Expressiveness and Decidability of Weighted Automata and Weighted Logics

Erik Paul

QuantLA

Max-Plus Tree Automata

A Weighted Feferman-Vaught Theorem

Monitor Logic

Max-Plus Automata

$\mathcal{A} = (Q, T, I, F)$ over $\Sigma = \{a, b\}$

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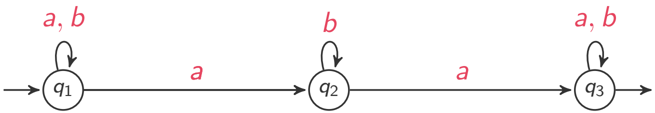

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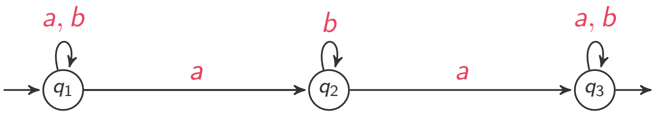

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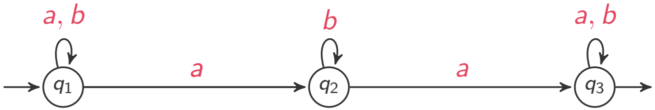

Weights in $\mathbb{R} \cup \{-\infty\}$

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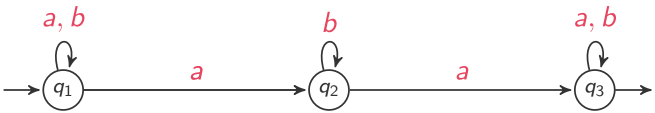

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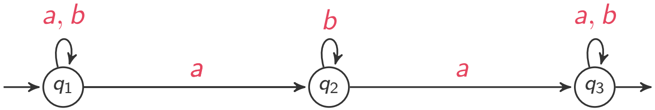

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Weight of run:

initial weight + transition weights + final weight

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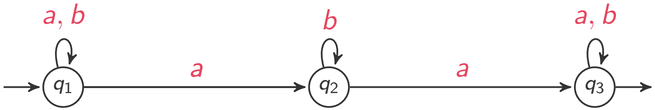

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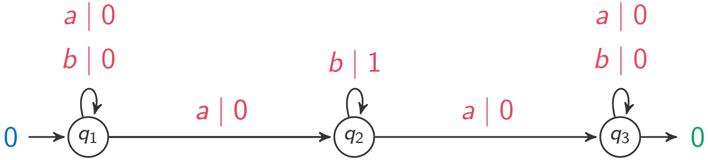

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Ambiguity

sequential / deterministic

one "initial state"
no two valid $p \xrightarrow{a} q_1$, $p \xrightarrow{a} q_2$

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$$\text{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq -\infty\}$$

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$$\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq -\infty\}$$

unambiguous — $|\mathrm{Run}(w)| \leq 1$

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unambiguous	$\lvert\text{Run}(w)\rvert \leq 1$
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Equivalence problem

Given $\mathcal{A}_1, \mathcal{A}_2$ Is $[\![\mathcal{A}_1]\!](w) = [\![\mathcal{A}_2]\!](w)$ for all w?

Four Decision Problems

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Finite Sequentiality problem

Given $\mathcal{A}$ — Is $[\![\mathcal{A}]\!] = \max_{i=1}^{n} [\![\mathcal{A}_i]\!]$ for some determ $\mathcal{A}_i$?

Decidability for max-plus automata on words

	Equivalence	Unambiguity	Sequentiality	Fin Seq
fin-amb				
poly-amb				
general				

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Krob

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fin-amb	yes	yes	yes	yes
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... on trees before the thesis

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fin-amb	?	?	?	?
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Four Decision Problems

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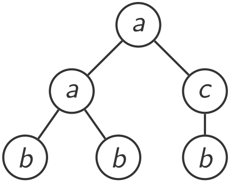
ranked alphabet Γ



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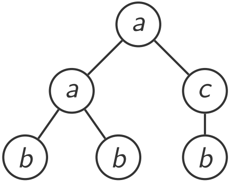

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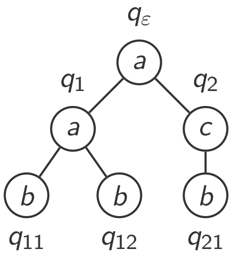

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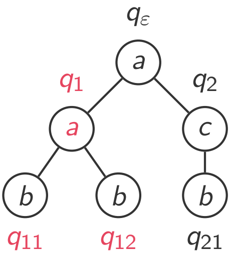

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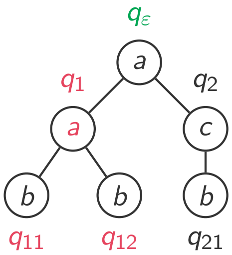

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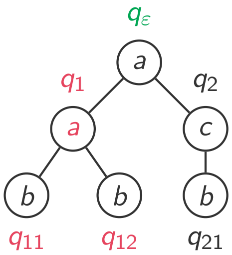

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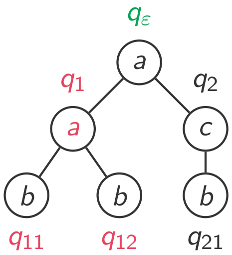

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unambiguity problem: decomposing runs

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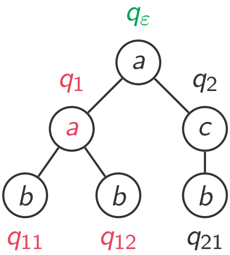

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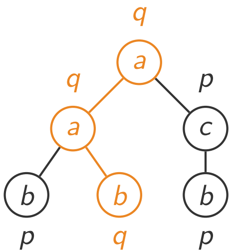

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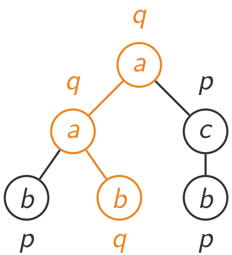

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equivalence problem:

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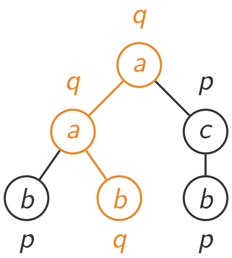

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equivalence problem: new approach

$\rightarrow$ uses Parikh's Theorem simplifies word case

$\mathcal{A} = (Q, \Gamma, \mu, \nu)$ over ranked alphabet Γ

weight of run $=$

transition weights $+$ final weight

(q_{11}, q_{12}, a, q_1)

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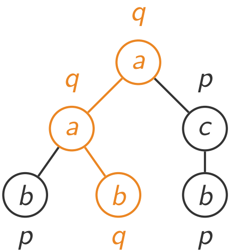

finite sequentiality problem:

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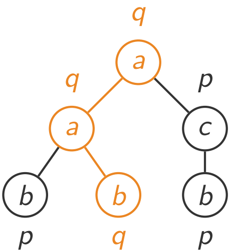

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finite sequentiality problem:

THM [Bala, Koniński]

$\mathcal{A}$ unamb ⇒ $[\![\mathcal{A}]\!]$ finitely sequential ↔ $\mathcal{A}$ satisfies X

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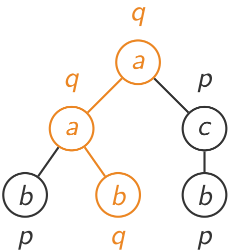

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finite sequentiality problem:

THM [Bala, Koniński]

$\mathcal{A}$ unamb $\Rightarrow$ $[\![\mathcal{A}]\!]$ finitely sequential $\leftrightarrow$ $\mathcal{A}$ satisfies X

New THM

$\mathcal{A}$ unamb $\Rightarrow$ $[\![\mathcal{A}]\!]$ finitely sequential $\leftrightarrow$ $\mathcal{A}'$ satisfies X and Y

A Weighted Feferman-Vaught Theorem

- finite automata $\leftrightarrow$ MSO logic [Büchi]

Weighted Logics — Motivation

- finite automata $\leftrightarrow$ MSO logic [Büchi]
- max-plus automata $\leftrightarrow$??? logic

WEIGHTED LOGICS — MOTIVATION

- finite automata $\leftrightarrow$ MSO logic [Büchi]

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  $\rightsquigarrow$ weighted automata $\leftrightarrow$ weighted MSO [Droste, Gastin]

■ finite automata	$\leftrightarrow$	MSO logic [Büchi]
■ max-plus automata	$\leftrightarrow$	??? logic
$\rightsquigarrow$ weighted automata	$\leftrightarrow$	weighted MSO [Droste, Gastin]
■ ambiguity subclasses	$\leftrightarrow$	fragments of weighted MSO

Weighted Logics — Motivation

- finite automata $\leftrightarrow$ MSO logic [Büchi]
- max-plus automata $\leftrightarrow$??? logic
  $\rightsquigarrow$ weighted automata $\leftrightarrow$ weighted MSO [Droste, Gastin]
- ambiguity subclasses $\leftrightarrow$ fragments of weighted MSO
- Feferman-Vaught theorem: MSO $\rightarrow$ weighted MSO

Feferman-Vaught Theorem

formula β $\overset{\text{satisfaction}}{\longleftrightarrow}$ structure $\mathcal{A}$

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Feferman-Vaught theorem

question about union of structures $\mathcal{A} \sqcup \mathcal{B}$

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questions about $\mathcal{A}$

questions about $\mathcal{B}$

formula β $\xleftrightarrow{\text{satisfaction}}$ structure $\mathcal{A}$

Feferman-Vaught theorem

question about union of structures $\mathcal{A} \sqcup \mathcal{B}$

$\Uparrow$

combine answers

$\nearrow$ $\nwarrow$

questions about $\mathcal{A}$ questions about $\mathcal{B}$

$\sigma = (\mathsf{Rel}, \mathsf{ar})$	signature
$\mathsf{Rel} = \{R_1, \dots, R_m\}$	relation symbols
$\mathsf{ar} \colon \mathsf{Rel} \to \mathbb{N}$	arity function

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$\mathcal{A} = (A, \mathcal{I})$	σ-structure
A	universe
$\mathcal{I}(R) \subseteq A^{\mathrm{ar}(R)}$ $(R \in \mathrm{Rel})$	interpretation

Disjoint union $\mathcal{A} \sqcup \mathcal{B}$ of σ-structures

$A \sqcup B$	universe
$\mathcal{I}_{\mathcal{A}}(R) \sqcup \mathcal{I}_{\mathcal{B}}(R)$	interpretation

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MSO(σ) logic

$$\beta ::= R(x_1, \ldots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$

Disjoint union $\mathcal{A} \sqcup \mathcal{B}$ of σ-structures

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Propositional formulas Prop

$$P ::= x_i \mid y_i \mid P \vee P \mid P \wedge P$$

Given

signature σ $\beta \in \mathrm{MSO}(\sigma)$

Classical Feferman-Vaught Theorem

Given

signature σ $\beta \in \mathrm{MSO}(\sigma)$

---

there exist

$n \geq 1$ $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{\mathbb{2}} \in \mathrm{MSO}(\sigma)^n$ $P \in \mathrm{Prop}$

Given

signature σ $\qquad$ $\beta \in \mathrm{MSO}(\sigma)$

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there exist

$n \geq 1$ $\qquad$ $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{\mathbb{2}} \in \mathrm{MSO}(\sigma)^n$ $\qquad$ $P \in \mathrm{Prop}$

---

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$\mathcal{A} \sqcup \mathcal{B} \models \beta$ iff $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \mathtt{true}$

where $x_i = \mathtt{true}$ iff $\beta_i^{\mathbb{1}} \models \mathcal{A}$

$y_i = \mathtt{true}$ iff $\beta_i^{\mathbb{2}} \models \mathcal{B}$

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$\langle\!\langle E \rangle\!\rangle : S^n \times S^n \to S$

$$\langle\!\langle x_1 \oplus y_2 \rangle\!\rangle(\bar{s}, \bar{t}) = s_1 \oplus t_2$$

Given

signature σ $\beta \in \mathrm{MSO}(\sigma)$

---

there exist

$n \geq 1$ $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{\mathbb{2}} \in \mathrm{MSO}(\sigma)^n$ $P \in \mathrm{Prop}$

---

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$\mathcal{A} \sqcup \mathcal{B} \models \beta$ iff $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \texttt{true}$

where $x_i = \texttt{true}$ iff $\beta_i^{\mathbb{1}} \models \mathcal{A}$

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Classical Feferman-Vaught Theorem

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Given

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there exist

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such that for all finite structures $\mathcal{A}$,$\mathcal{B}$

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Weighted Feferman-Vaught Theorem

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Example $\quad \text{label}_a(\cdot), \text{label}_b(\cdot), \text{edge}(\cdot, \cdot) \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

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$\bar{\varphi}^{1} = \bar{\varphi}^{2} = (\varphi_{|b-b|}, \varphi_{|a|})$ $\qquad E = (x_1 \oplus y_1) \otimes (x_2 \oplus y_2)$

Results of the Thesis: Overview

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    counterexamples exist

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item x restocked (in a shop) every Monday

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Quantitative Monitor Automata $\leftrightarrow$ Monitor Logic

$\rightarrow$ extension of logic for weighted Büchi automata

Equivalence, Unambiguity, (Finite) Sequentiality of Max-Plus Tree Automata

A Weighted Feferman-Vaught Theorem

Monitor Logic for Quantitative Monitor Automata