Run}(t_n)| \geq n^k$$



Polynomial Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

Theorem

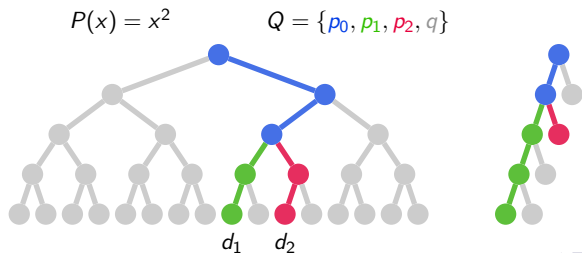
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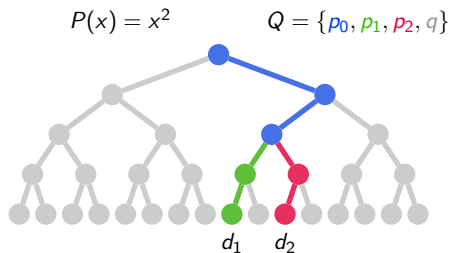
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$$|t_n| = 4n + 1$$

Polynomial Ambiguity: $\llbracket \mathcal{A} \rrbracket = \llbracket \mathcal{A}_1 \rrbracket \oplus \dots \oplus \llbracket \mathcal{A}_n \rrbracket$

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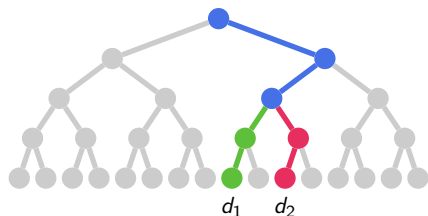
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$$|t_n| = 4n + 1$$

$$|\text{Run}(t_n)| \geq 2n^2$$

Polynomial Ambiguity: $[[\mathcal{A}]] = [[\mathcal{A}_1]] \oplus \dots \oplus [[\mathcal{A}_n]]$

Theorem

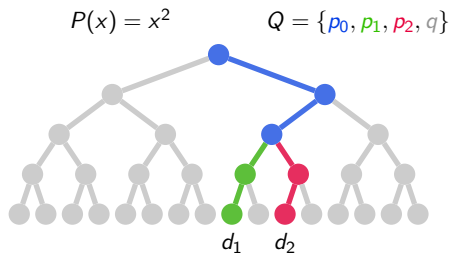
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■ \exists transitions $d_1 \dots d_k \exists C$

$$\forall t \forall \vec{w} : |\text{Run}(t; \vec{w}, \vec{d})| \leq C \quad \wedge \quad \forall r \forall i : d_i \in r$$

■ $\exists t_n \exists C$

$$|t_n| \leq Cn \quad \wedge \quad |\text{Run}(t_n)| \geq n^k$$



$$|t_n| = 4n + 1$$

$$|\text{Run}(t_n)| \geq 2n^2$$