A Feferman-Vaught Decomposition Theorem for Weighted MSO Logic

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$\sigma = (\mathsf{Rel}, \mathsf{ar})$	signature
$\mathsf{Rel} = \{R_1, \ldots, R_m\}$	relation symbols
$\mathsf{ar}\colon \mathsf{Rel} \to \mathbb{N}$	arity function

Model Theory

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$\mathfrak{A} = (A, \mathcal{I})$	σ-structure
A	finite universe
$\mathcal{I}(R) \subseteq A^{\mathsf{ar}(R)}$	interpretation

Coproduct $\mathfrak{A} \sqcup \mathfrak{B}$ of σ-structures

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MSO(σ) Logic

$$\beta ::= R(x_1, \ldots, x_n) \;\mid\; x \in X \;\mid\; \neg\beta \;\mid\; \beta \vee \beta \;\mid\; \exists x.\beta \;\mid\; \exists X.\beta$$

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Propositional formulas Prop

$$P ::= x_i \;\mid\; y_i \;\mid\; P \vee P \;\mid\; P \wedge P$$

Classical Feferman-Vaught Theorem

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signature σ $\qquad$ $\beta \in \mathrm{MSO}(\sigma)$

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$$\mathfrak{A} \sqcup \mathfrak{B} \models \beta \quad \text{iff} \quad \mathtt{true} = P(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

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$$x_i = \mathtt{true} \text{ iff } \beta_i^{1} \models \mathfrak{A} \quad \text{and} \quad y_i = \mathtt{true} \text{ iff } \beta_i^{2} \models \mathfrak{B}$$

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wMSO(σ, S) Logic

$$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \otimes \psi$$

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$[\![\varphi]\!] : \mathrm{Str}(\sigma) \to S$

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Example $(\mathbb{N}_0, +, \cdot, 0, 1)$

$[\![\bigoplus x. \bigoplus y.\mathrm{edge}(x, y)]\!]$ = number of edges

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$\langle\!\langle E \rangle\!\rangle : S^n \times S^n \to S$

$$\langle\!\langle x_i \rangle\!\rangle(\bar{s}, \bar{t}) = s_i$$

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signature σ $\qquad$ $\beta \in \mathrm{MSO}(\sigma)$

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- restriction $\bigotimes x.\psi$ necessary

WEIGHTED FEFERMAN-VAUGHT THEOREM

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- restriction $\bigotimes x.\psi$ necessary
- "better" coproducts: translation schemes

Restriction

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fails for

$\bigotimes x. \bigoplus y.1$ $\qquad$ $(\mathbb{N}_0, +, \cdot, 0, 1)$

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$\bigotimes x. \bigotimes y.1$	$(\mathbb{N}_0 \cup \{-\infty\}, \max, +, -\infty, 0)$
$\bigotimes x. \bigotimes y.1$	$(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

Restriction: $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

assume

$$[\![\bigotimes x. \bigotimes y.1]\!](\mathfrak{A} \sqcup \mathfrak{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{A}), [\![\bar{\varphi}^2]\!](\mathfrak{B})) \qquad \forall \mathfrak{A}, \mathfrak{B}$$

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$$\mathfrak{S}_l = (\{1, \ldots, l\}, \emptyset)$$

Restriction: $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

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$$[\![\bigotimes x. \bigotimes y.1]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{S}_l), [\![\bar{\varphi}^2]\!](\mathfrak{S}_m)) \qquad \forall l, m$$

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$$\mathfrak{S}_l = (\{1, \ldots, l\}, \emptyset)$$

$$(l + m)^2 \qquad = [\![\bigotimes x. \bigotimes y.1]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m)$$

assume

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$$\mathfrak{S}_l = (\{1, \ldots, l\}, \emptyset)$$

$$E = \bigoplus_{i=1}^{k} \left(x_1^{g_{i,1}} \otimes \ldots \otimes x_n^{g_{i,n}} \otimes y_1^{h_{i,1}} \otimes \ldots \otimes y_n^{h_{i,n}} \right) \qquad \text{wlog}$$

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$$a_{li} = ([\![\bar{\varphi}_1^1]\!](\mathfrak{S}_l))^{g_{i,1}} \otimes \ldots \otimes ([\![\bar{\varphi}_n^1]\!](\mathfrak{S}_l))^{g_{i,n}}$$
$$b_{mi} = ([\![\bar{\varphi}_1^2]\!](\mathfrak{S}_m))^{h_{i,1}} \otimes \ldots \otimes ([\![\bar{\varphi}_n^2]\!](\mathfrak{S}_m))^{h_{i,n}}$$

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$$(l+m)^2 \qquad = [\![\bigotimes x. \bigotimes y.1]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \qquad \min_{i=1}^{k} \; a_{li} + b_{mi}$$

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Restriction: $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

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RESTRICTION: $(\mathbb{N}_0 \cup \{\infty\}, \textsc{min}, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{lj_{lm}} + b_{mj_{lm}}$

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$$J \qquad \begin{array}{c|cccccc} & & & m \rightarrow & & & \\ \hline l \downarrow & 2 & 2 & 1 & 1 & 3 & 1 \\ & 2 & 1 & 2 & 3 & 2 & 2 \\ & 1 & 3 & 1 & 2 & 1 & 2 \\ & 3 & 2 & 1 & 2 & 1 & 3 \\ & 1 & 2 & 3 & 2 & 2 & 1 \\ & 1 & 2 & 1 & 3 & 1 & 2 \end{array}$$

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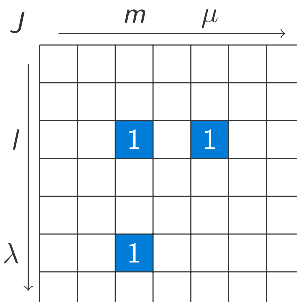

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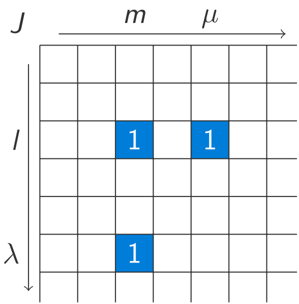

$$(l+m)^2 = a_{l1} + b_{m1}$$

Restriction: $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

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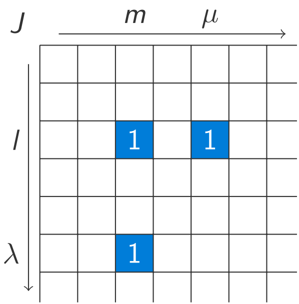
$$\text{choose} \quad j_{lm} \quad \text{with} \quad (l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$$



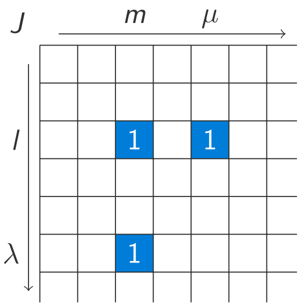
$$(l+m)^2 = a_{l1} + b_{m1}$$

$$(\lambda+m)^2 = a_{\lambda 1} + b_{m1}$$

Restriction: $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} \; a_{li} + b_{mi} \qquad \forall l, m$$

choose $\quad j_{lm} \quad$ with $\quad (l+m)^2 = a_{lj_{lm}} + b_{mj_{lm}}$



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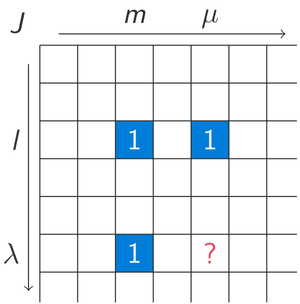

$(\lambda+\mu)^2$

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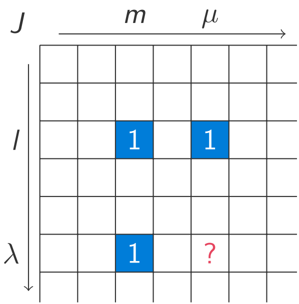

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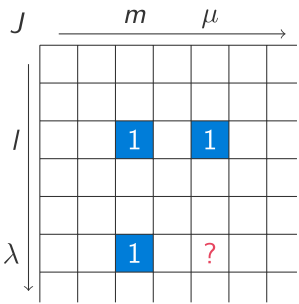

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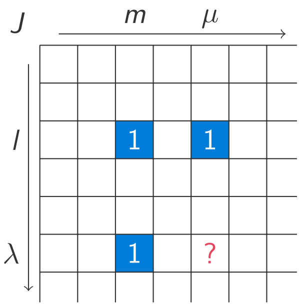

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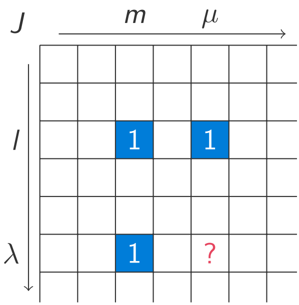

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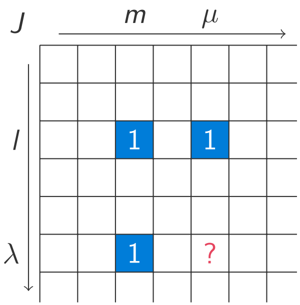

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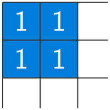
finitely many colors not sufficient

finitely many colors not sufficient

1 color

2×2 matrix



finitely many colors not sufficient

1 color

2×2 matrix

1	1
1	1

2 colors

9×9 matrix

?	?	?	?	?	?	?	?	2
?	?	?	?	?	?	?	1	?
?	?	?	?	?	?	2	?	?
?	?	?	?	?	1	?	?	?
?	?	?	?	2	?	?	?	?
?	?	?	1	?	?	?	?	?
?	?	1	?	?	?	?	?	?
?	2	?	?	?	?	?	?	?
2	?	?	?	?	?	?	?	?

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5 times same color on counter diagonal

?	?	?	?	?	?	?	?	2
?	?	?	?	?	?	?	1	?
?	?	?	?	?	?	2	?	?
?	?	?	?	?	1	?	?	?
?	?	?	?	2	?	?	?	?
?	?	?	1	?	?	?	?	?
?	?	1	?	?	?	?	?	?
?	2	?	?	?	?	?	?	?
2	?	?	?	?	?	?	?	?

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?	?	?	?	?	?	?		2	
?	?	?	?	?	?	2		?	
?	?	?	?	?	1	?		?	
?	?	?	?	2	?	?		?	
?	?	?	1	?	?	?		?	
?	?	1	?	?	?	?		?	
?	2	?	?	?	?	?		?	
2	?	?	?	?	?	?		?	

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?	?	?	?	2		?		?
?	?	?	1	?		?		?
?	?	1	?	?		?		?
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2	?	?	?	?		?		?

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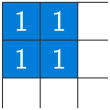

2 colors

9 × 9 matrix

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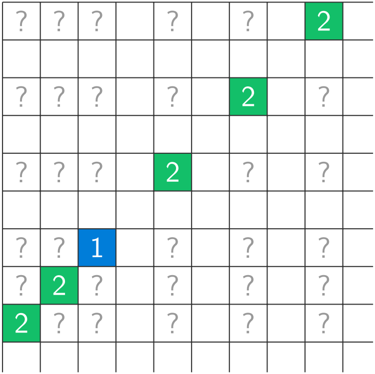

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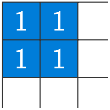

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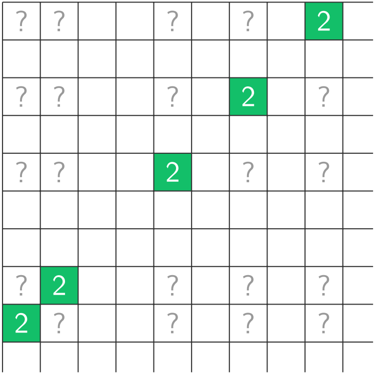

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1	1	
1	1	

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?	?	?	?	2					
?	?	?	2	?					
?	?	2	?	?					
?	2	?	?	?					
2	?	?	?	?					

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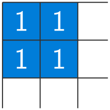

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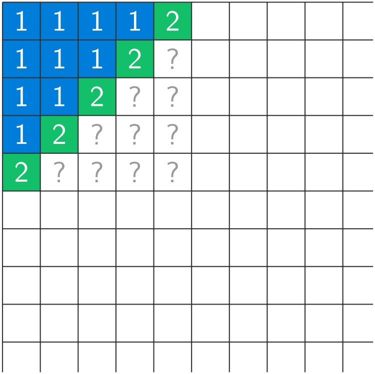

Restriction: $(\mathbb{N}_0 \cup \{\infty\}, \textsc{min}, +, \infty, 0)$

if solution exists for

$$[\![\bigotimes x. \bigotimes y.1]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^{1}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^{2}]\!](\mathfrak{S}_m)) \quad \forall l, m$$

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$\Rightarrow$ solution for coloring problem

σ, τ signatures

σ, τ	signatures
$\Phi = (\phi_{\mathcal{U}}, (\phi_R)_{R \in \mathsf{Rel}(\tau)})$	σ-τ-translation scheme

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$R \mapsto \{\bar{a} \in U^{\mathrm{ar}(R)} \mid (\mathfrak{A}, \bar{z} \to \bar{a}) \models \phi_R\}$	interpretation

Example 1

σ $\mathrm{succ}(\cdot, \cdot)$

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τ	$<(\cdot,\cdot)$
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$\phi_<$	$\exists X.(z_1 \in X \wedge z_2 \in X \wedge \dots$

Example 2 - Subtree

$\sigma = \tau$ edge$(\cdot, \cdot)$, marked$(\cdot)$

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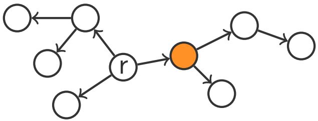

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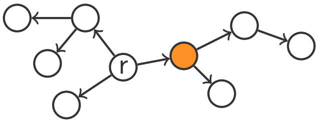

ϕ_{edge} edge(z_1, z_2)

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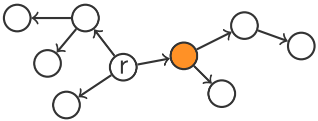

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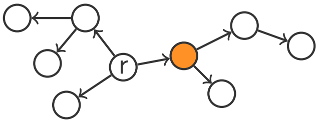

ϕ_{edge}	edge(z_1, z_2)
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$\Phi^*(\mathfrak{T})$	subtree at marked

Given

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σ-τ-translation scheme Φ

Translation Schemes

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such that for all σ-structures $\mathfrak{A}$

$$[\![\zeta]\!](\Phi^*(\mathfrak{A})) = [\![\varphi]\!](\mathfrak{A})$$

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$R(x_1, \ldots, x_n) \rightsquigarrow \phi_R(x_1, \ldots, x_n)$ $\exists x \ldots \rightsquigarrow \exists x.(\phi_{\mathcal{U}}(x) \wedge \ldots$

$$[\![\zeta]\!](\Phi^*(\mathfrak{A} \sqcup \mathfrak{B})) = [\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B})$$

$$\begin{aligned} [\![\zeta]\!](\Phi^*(\mathfrak{A} \sqcup \mathfrak{B})) &= [\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B}) \\ &= \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{A}), [\![\bar{\varphi}^2]\!](\mathfrak{B})) \end{aligned}$$

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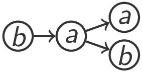
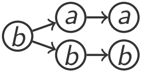
&= \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{A}), [\![\bar{\varphi}^2]\!](\mathfrak{B}))
\end{aligned}
$$

Example $\quad \sigma = \tau = \mathsf{edge}(\cdot,\cdot), \mathsf{label}_a(\cdot), \mathsf{label}_b(\cdot)$

$\mathfrak{T}_1$





$\mathfrak{T}_2$

$$
\begin{aligned}
[\![\zeta]\!](\Phi^*(\mathfrak{A} \sqcup \mathfrak{B})) &= [\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B}) \\
&= \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{A}), [\![\bar{\varphi}^2]\!](\mathfrak{B}))
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$$

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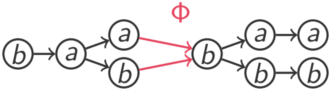

$\mathfrak{T}_2$

$$\llbracket \zeta \rrbracket(\Phi^*(\mathfrak{A} \sqcup \mathfrak{B})) = \llbracket \varphi \rrbracket(\mathfrak{A} \sqcup \mathfrak{B})$$
$$= \langle\!\langle E \rangle\!\rangle(\llbracket \bar{\varphi}^1 \rrbracket(\mathfrak{A}), \llbracket \bar{\varphi}^2 \rrbracket(\mathfrak{B}))$$

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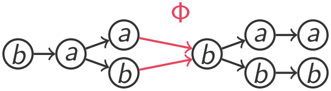

$\mathfrak{T}_2$

$$\zeta = \exists x.\exists y.(\mathrm{edge}(x,y) \wedge \mathrm{label}_a(x) \wedge \mathrm{label}_b(y))$$

$$\begin{aligned}[\![\zeta]\!](\Phi^*(\mathfrak{A} \sqcup \mathfrak{B})) &= [\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B}) \\ &= \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}))\end{aligned}$$

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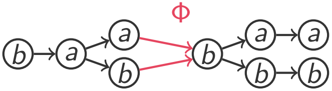
$\mathfrak{T}_1$



$\mathfrak{T}_2$

$$\zeta = \exists x.\exists y.(\text{edge}(x, y) \wedge \text{label}_a(x) \wedge \text{label}_b(y))$$

$$\varphi^{\mathbb{1}} = (\zeta, \exists x.(\text{leaf}(x) \wedge \text{label}_a(x)))$$
$$\varphi^{2} = (\zeta, \exists y.(\text{root}(y) \wedge \text{label}_b(y)))$$

$$E = x_1 \vee y_1 \vee (x_2 \wedge y_2)$$