On aperiodic and star-free formal power series in partially commuting variables*

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Abstract. Formal power series over non-commuting variables have been investigated as representations of the behavior of automata with multiplicities. Here we introduce and investigate the concepts of aperiodic and of star-free formal power series over semirings and partially commuting variables. We prove that if the semiring K is idempotent and commutative, or if K is idempotent and the variables are non-commuting, then the product of any two aperiodic series is again aperiodic. We also show that if K is idempotent and the matrix monoids over K have a Burnside property (satisfied, e.g. by the tropical semiring), then the aperiodic and the star-free series coincide. This generalizes a classical result of Schützenberger (1961) for aperiodic regular languages and subsumes a result of Guaiana, Restivo and Salemi (1992) on aperiodic trace languages.

1 Introduction

In the theory of automata, Kleene's fundamental theorem on the coincidence of regular and rational languages in free monoids has been extended in many ways. Schützenberger [26] investigated formal power series over arbitrary semirings (e.g., like the natural numbers) and with non-commuting variables and showed that the recognizable formal power series, which represent precisely the behavior of automata with multiplicities (cf. Eilenberg [10]), coincide with the rational series. This was the starting point for a large amount of work on formal power series, cf. [25, 16, 1, 15] for surveys. Special cases of automata with multiplicities are networks with capacities (weights) and have been also investigated in operations research for algebraic optimization problems, cf. [32] and in the 'max-plus-community' [11].

Schützenberger [27] also showed that in free monoids the aperiodic regular languages coincide with the star-free languages. Such languages are important for

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and arise from counter-free automata, and have been also investigated intensively due to characterizations using first order logic (McNaughton and Papert [21]) or temporal logic (Kamp [14]).

An algebraic characterization of the sub-algebra of series over commutative fields generated by letters and geometric series was announced by Reutenauer [24] as an analogue of Schützenberger's theorem. However, in the case of the boolean semiring, his class of series restricts to a proper subclass of the aperiodic languages (dot-depth 3/2).

It is the aim of this paper to introduce and investigate the concepts of aperiodicity and star-freeness for formal power series over arbitrary semirings. In fact, we will allow the variables to be partially commutative. A recognizable series is called aperiodic if, when iterating any complex task in a representing automaton with multiplicities, there is a fixed bound of iterations after which the computed value (weight) remains stable. This is an assumption often made in optimization problems, cf. [32]. A series is star-free, if it can be constructed from finitely many polynomials using the operations sum, product and complement, with the latter being applied only to characteristic series. This generalizes the concepts of aperiodic and of star-free languages, respectively.

Before stating our results, let us recall the notion of partially commuting variables. A trace alphabet (Σ, I) consists of a finite alphabet Σ and an irreflexive symmetric relation I indicating when two elements a, b of Σ commute, e.g. can occur independently of each other in a given concurrent system. A trace monoid $\mathbb{M}$ is therefore defined as the quotient of the free monoid Σ^* modulo the congruence generated by the relations $ab \sim ba$ if $a\ I\ b$. These monoids were introduced by Mazurkiewicz [19, 20] as an important mathematical model for the behavior of concurrent systems, see also [3, 5, 4] for their well-developed theory.

Now let K be an arbitrary semiring, and let $K\langle\langle\mathbb{M}\rangle\rangle$ be the collection of all formal power series $S = \sum_{m\in\mathbb{M}}(S, m) \cdot m$. These can also be regarded as series with entries (S, m) from K in which certain of the variables (= elements of Σ) are allowed to commute, as indicated by the relation I.

Whereas formal power series over non-commuting variables represent the behavior of sequential systems with weights, series over partially commuting variables can be viewed as the behavior of concurrent systems with multiplicities ('weights' for the actions). For an investigation of recognizable and rational formal power series over trace monoids, we refer the reader to [7].

Let us now give a summary of our main results. They all require that the semiring K of coefficients be idempotent. In applications, this is satisfied when 'addition' means the operation of taking minimum or maximum. We first show that the product of any two aperiodic series with non-commuting variables is again aperiodic. This means that the sequentialization of two such aperiodic systems stays aperiodic. We then show that this remains true even with partially commuting variables, but under the additional assumption that K be commutative. With an example, we show that the idempotence of K is necessary.

For our further results we need that the matrix monoids $K^{n\times n}$ of $(n \times n)$-matrices over K have a Burnside property: each finitely generated torsion submonoid of $K^{n\times n}$ be finite. We note that since Burnside's question in 1902, this property has been deeply investigated in group and semigroup theory, cf. the surveys of Simon [29] and Pin [23] for its relevance to automata theory. By a deep result of Simon [28], the important tropical semiring $(\mathbb{N}\cup\{\infty\},\min,+)$ (and several others) satisfies this property. We can then show that any series over K and partially commuting variables is aperiodic if and only if it is star-free. If here $K=\mathbb{B}$, the Boolean semiring, we obtain as a consequence that in trace monoids the aperiodic languages are precisely the star-free ones, a result of Guaiana, Restivo and Salemi [13] which in turn contains Schützenberger's classical result for aperiodic languages of words.

Note that this provides a syntactic construction of how to obtain the behavior of a given aperiodic concurrent system by combining singleton automata using the operations parallel sum, sequentialization, and complementation.

A preliminary version of this work appeared in the extended abstract [8].

2 Preliminaries

Here we recall the necessary notation and background for formal power series and for trace theory. For more details, we refer the reader to [25, 1, 3, 5].

Let M be any monoid and $(K,+,\cdot,0,1)$ any semiring, i.e., $(K,+,0)$ is a commutative monoid, $(K,\cdot,1)$ is a monoid, multiplication distributes over addition, and $0\cdot x=x\cdot 0=0$ for each $x\in K$. If multiplication is commutative, we say that K is *commutative*. If the addition is idempotent, then the semiring is called *idempotent*. For instance, the Boolean semiring $\mathbb{B}=(\{0,1\},+,\cdot,0,1)$ is both commutative and idempotent. The semiring of natural numbers $(\mathbb{N},+,\cdot,0,1)$ is commutative but not idempotent. The semiring $(\mathcal{P}(\Sigma^*),\cup,\cdot,\emptyset,\{1\})$ of languages over some alphabet Σ is idempotent but not commutative. The semiring $\mathbb{N}^{n\times n}$ of $(n\times n)$-matrices is neither commutative nor idempotent. Other semirings useful in computer science (and also in optimization problems of operations research [32]) are the min-plus (or max-plus or min-max) semirings over the integers or the reals. For instance the min-plus semiring over the reals $(\mathbb{R}\cup\{\infty\},\min,+,\infty,0)$ is both commutative and idempotent.

A formal power series is a mapping

$$\begin{aligned} S: M &\longrightarrow K \\ m &\longmapsto (S,m) \end{aligned}$$

It is usually denoted as a formal sum $S=\sum_{m\in M}(S,m).m$. The set $\mathrm{Im}(S)=\{(S,m)\mid m\in M\}$ is called the *image* of S. The set $\mathrm{Supp}(S)=\{m\in M\mid (S,m)\neq 0\}$ is called the *support* of S. If $\mathrm{Supp}(S)$ is finite, then S is called a *polynomial*. We consider elements of K also as polynomials in the natural way, having a non-zero entry only at $1\in M$. If $L\subseteq M$, we define the *characteristic series* 1_L of L by letting $1_L(m)=1$ if $m\in L$ and $1_L(m)=0$ otherwise. The collection of all formal power series is denoted by $K\langle\langle M\rangle\rangle$.

The *external product* of $k\in K$ and $S\in K\langle\langle M\rangle\rangle$, the *sum* $S+S'$ and the *Hadamard product* $S\odot S'$ of two series $S,S'\in K\langle\langle M\rangle\rangle$ are defined componentwise, that is, for $m \in M$ let

$$\begin{aligned}(k \cdot S, m) &= k(S, m),\\ (S + S', m) &= (S, m) + (S', m),\\ (S \odot S', m) &= (S, m) \cdot (S', m).\end{aligned}$$

With sum and Hadamard product, $K\langle\langle M\rangle\rangle$ is a semiring. In order to define the *(Cauchy) product* of two series, we assume that each element $m \in M$ has only finitely many factorizations $m = m_1 \cdot m_2$. Then, the *(Cauchy) product* of two series S, S' in $K\langle\langle M\rangle\rangle$ is the series defined for $m \in M$ by

$$(S \cdot S', m) = \sum_{m=m_1 \cdot m_2} (S, m_1) \cdot (S', m_2).$$

Without any assumptions on M, a product $S_1 \cdot S_2$ of two series $S_1, S_2 \in K\langle\langle M\rangle\rangle$ can also be naturally defined as above if S_1, S_2 have finite image and K is idempotent, using the convention that an infinite sum of a constant value $k \in K$ equals k. Note that if $K = \mathbb{B}$, the Boolean semiring, the mapping $L \mapsto 1_L$ constitutes a bijection between $\mathcal{P}(M)$ and $\mathbb{B}\langle\langle M\rangle\rangle$ with inverse $S \mapsto \mathrm{supp}(S)$. Under this bijection, the operations union, intersection and product for languages correspond to sum, Hadamard product and Cauchy product for series. With sum and Cauchy product, $K\langle\langle M\rangle\rangle$ is again a semiring.

Next we define inverse images and left and right quotients. Let $S \in K\langle\langle M\rangle\rangle$ be a series and let $h : N \to M$ be a mapping. The inverse image of S by h is the series $h^{-1}(S) \in K\langle\langle N\rangle\rangle$ defined for $n \in N$ by $(h^{-1}(S), n) = (S, h(n))$. If h is a monoid morphism then $h^{-1}(S)$ is called an inverse homomorphic image of S. Now, if $p \in M$ then the left and right quotients of S by p are the series $p^{-1}S, Sp^{-1} \in K\langle\langle M\rangle\rangle$ defined respectively for $m \in M$ by $(p^{-1}S, m) = (S, pm)$ and $(Sp^{-1}, m) = (S, mp)$. Note that $p^{-1}S$ is the inverse image of S by the mapping $h : M \to M$ defined by $h(x) = px$, and similarly for Sp^{-1}.

Let $n \geq 1$ and $[n] = \{1, \ldots, n\}$. Note that $K^{n \times n}$ is a monoid (with matrix multiplication as usual). A series $S \in K\langle\langle M\rangle\rangle$ is called *recognizable*, if there exists an integer $n \geq 1$, a monoid morphism $\mu : M \to K^{n \times n}$ and vectors $\lambda \in K^{1 \times n}, \gamma \in K^{n \times 1}$ such that

$$(S, m) = \lambda \cdot \mu(m) \cdot \gamma = \sum_{i,j \in [n]} \lambda_i \mu(m)_{ij} \gamma_j$$

for each $m \in M$. In this case, the triple (λ, μ, γ) is called a *representation* of dimension n of the series S, and we often shortly write $S = (\lambda, \mu, \gamma)$ to denote this. We also call (λ, μ, γ) a weighted automaton recognizing the series S. With this terminology, the set of states is $[n]$, the weight for entering the automaton in state i is λ_i and similarly the weight for leaving the automaton in state j is γ_j. Also, $\mu(m)_{i,j}$ denotes the weight (cost) for going from i to j reading m in the automaton. If $M = \Sigma^*$ is a free monoid then μ can be defined by the matrices $(\mu(a))_{a \in \Sigma}$ and $\mu(a)_{i,j}$ is the weight of the transition from i to j which is labeled a. Then, the weight of a path $P = i_0, a_1, i_1, \ldots, a_n, i_n$ is the product $\lambda_{i_0}\mu(a_1)_{i_0,i_1} \cdots \mu(a_n)_{i_{n-1},i_n}\gamma_{i_n}$ and one can check that for $w \in \Sigma^*$ the value (S, w) is the sum over all paths P labeled w of the weight of P. We let $K^{\text{rec}}\langle\langle M\rangle\rangle$ denote the set of all recognizable formal power series.

In some proofs, we will use the Kronecker product, which is defined for any two matrices $A = (a_{ij}) \in K^{m\times n}$ and $B = (b_{i'j'}) \in K^{p\times q}$ as the matrix $A \otimes B \in K^{mp\times nq}$ whose block representation is

$$A \otimes B = \left(\begin{array}{c|c|c} a_{11}B & \cdots & a_{1n}B \\ \hline \vdots & & \vdots \\ \hline a_{m1}B & \cdots & a_{mn}B \end{array}\right).$$

More formally, we have $(A \otimes B)_{(i-1)p+i',(j-1)q+j'} = a_{ij}b_{i'j'}$. Using a standard bijection, we could also view $A\otimes B$ as a $([m]\times[p])\times([n]\times[q])$-matrix over K with entries $(A \otimes B)_{(i,i'),(j,j')} = a_{ij}b_{i'j'}$. It is easy to verify that if K is commutative and if $A' \in K^{n\times \ell}$ and $B' \in K^{q\times r}$ then we have $(A\otimes B)(A'\otimes B') = (AA')\otimes(BB')$. In particular, if $m = n$ and $p = q$, then $(A \otimes B)^k = A^k \otimes B^k$ for each $k \geq 1$.

Next we recall basic notions from trace theory. A pair (Σ, I) is called a *trace alphabet*, if Σ is a finite set and I is an irreflexive symmetric binary *independence* relation on Σ. Let $\sim$ denote the smallest congruence on Σ^* containing $\{(ab, ba) \mid a\ I\ b\}$. The quotient monoid $\mathbb{M} = \mathbb{M}(\Sigma, I) := \Sigma^*/\sim$ is called the *trace monoid* (or free partially commutative monoid) over (Σ, I) and its elements are called traces. Note that in a trace monoid, each element has only finitely many factorizations.

If $w \in \Sigma^*$, we let $[w]$ denote the equivalence class of w in $\mathbb{M}$. Also, let $\alpha(w)$ be the set of all letters of Σ occurring in w, called the *alphabet* of w. Since equivalent words have the same alphabet, we may put $\alpha([w]) = \alpha(w)$.

We say that two subsets A, B of Σ are independent and we write $A\ I\ B$ if $A \times B \subseteq I$. We also say that two words w, w' of Σ^* are independent, denoted by $w\ I\ w'$, if $\alpha(w)\ I\ \alpha(w')$. Similarly, we define $[w]\ I\ [w']$, $w\ I\ A$, etc.

The following generalized Levi's factorization is a very useful and classical result in trace theory.

Lemma 2.1 ([2, 3]). *Let $u, v, w_1, \ldots, w_n \in \mathbb{M}$. Then, $uv = w_1 \cdots w_n$ if and only if there are $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{M}$ such that $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n$, $w_i = u_iv_i$ for all $1 \leq i \leq n$ and $v_i\ I\ u_j$ for all $1 \leq i < j \leq n$. Moreover, the traces $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{M}$ with this property are unique.*

A monoid N is said to be *aperiodic* if there exists some integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$. The *index* of an aperiodic monoid N is the smallest integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$.

A language $L \subseteq M$ is *aperiodic* if there exists a morphism $\varphi : M \to N$ into some finite aperiodic monoid N such that $L = \varphi^{-1}(\varphi(L))$. When this holds, we say that the morphism φ recognizes L. We denote by $\mathrm{AP}(M)$ the family of aperiodic languages of M; it contains $\emptyset$ and M and is closed under the operations union and complement.

Note that by definition, any aperiodic language is recognizable.The collection $\mathrm{SF}(M)$ of all *star-free* languages in M is defined as the smallest set of languages of M containing all finite languages and which is closed under the operations union, product and complement.

A fundamental theorem by Schützenberger [27] states that in free monoids aperiodic languages coincide with star-free languages. It was extended by Guaiana, Restivo and Salemi [13] to trace monoids. It is the aim of this paper to generalize this theorem to formal power series both for free monoids and for trace monoids.

3 Aperiodic series

In this section we introduce *aperiodic* and *weakly aperiodic* series and we study closure properties of aperiodic series.

Definition 3.1. *A recognizable series $S \in K\langle\langle M\rangle\rangle$ is* aperiodic *if there exists a representation $S = (\lambda, \mu, \gamma)$ with $\mu(M)$ aperiodic, i.e. there is some integer $m \geq 0$ such that $\mu(u^m) = \mu(u^{m+1})$ for all $u \in M$. In this case, we say that the morphism μ is aperiodic and that (λ, μ, γ) is an aperiodic representation of S. The collection of aperiodic series in $K\langle\langle M\rangle\rangle$ is denoted by $K^{\mathrm{ap}}\langle\langle M\rangle\rangle$.*

Also we say that a recognizable series is *weakly aperiodic* if there exists some integer $m \geq 0$ such that $(S, uv^m w) = (S, uv^{m+1}w)$ for all $u, v, w \in M$. We let $K^{\mathrm{wap}}\langle\langle M\rangle\rangle$ denote the collection of all weakly aperiodic series. Clearly, all aperiodic series are also weakly aperiodic. The converse does not hold, even for idempotent semirings, as shown in the next section. We will see in Sections 4 and 5 that the converse is true when the semiring K is a field or locally finite.

First we show closure properties of aperiodic series.

Proposition 3.2. *Let K be an arbitrary semiring and M be any monoid. The collection $K^{\mathrm{ap}}\langle\langle M\rangle\rangle$ is closed under left and right quotients, external product and sum. Inverse homomorphic images of aperiodic series are again aperiodic. If K is commutative, $K^{\mathrm{ap}}\langle\langle M\rangle\rangle$ is also closed under Hadamard product.*

Proof. Let $S \in K\langle\langle M\rangle\rangle$ be an aperiodic series and let $(\lambda, \mu, \gamma) = S$ with $\mu(M)$ aperiodic.

Let $p \in M$. We have $p^{-1}S = (\lambda\mu(p), \mu, \gamma)$ and $Sp^{-1} = (\lambda, \mu, \mu(p)\gamma)$ hence both quotients are aperiodic. Further, for $b \in K$ the series $b \cdot S = (b\lambda, \mu, \gamma)$ is also aperiodic.

Let N be a monoid and $h : N \to M$ a morphism. We have $h^{-1}(S) = (\lambda, \mu \circ h, \gamma)$ and $\mu \circ h(N)$ is a submonoid of $\mu(M)$. Therefore, $h^{-1}(S)$ is aperiodic.

For $i = 1, 2$, let $S_i = (\lambda^i, \mu^i, \gamma^i)$ be an aperiodic series of dimension n_i. Then $S_1 + S_2 = (\lambda, \mu, \gamma)$ with $\lambda = (\lambda^1, \lambda^2)$, $\gamma = \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix}$, and $\mu = \begin{pmatrix} \mu^1 & 0 \\ 0 & \mu^2 \end{pmatrix}$. Now, since μ is a block matrix, it is immediate to check that if $\mu^1(M)$ and $\mu^2(M)$ are aperiodic with indexes m_1 and m_2 then $\mu(M)$ is also aperiodic with index $m = \max(m_1, m_2)$.

Now assume K is commutative and S_1, S_2 are aperiodic series as above. Let $n = n_1 n_2$. Using the Kronecker product of matrices we define $S = (\lambda, \mu, \gamma)$ by $\lambda = \lambda^1 \otimes \lambda^2 \in K^{1\times n}$, $\gamma = \gamma^1 \otimes \gamma^2 \in K^{n\times 1}$ and $\mu : M \to K^{n\times n}$ by $\mu(w) = \mu^1(w) \otimes \mu^2(w)$. The intuition is that S is the synchronized product of the two weighted automata S_1 and S_2. Identifying $[n]$ with $[n_1] \times [n_2]$ we have $\lambda_{(i_1,i_2)} = \lambda^1_{i_1}\lambda^2_{i_2}$, $\gamma_{(j_1,j_2)} = \gamma^1_{j_1}\gamma^2_{j_2}$, and $\mu(w)_{(i_1,i_2),(j_1,j_2)} = \mu^1(w)_{i_1,j_1}\mu^2(w)_{i_2,j_2}$.

Since K is commutative, μ is a morphism and $S = S_1 \odot S_2$ [25, Thm. II.4.4] (this is easy to check using the property of the Kronecker product recalled in the preliminaries). As above, if $\mu^1(M)$ and $\mu^2(M)$ are aperiodic with indexes m_1 and m_2 then $\mu(M)$ is also aperiodic with index $m = \max(m_1, m_2)$. □

It is well-known [25] that if a language $L \subseteq M$ is recognizable then so is its characteristic series $1_L \in K\langle\langle M\rangle\rangle$. The converse seems to be open for arbitrary semirings but it holds for a wide class of semirings such as commutative semirings [31] or locally finite semirings (see next section). We show now that assuming recognizability, the equivalence is true for aperiodic languages and aperiodic characteristic series.

Proposition 3.3. *Let K be an arbitrary semiring, let M be an arbitrary monoid and let $L \subseteq M$ be recognizable. Then L is aperiodic iff its characteristic series $1_L \in K\langle\langle M\rangle\rangle$ is aperiodic.*

Proof. First, let N be a finite aperiodic monoid and let $\varphi : M \to N$ be a morphism recognizing L, i.e. $L = \varphi^{-1}(\varphi(L))$. Let $n = |N|$. We identify N with $[n] = \{1, \ldots, n\}$, 1 being indeed the neutral element of N. We define $\mu : M \to K^{n\times n}$, $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ by

$$\mu(u)_{i,j} = \begin{cases} 1 & \text{if } j = i \cdot \varphi(u) \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_j = \begin{cases} 1 & \text{if } j \in \varphi(L) \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that μ is a morphism and that $1_L = (\lambda, \mu, \gamma)$. Indeed,

$$\begin{aligned}(\mu(u) \cdot \mu(v))_{i,j} &= \sum_k \mu(u)_{i,k}\mu(v)_{k,j} = \mu(u)_{i,i\cdot\varphi(u)}\mu(v)_{i\cdot\varphi(u),j} = \mu(v)_{i\cdot\varphi(u),j} \\ &= \begin{cases} 1 & \text{if } j = i \cdot \varphi(u) \cdot \varphi(v) \\ 0 & \text{otherwise} \end{cases} = \mu(uv)_{i,j}\end{aligned}$$

and

$$\begin{aligned}\lambda\mu(w)\gamma &= \sum_{i,j} \lambda_i \mu(w)_{i,j}\gamma_j = \sum_j \mu(w)_{1,j}\gamma_j = \mu(w)_{1,1\cdot\varphi(w)}\gamma_{1\cdot\varphi(w)} = \gamma_{1\cdot\varphi(w)} \\ &= \begin{cases} 1 & \text{if } 1 \cdot \varphi(w) \in \varphi(L) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } w \in \varphi^{-1}\varphi(L) \\ 0 & \text{otherwise} \end{cases} = 1_L(w)\end{aligned}$$

Finally, it remains to show that the morphism μ is aperiodic. Since N is aperiodic, there exists some integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$. Then,

$$\mu(u^{m+1})_{i,j} = \begin{cases} 1 & \text{if } j = i \cdot \varphi(u)^{m+1} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } j = i \cdot \varphi(u)^{m} \\ 0 & \text{otherwise} \end{cases} = \mu(u^m)_{i,j}$$

Conversely, assume that 1_L is aperiodic, then it is also weakly aperiodic which implies the existence of an integer $m \geq 0$ such that $uv^m w \in L$ iff $uv^{m+1}w \in L$ for all $u, v, w \in M$. Therefore, the syntactic monoid of L is aperiodic. Since we assumed L recognizable, the syntactic monoid of L is also finite and we obtain that L is aperiodic. □

From the above result, we deduce that a semiring for which L is recognizable whenever 1_L is recognizable satisfies also that L is aperiodic whenever 1_L is aperiodic. In the next section (Corollary 5.6) we will see semirings for which this last statement is true.

Recall that polynomials are the series with finite supports and they correspond to finite languages when the semiring is $\mathbb{B}$. In arbitrary monoids, finite languages are not always recognizable. But finite languages are aperiodic in particular for trace monoids. We show that polynomials are aperiodic series whenever finite languages are aperiodic.

Corollary 3.4. *Let K be an arbitrary semiring and let M be a monoid such that singletons are aperiodic languages. Then, all polynomials in $K\langle\langle M\rangle\rangle$ are aperiodic.*

Proof. Let $S \in K\langle\langle M\rangle\rangle$ be a polynomial and let $L = \{w \in M \mid (S, w) \neq 0\}$ be its finite support. We may write $S = \sum_{w \in L}(S, w) \cdot 1_{\{w\}}$. We can conclude immediately using Propositions 3.3 and 3.2. □

Our aim is now to show that aperiodic series over free monoids are closed under Cauchy product when the semiring K is idempotent. For this, we will use special representations whose existence are proved in the next two lemmas.

When (λ, μ, γ) is a representation of dimension n, we call each $i \in [n]$ with $\lambda_i \neq 0$ an *initial* state and each $j \in [n]$ with $\gamma_j \neq 0$ a *final* state.

We say that M is a monoid *without divisors of the identity*, if $x \cdot y = 1$ implies $x = y = 1$ for any $x, y \in M$.

Lemma 3.5. *Let K be an arbitrary semiring, M be an arbitrary monoid without divisors of the identity and $S \in K\langle\langle M\rangle\rangle$ be a recognizable series. Then there exists a representation $S = (\lambda, \mu, \gamma)$ of dimension n such that there is exactly one final state, say j, and moreover, $\gamma_j = 1$ and $\mu(w)_{ji} = 0$ for any $i \in [n]$ and $w \in M \setminus \{1\}$. Moreover, we may also require that μ is aperiodic if S is an aperiodic series.*

Proof. Let (λ, μ, γ) be a representation of S of dimension n. We define a representation $(\lambda', \mu', \gamma')$ of dimension $n+1$ by the block matrices and vectors

$$\lambda' = \left(\begin{array}{c|c} \quad \lambda \quad & \lambda\gamma \end{array} \right), \qquad \mu'(w) = \left(\begin{array}{c|c} \mu(w) & \mu(w)\gamma \\ \hline 0 & 0 \end{array} \right) \qquad \gamma' = \left(\begin{array}{c} 0 \\ \hline 1 \end{array} \right)$$

where $w \in M \setminus \{1\}$ and we let $\mu'(1)$ be the unit matrix.

We first show that μ' is indeed a morphism so that $(\lambda', \mu', \gamma')$ is a well-defined representation. Let $u, v \in M \setminus \{1\}$. Then $uv \neq 1$ by the assumption on M, and we have

$$\mu'(u) \cdot \mu'(v) = \left(\begin{array}{c|c} \mu(u)\mu(v) & \mu(u)\mu(v)\gamma \\ \hline 0 & 0 \end{array} \right) = \mu'(uv).$$

Clearly, the representation $(\lambda', \mu', \gamma')$ fulfills the requirements of the lemma and we have $\lambda'\gamma' = \lambda\gamma = (S, 1)$ and for all $w \in M \setminus \{1\}$, $\lambda'\mu'(w)\gamma' = \lambda\mu(w)\gamma = (S, w)$ which proves that $S = (\lambda', \mu', \gamma')$.

Finally, if S is aperiodic then we may assume that there exists $m \geq 0$ such that $\mu(w^m) = \mu(w^{m+1})$ for all $w \in M$. From the definition of μ' above, it is clear that $\mu'(w^m) = \mu'(w^{m+1})$ for all $w \in M$ showing that μ' is also aperiodic. $\square$

Using a similar proof, we also obtain

Lemma 3.6. *Let K be an arbitrary semiring, M be an arbitrary monoid without divisors of the identity and $S \in K\langle\langle M \rangle\rangle$ be a recognizable series. Then there exists a representation $S = (\lambda, \mu, \gamma)$ of dimension n such that there is exactly one initial state, say i, and moreover, $\lambda_i = 1$ and $\mu(w)_{ji} = 0$ for any $j \in [n]$ and $w \in M \setminus \{1\}$. Moreover, we may also require that μ is aperiodic if S is an aperiodic series.*

Note that it is also possible to find a representation which satisfies both requirements of Lemmas 3.5 and 3.6 if the series S is proper, i.e., if $(S, 1) = 0$.

We are now ready to prove that aperiodic series over the free monoid Σ^* are closed under Cauchy product when the semiring K is idempotent.

Theorem 3.7. *Assume that the semiring K is idempotent. Let $S_1, S_2 \in K\langle\langle \Sigma^* \rangle\rangle$ be aperiodic series, then their product $S = S_1 \cdot S_2$ is also aperiodic.*

Proof. Let $(\lambda^1, \mu^1, \gamma^1)$ be an aperiodic representation of S_1 of dimension $n_1 + 1$ which satisfies the requirements of Lemma 3.5 with $n_1 + 1$ as unique final state so that we have the following block matrix representation (for $u \in \Sigma^+$):

$$\lambda^1 = \left(\begin{array}{c|c} \quad \alpha \quad & \beta \end{array} \right) \qquad \mu^1(u) = \left(\begin{array}{c|c} A(u) & B(u) \\ \hline 0 & 0 \end{array} \right) \qquad \gamma^1 = \left(\begin{array}{c} 0 \\ \hline 1 \end{array} \right).$$

Let $(\lambda^2, \mu^2, \gamma^2)$ be an aperiodic representation of S_2 of dimension $1 + n_2$ which satisfies the requirements of Lemma 3.6 with 1 as unique initial state so that we have the following block matrix representation (for $u \in \Sigma^+$):

$$\lambda^2 = \left(\begin{array}{c|c} 1 & 0 \end{array}\right) \qquad \mu^2(u) = \left(\begin{array}{c|c} 0 & C(u) \\ \hline 0 & D(u) \end{array}\right) \qquad \gamma^2 = \left(\begin{array}{c} \varphi \\ \hline \delta \end{array}\right).$$

Now we follow the classical construction of a weighted automaton recognizing $S_1 \cdot S_2$ as indicated in [10, proof of Proposition VI.7.8]. For this, we take a disjoint union of the two weighted automata merging the final state of S_1 with the initial state of S_2. Formally, let $n = n_1 + 1 + n_2$ and define the block matrices (for $a \in \Sigma$)

$$\lambda = \left(\begin{array}{c|c|c} \alpha & \beta & 0 \end{array}\right) \quad \mu(a) = \left(\begin{array}{c|c|c} A(a) & B(a) & 0 \\ \hline 0 & 0 & C(a) \\ \hline 0 & 0 & D(a) \end{array}\right) \quad \gamma = \left(\begin{array}{c} 0 \\ \hline \varphi \\ \hline \delta \end{array}\right).$$

We can check by induction on the length of a word $w \in \Sigma^+$ that

$$\mu(w) = \left(\begin{array}{c|c|c} A(w) & B(w) & E(w) \\ \hline 0 & 0 & C(w) \\ \hline 0 & 0 & D(w) \end{array}\right) \qquad \text{where } E(w) = \sum_{u,v \in \Sigma^+ \mid w = uv} B(u)C(v).$$

An easy computation yields for $w \in \Sigma^+$

$$\begin{aligned} \lambda\mu(w)\gamma &= \alpha B(w)\varphi + \alpha E(w)\delta + \beta C(w)\delta \\ &= (S_1, w)(S_2, 1) + \sum_{u,v \in \Sigma^+ \mid w = uv} (S_1, u)(S_2, v) + (S_1, 1)(S_2, w) \\ &= (S_1 \cdot S_2, w). \end{aligned}$$

Since $\lambda\gamma = \beta\varphi = (S_1, 1)(S_2, 1)$, we deduce that (λ, μ, γ) is a representation of the Cauchy product $S_1 \cdot S_2$.

Finally, we show that if $\mu^1(\Sigma^*)$ and $\mu^2(\Sigma^*)$ are aperiodic with indexes m_1 and m_2 and K is idempotent, then $\mu(\Sigma^*)$ is aperiodic with index $m = m_1 + m_2$. Clearly, we have $X(w^{m+1}) = X(w^m)$ for all $X \in \{A, B, C, D\}$ and $w \in \Sigma^*$ and it remains to show the same equality for $X = E$. We introduce the set

$$\mathcal{E}(w) = \{B(u)C(v) \mid u, v \in \Sigma^+ \text{ and } w = uv\}$$

and we show that $\mathcal{E}(w^{m+1}) = \mathcal{E}(w^m)$. Note that, if $w^m = uv$ then either w^{m_1} is a prefix of u or w^{m_2} is a suffix of v. In the first case, let $u = w^{m_1}u'$. Since μ is a morphism, for any $u'' \in \Sigma^+$ we have $B(u''u') = A(u'')B(u')$. Hence we have $B(u) = A(w^{m_1})B(u') = A(w^{1+m_1})B(u') = B(wu)$ Hence, $B(u)C(v) = B(wu)C(v) \in \mathcal{E}(w^{m+1})$. Similarly, in the second case, we have $B(u)C(v) = B(u)C(vw) \in \mathcal{E}(w^{m+1})$. Therefore, $\mathcal{E}(w^m) \subseteq \mathcal{E}(w^{m+1})$ and we can show the converse inclusion similarly. Using the fact that sum is idempotent, we deduce $E(w^{m+1}) = E(w^m)$. □

The idempotence of sum is needed to obtain this result, even for the free monoid. Indeed, let $K = (\mathbb{N}, +, \cdot)$ and $\Sigma = \{a\}$. The language Σ^* is clearly aperiodic and therefore its characteristic series $S = 1_{\Sigma^*}$ is aperiodic as well (Proposition 3.3). Now, for all $m \geq 0$, $(S \cdot S, a^m) = \sum_{a^m = uv} (S, u)(S, v) = m + 1$. Therefore, the series S^2 is not weakly aperiodic, whence is not aperiodic.

We aim now at establishing a similar result for arbitrary trace monoids. Alphabetic representations have been introduced in [7] in order to study the closure of recognizable series over trace monoids under product and star. We will use a simpler form of alphabetic representations which is sufficient to show that aperiodic series are also closed under product.

We say that a representation (λ, μ, γ) of dimension n is *(past-)alphabetic*, if there exists a function $\overleftarrow{\alpha} : [n] \to \mathcal{P}(\Sigma)$ such that for all $u \in \mathbb{M}$, the following two conditions are satisfied:
(1) Whenever $\mu(u)_{ij} \neq 0$, then $\overleftarrow{\alpha}(j) = \overleftarrow{\alpha}(i) \cup \alpha(u)$
(2) whenever $\lambda_i \neq 0$, then $\overleftarrow{\alpha}(i) = \emptyset$.
We call $(\lambda, \mu, \gamma; \overleftarrow{\alpha})$ an alphabetic representation of S.

Proposition 3.8. *Let K be an arbitrary semiring and $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be an aperiodic series over traces. Then there exists an alphabetic representation $S = (\lambda, \mu, \gamma; \overleftarrow{\alpha})$ with μ aperiodic.*

Proof. The construction of the representation follows the same line as in [7, Proposition 6]. Assume that $S = (\lambda', \mu', \gamma')$ with $\mu' : \mathbb{M} \to K^{n' \times n'}$ an aperiodic morphism. Let $n = n' \cdot 2^{|\Sigma|}$. Subsequently, we identify $[n]$ with $[n'] \times \mathcal{P}(\Sigma)$. We define $\mu : \mathbb{M} \to K^{n \times n}$ and $\lambda \in K^{1 \times n}$, $\gamma \in K^{n \times 1}$ by

$$\mu(u)_{(i,X)(j,Y)} = \begin{cases} \mu'(u)_{ij} & \text{if } Y = X \cup \alpha(u) \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{(i,X)} = \begin{cases} \lambda'_i & \text{if } X = \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_{(i,X)} = \gamma'_i$$

Also, we put $\overleftarrow{\alpha}(i, X) = X$.

It can be shown as in [7, Proposition 6] that $(\lambda, \mu, \gamma; \overleftarrow{\alpha})$ is an alphabetic representation of S. Here, we only have to show that μ is aperiodic. Indeed, let $m > 0$ be such that $\mu'(u^m) = \mu'(u^{m+1})$ for all $u \in \mathbb{M}$. Note that since we have taken $m > 0$, we have $\alpha(u^{m+1}) = \alpha(u) = \alpha(u^m)$ for all $u \in \mathbb{M}$. It follows directly from the definition of μ that $\mu(u^m) = \mu(u^{m+1})$ for all $u \in \mathbb{M}$. □

We will now prove that aperiodic series over trace monoids are closed under Cauchy product when the semiring K is idempotent and commutative.

Theorem 3.9. *Assume that the semiring K is idempotent and commutative. Let $S_1, S_2 \in K\langle\langle\mathbb{M}\rangle\rangle$ be aperiodic series, then their product $S = S_1 \cdot S_2$ is also aperiodic.*

Proof. Again, we use the construction of [7, Theorem 7]. Let $(\lambda^1, \mu^1, \gamma^1)$ be an aperiodic representation of S_1 of dimension n_1 and let $(\lambda^2, \mu^2, \gamma^2; \overleftarrow{\alpha})$ be an alphabetic representation of S_2 of dimension n_2 with μ_2 aperiodic (Proposition 3.8). Let $n = n_1 \cdot n_2$. For each $u \in \Sigma^*$ we define the diagonal matrix

$$I(u) = \begin{pmatrix} I(u,1) & & 0 \\ & \ddots & \\ 0 & & I(u,n_2) \end{pmatrix} \in K^{n_2 \times n_2}$$

by letting

$$I(u,i) = \begin{cases} 1 & \text{if } u \; I \; \overleftarrow{\alpha}(i) \\ 0 & \text{otherwise.} \end{cases}$$

Now define $\mu : \mathbb{M} \to K^{n \times n}$ by putting

$$\mu(w) = \sum_{w=uv} \mu^1(u) \otimes \big(I(u) \cdot \mu^2(v)\big)$$

In other words, if we identify $[n]$ with $[n_1] \times [n_2]$, we have

$$\mu(w)_{(i_1,i_2)(j_1,j_2)} = \sum_{w=uv} I(u,i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2}$$

It was shown in [7, Theorem 7] that μ is a morphism and $S = (\lambda, \mu, \gamma)$ where $\lambda \in K^{1\times n}$ and $\gamma \in K^{n \times 1}$ are defined by $\lambda = \lambda^1 \otimes \lambda^2$ and $\gamma = \gamma^1 \otimes \gamma^2$.

We have assumed μ^1 and μ^2 aperiodic so let $q > 1$ be such that $\mu^1(w^q) = \mu^1(w^{q-1})$ and $\mu^2(w^q) = \mu^2(w^{q-1})$ for all $w \in \mathbb{M}$. We will show that μ is aperiodic. For $w \in \mathbb{M}$ and $m \geq 0$, we define

$$X_m(w) = \{(u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{M}^{2m} \mid \forall i \in [m], w = u_i v_i \\ \text{and } \forall 1 \leq i < j \leq m, v_i \; I \; u_j\}.$$

Also, for $x = (u_1, \ldots, u_m, v_1, \ldots, v_m)$, we set $\varphi(x) = \mu^1(u) \otimes (I(u) \cdot \mu^2(v))$ where $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_m$. Using the generalized Levi's factorization (Lemma 2.1) and the definition of μ we deduce immediately that

$$\mu(w^m) = \sum_{x \in X_m(w)} \varphi(x).$$

Note that, thanks to the unicity of the factorization in Levi's Lemma, we do not use that sum is idempotent for this result.

Claim. If $m > q(|\Sigma|+1)$ then $\varphi(X_m(w)) = \varphi(X_{m+1}(w))$. Actually, we will prove that for all $x \in X_m(w)$, there exist $y \in X_{m-1}(w)$ and $z \in X_{m+1}(w)$ such that $\varphi(x) = \varphi(y) = \varphi(z)$.

Let $x = (u_1, \ldots, u_m, v_1, \ldots, v_m) \in X_m(w)$. For $1 \leq i < j \leq m$ we have $v_i \; I \; u_j$, which implies that $\alpha(v_i) \subseteq \alpha(w) \setminus \alpha(u_j) \subseteq \alpha(v_j)$. Therefore we have $\alpha(v_1) \subseteq \cdots \subseteq \alpha(v_m)$ and from the hypothesis on m we deduce that there exists $1 \leq p \leq m - q$ such that $\alpha(v_p) = \cdots = \alpha(v_{p+q})$.

Let $p < k \leq p + q$, we have $\alpha(w) = \alpha(u_k) \cup \alpha(v_k)$ and $\alpha(u_k) \; I \; \alpha(v_{k-1}) = \alpha(v_k)$. Hence we obtain $\alpha(u_k) = \alpha(w) \setminus \alpha(v_k)$. Since there is at most one way to split the trace w into two independent traces whose alphabet are fixed, we deduce that $u_{p+1} = \cdots = u_{p+q} = \bar{u}$ and $v_{p+1} = \cdots = v_{p+q} = \bar{v}$.

Let $u' = u_1 \cdots u_p$, $u'' = u_{p+q+1} \cdots u_m$, $v' = v_1 \cdots v_p$ and $v'' = v_{p+q+1} \cdots v_m$. Then we have

$$\begin{aligned}
\mu^1(u) = \mu^1(u'\bar{u}^q u'') = \mu^1(u'\bar{u}^{q+1} u'') = \mu^1(u'\bar{u}^{q-1} u'') \\
\mu^2(v) = \mu^2(v'\bar{v}^q v'') = \mu^2(v'\bar{v}^{q+1} v'') = \mu^2(v'\bar{v}^{q-1} v'') \\
\alpha(u) = \alpha(u'\bar{u}^q u'') = \alpha(u'\bar{u}^{q+1} u'') = \alpha(u'\bar{u}^{q-1} u'')
\end{aligned}$$

The claim follows since this implies that $\varphi(x) = \varphi(y) = \varphi(z)$ for

$$\begin{aligned}
y &= (u_1, \ldots, u_{p+q-1}, u_{p+q+1}, \ldots, u_m, v_1, \ldots, v_{p+q-1}, v_{p+q+1}, \ldots, v_m) \in X_{m-1}(w) \\
z &= (u_1, \ldots, u_{p+q}, u_{p+q}, \ldots, u_m, v_1, \ldots, v_{p+q}, v_{p+q}, \ldots, v_m) \in X_{m+1}(w).
\end{aligned}$$

Using the claim, we deduce immediately that for $m > q(|\Sigma| + 1)$ it holds

$$\mu(w^m) = \sum_{x \in X_m(w)} \varphi(x) = \sum_{x \in X_{m+1}(w)} \varphi(x) = \mu(w^{m+1}).$$

Here we have used that sum is idempotent in K since $|X_m(w)| < |X_{m+1}(w)|$ as soon as $w \neq 1$. □

Commutativity is needed because we are dealing with trace monoids. An example was given in [7, Section 5] showing that recognizable series over trace monoids are not closed under product in general.

As usual, in the special case of the boolean semiring we obtain the result on trace languages as a corollary.

Corollary 3.10 ([13]). *The product of two aperiodic trace languages is again aperiodic.*

Proof. Let $L_1, L_2 \subseteq \mathbb{M}$ be two aperiodic trace languages. By Proposition 3.3 the characteristic series $1_{L_1}, 1_{L_2} \in \mathbb{B}\langle\langle\mathbb{M}\rangle\rangle$ are aperiodic. Then by Theorem 3.9 we deduce that $1_{L_1} \cdot 1_{L_2} = 1_{L_1 \cdot L_2} \in \mathbb{B}\langle\langle\mathbb{M}\rangle\rangle$ is also aperiodic. Since we are in the boolean semiring this implies that $L_1 \cdot L_2$ is recognizable. Applying again Proposition 3.3 we deduce that $L_1 \cdot L_2$ is aperiodic. □

4 Weakly aperiodic series

In this section, we will investigate weakly aperiodic series and their relationship with aperiodic series. As noted before, any aperiodic series is weakly aperiodic. The converse does not hold, even for idempotent semirings, as shown by the following example due to an anonymous referee.

Example 4.1. We will define an idempotent semiring K and a series $S \in K\langle\langle \Sigma^* \rangle\rangle$ with $\Sigma = \{a, b\}$ which is weakly aperiodic but not aperiodic. We start with the definition of the semiring. Let $M' = \{a, b, \#\}^* \uplus \{0\}$ be the free monoid over the alphabet $\{a, b, \#\}$ equipped with a zero. Let $M = M'/\!\approx$ be the quotient of M' by the congruence generated by $\#uv^3w\# \approx 0$ for $u, v, w \in \Sigma^*$ with v nonempty. Let K be the set of finite subsets of M. Using union as addition and concatenation as multiplication we obtain a semiring with $\emptyset$ as zero and $\{[1]_\approx\}$ as unity.

The series $S \in K\langle\langle \Sigma^* \rangle\rangle$ is defined by $(S, w) = \{[\#w\#]_\approx\}$. It is recognizable by the following representation of dimension 1: $(\{[\#]_\approx\}, \mu, \{[\#]_\approx\})$ where $\mu(x) = \{[x]_\approx\}$ for all $x \in \Sigma$. Clearly, S is weakly aperiodic.

We show now that the series S is not aperiodic. Let (λ, μ, γ) be a representation of S with dimension n. By a result of Morse and Hedlund from 1944, infinitely many cube-free words exist in Σ^*. Hence, we can choose a cube-free word $w \in \Sigma^*$ of length $N > 3n$. By definition of S, we have $(S, w) = \{[\#w\#]_\approx\}$ hence we can find a path P in the automaton with weight$(P) = \{[\#w\#]_\approx\}$. Since $N > 3n$ we find a state q occurring at least 4 times in the path P so that we can write

$$P = q_0 \xrightarrow{v_0} q \xrightarrow{v_1} q \xrightarrow{v_2} q \xrightarrow{v_3} q \xrightarrow{v_4} q_N$$

with $v_1, v_2, v_3 \in \Sigma^+$. Now, let $[u_0]_\approx \in \lambda_{q_0}\mu(v_0)_{q_0,q}$, $[u_1]_\approx \in \mu(v_1)_{q,q}$, $[u_2]_\approx \in \mu(v_2)_{q,q}$, $[u_3]_\approx \in \mu(v_3)_{q,q}$ and $[u_4]_\approx \in \mu(v_4)_{q,q_N}\gamma_{q_N}$. We have $[u_0u_1u_2u_3u_4]_\approx = [\#w\#]_\approx \neq [0]_\approx$, hence $\#w\# = u_0u_1u_2u_3u_4$. None of u_1, u_2, u_3 may be empty since otherwise, iterating 3 times the corresponding loop we would get a path P' with $[u_0u_1u_2u_3u_4]_\approx \in$ weight(P'), a contradiction with the fact that the label of P' contains a cube. Therefore, $u_2 \in \Sigma^+$ and the classes $([u_2^k]_\approx)_{k>0}$ are pairwise different. Since $[u_2^k]_\approx \in \mu(v_2^k)_{q,q}$ for all $k > 0$, we deduce that $\mu(\Sigma^*)$ is not aperiodic. □

On the other hand, if the semiring is a field we show the following.

Theorem 4.2. *Let K be a field and M be any monoid. Then, $K^{\mathrm{wap}}\langle\langle M \rangle\rangle = K^{\mathrm{ap}}\langle\langle M \rangle\rangle$.*

Proof. Let $S \in K^{\mathrm{wap}}\langle\langle M \rangle\rangle$. We will show that the standard construction of a minimal automaton for S yields an aperiodic representation. Note that $K\langle\langle M \rangle\rangle$ with addition and multiplication with scalars from K as usual is vector space. Let $Z = \langle u^{-1}S : u \in M \rangle$, the subspace of $K\langle\langle M \rangle\rangle$ generated by the rows of $H(S)$. Since S is recognizable, by [25, Ch.II.3], Z is finitely generated and thus Z has a finite basis $B = \{F_1, \ldots, F_n\}$, say. Now define a mapping $\mu : M \to K^{n \times n}$

such that for each $v \in M$ and $i \in \{1, \ldots, n\}$, we have $v^{-1}F_i = \sum_{j=1}^{n} \mu(v)_{ij} \cdot F_j$ with uniquely determined scalars $\mu(v)_{ij} \in K$. Furthermore, $S = \sum_{i=1}^{n} \lambda_i F_i$ with $\lambda_i \in K$, and let $\gamma_j = F_j(1)$. Put $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)'$. Then μ is a morphism, and (λ, μ, γ) is a representation of S [25, Ch.II.3].

Let $u_i \in M$ such that $F_i = u_i^{-1}S$ $(i = 1, \ldots, n)$. Since S is weakly aperiodic, there is $m \in \mathbb{N}$ such that for all $v \in M$ and $i = 1, \ldots, n$ we have $v^{-m}(u_i^{-1}S) = v^{-(m+1)}(u_i^{-1}S)$. Hence

$$\sum_{j=1}^{n} (\mu v^m)_{ij} \cdot F_j = v^{-m}F_i = v^{-(m+1)}F_i = \sum_{j=1}^{n} (\mu v^{m+1})_{ij} \cdot F_j.$$

Since B is a basis, comparison of coefficients implies $\mu(v^m) = \mu(v^{m+1})$. Thus S is aperiodic. $\square$

Next we wish to show that the collection of all weakly aperiodic series has similar closure properties with respect to the operations inverse morphims, left and right quotients, sum, external product, Hadamard and Cauchy product as established above for the aperiodic series. The proofs are slightly easier than for aperiodic series.

Theorem 4.3.

(a) Let K be any semiring and M any monoid. Then, $K^{\mathrm{wap}}\langle\langle M\rangle\rangle$ is closed under left and right quotients, sum and external product. Inverse homomorphic images of weakly aperiodic series of $K\langle\langle M\rangle\rangle$ are again weakly aperiodic. If K is commutative, then $K^{\mathrm{wap}}\langle\langle M\rangle\rangle$ is also closed under the Hadamard product.
(b) Let K be an idempotent semiring. If $S_1, S_2 \in K\langle\langle \Sigma^*\rangle\rangle$ are weakly aperiodic, then so is the product $S_1 \cdot S_2$.
(c) Let K be idempotent and commutative. If $S_1, S_2 \in K\langle\langle \mathbb{M}\rangle\rangle$ are weakly aperiodic, then so is the product $S_1 \cdot S_2$.

Proof. (a) Observe that $K^{\mathrm{rec}}\langle\langle M\rangle\rangle$ is closed under these operations. The rest is straightforward from the definitions.

(b) and (c). By [26], $K^{\mathrm{rec}}\langle\langle \Sigma^*\rangle\rangle$ is closed under the Cauchy product. By [7], if K is commutative, $K^{\mathrm{rec}}\langle\langle \mathbb{M}\rangle\rangle$ is also closed under the Cauchy product. Therefore, it remains to show that if K is idempotent and S_1, S_2 are weakly aperiodic, there is some $m \geq 0$ such that $(S_1 \cdot S_2, uv^m w) = (S_1 \cdot S_2, uv^{m+1}w)$ for all words $u, v, w \in \mathbb{M}$.

We proceed as in the proof of Theorem 3.9. So let $q > 1$ be such that $(S_i, uw^q v) = (S_i, uv^{q+1}w)$ for all $u, v, w \in \mathbb{M}$ and $i = 1, 2$. For $m \geq 0$ consider the set

$$\begin{aligned} X_m = \{(u_0, \ldots, u_{m+1}, v_0, \ldots, v_{m+1}) \in \mathbb{M}^{2m+4} \mid \\ u = u_0 v_0, w = u_{m+1} v_{m+1}, v = u_i v_i \text{ for all } i \in [m], \\ \text{and } v_i I u_j \text{ for all } 0 \leq i < j \leq m+1\} \end{aligned}$$

Also, for $x = (u_0, \ldots, u_{m+1}, v_0, \ldots, v_{m+1})$, we put

$$\psi(x) = (S_1, u_0 \ldots u_{m+1})(S_2, v_0 \ldots v_{m+1}).$$

Using the generalized Levi's factorization (Lemma 2.1), we obtain that

$$(S_1 \cdot S_2, uv^m w) = \sum_{\substack{y,z \in \Sigma^* \\ uv^m w = yz}} (S_1, y)(S_2, z) = \sum_{x \in X_m} \psi(x)$$

It follows that if $m > q \cdot (|\Sigma| + 1)$, then $\psi(X_m) = \psi(X_{m+1})$. Since K is idempotent, we obtain $(S_1 \cdot S_2, uv^m w) = \sum_{x \in X_m} \psi(x) = \sum_{x \in X_{m+1}} \psi(x) = (S_1 \cdot S_2, uv^{m+1} w)$ for such m, and the result follows. $\square$

In spite of these similarities, it will turn out that the class of star-free series introduced in section 6 for many natural semirings better compares to the class of aperiodic series than to weakly aperiodic series, see Theorem 6.7.

5 Aperiodic and recognizable step functions

In this section we will investigate series which take on only finitely many values. For this, we will introduce semirings with local finiteness conditions which will be crucial later on.

Let K be any semiring and M any monoid. We will call a series $S : M \to K$ a *recognizable* (respectively, *aperiodic*) *step function*, if $S = \sum_{i=1}^{n} k_i 1_{L_i}$ for some $n \geq 1$, $k_i \in K$ and recognizable (resp., aperiodic) languages $L_i \subseteq M$ for $i = 1, \ldots, n$. That is, S is a finite linear combination of characteristic series of recognizable (resp., aperiodic) languages. Since the classes of recognizable (resp. aperiodic) languages of M form a Boolean algebra, an equivalent condition is that $\mathrm{Im}(S)$ is finite and each language $S^{-1}(k) = \{m \in M \mid (S, m) = k\}$ ($k \in K$) is recognizable (resp., aperiodic). The collection of all recognizable (resp., aperiodic) step functions in $K\langle\langle M\rangle\rangle$ is denoted by $K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$ (resp., $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$). First we have:

Proposition 5.1. *Let K be any semiring and M be any monoid. Then, $K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle \subseteq K^{\mathrm{rec}}\langle\langle M\rangle\rangle$ and $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle \subseteq K^{\mathrm{ap}}\langle\langle M\rangle\rangle$.*

Proof. For aperiodic step functions, apply Propositions 3.3 and 3.2, and for recognizable step functions apply the corresponding closure properties which are well known. $\square$

Note that we can represent any recognizable step function S by a classical deterministic complete M-automaton with weights attached only to the final states. That is, there is a representation (λ, μ, γ) of S with a unique initial state (which has weight 1) such that for each $w \in M$ and each i there is a unique j with $\mu(w)_{ij} \neq 0$, and then $\mu(w)_{ij} = 1$.

Next we wish to derive under suitable hypothesis on K a converse of Proposition 5.1. In group theory, a group is said to be locally finite, if each finitely generated subgroup is finite. A monoid is called *locally finite*, if each finitely generated submonoid is finite. Clearly, any commutative idempotent monoid is locally finite. Here, we will call a semiring K *locally finite*, if each finitely generated subsemiring is finite. We note that, a semiring $(K,+,\cdot,0,1)$ is locally finite iff both monoids $(K,+,0)$ and $(K,\cdot,1)$ are locally finite. Indeed, if X is a finite subset of K then the submonoid Y of $(K,\cdot,1)$ generated by X is finite and the submonoid Z of $(K,+,0)$ generated by Y is also finite. Now, it is easy to check that $Z\cdot Z\subseteq Z$ and we deduce that the subsemiring of $(K,+,\cdot,0,1)$ generated by X is the finite set Z.

For instance, if both sum and product are commutative and idempotent then the semiring is locally finite. For example, the max-min semiring $\mathcal{R}_{\max,\min} = (\mathbb{R}_+ \cup \{\infty\}, \max, \min, 0, \infty)$ of positive reals, used in operations research for maximum capacity problems of networks, is locally finite. An isomorphic semiring, namely $\mathbb{F} = ([0,1], \max, \min, 0, 1)$ sometimes called the fuzzy semiring, is also used in the theory of fuzzy languages (see e.g. [22]). Also, semirings which are boolean algebras like $(\mathcal{P}(\Sigma^*), \cup, \cap, \emptyset, \{1\})$ are locally finite. In fact, any distributive lattice $(L, \vee, \wedge, 0, 1)$ with smallest element 0 and largest element 1 is a locally finite semiring.

Note that if K is a locally finite semiring, then the matrix monoids $K^{n\times n}$ are locally finite for all n. Indeed, let $Y \subseteq K^{n\times n}$ be a finite set of matrices and let X be the subsemiring of K generated by the finite set $\{A_{ij} \mid A \in Y, i, j \in [n]\}$. Then the submonoid of $K^{n\times n}$ generated by Y is contained in $X^{n\times n}$ which is a finite submonoid of $K^{n\times n}$.

Conversely, if $K^{2\times 2}$ is a locally finite monoid then K is a locally finite semiring. Indeed, let $X \subseteq K$ be a finite set. By considering the submonoid generated by the matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in X \right\}$$

we obtain that $(K,\cdot,1)$ is locally finite. Now, by considering the submonoid generated by the matrices

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in X \right\}$$

we obtain that $(K,+,0)$ is locally finite. For this, note that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}.$$

Proposition 5.2. *Let K be a locally finite semiring and M be a finitely generated monoid. Then, $K^{\mathrm{rec}}\langle\langle M\rangle\rangle = K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$.*

Proof. Let $S = (\lambda, \mu, \gamma) \in K^{\mathrm{rec}}\langle\langle M\rangle\rangle$ with $\mu : M \to K^{n\times n}$. Let Σ be a finite set of generators of M. Then, $\mu(M)$ is the submonoid of $K^{n\times n}$ generated by the finite set $\mu(\Sigma)$. Since K is locally finite we deduce that $\mu(M)$ is finite. Therefore, $\mathrm{Im}(S) = \{\lambda \cdot A \cdot \gamma \mid A \in \mu(M)\}$ is also finite.

It remains to show for each $k \in K$ that $S^{-1}(k)$ is recognizable. We show that $\mu : M \to \mu(M)$ recognizes $S^{-1}(k)$. Indeed, if $u, v \in M$ with $\mu(u) = \mu(v)$ and $u \in S^{-1}(k)$, then $(S, v) = \lambda \cdot \mu(v) \cdot \gamma = \lambda \cdot \mu(u) \cdot \gamma = (S, u) = k$. Therefore, $S^{-1}(k)$ is recognizable in M. □

For applications on aperiodic series, we use weaker assumptions on K. A monoid N is called *torsion* (or *periodic*), if each of its cyclic submonoids $\{x^n \mid n \geq 0\}$ $(x \in N)$ is finite. Each locally finite monoid is torsion. The converse, which was posed as a problem for groups by Burnside in 1902 (and answered negatively by Golod in 1964), leads to deep problems in semigroup theory, see the surveys by Simon [29] and Pin [23] for its relevance to automata theory. We will say that a semiring K has *Burnside matrix monoids*, if in the monoids $K^{n \times n} (n \in \mathbb{N})$ each torsion submonoid is locally finite.

Clearly, if K is locally finite then K has Burnside matrix monoids. But also all of the following important (not locally finite) semirings have been shown to have Burnside matrix monoids:

Theorem 5.3. *Each of the following semirings has Burnside matrix monoids:*

- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ *and its completion* $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ *(Mandel and Simon [17]),*
- $\mathcal{M} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ *(Simon [28]),*
- $\mathcal{P} = (\mathbb{N} \cup \{-\infty, \infty\}, \max, +, -\infty, 0)$ *(Mascle [18]),*
- $\mathcal{R}at = (Rat(a^*), \cup, \cdot, \emptyset, a^*)$ *(Mascle [18]).*
- *any commutative ring or PI-ring (a ring satisfying a polynomial identity), cf. [23] and [30]*

Semirings with Burnside matrix monoids will be important for us because of the following easy but crucial observation.

Proposition 5.4. *Let K be a semiring with Burnside matrix monoids and M be a finitely generated monoid. Then,* $K^{\mathrm{ap}}\langle\langle M \rangle\rangle = K^{\mathrm{ap-step}}\langle\langle M \rangle\rangle$.

Proof. Let $S = (\lambda, \mu, \gamma) \in K\langle\langle M \rangle\rangle$ with $\mu : M \to K^{n \times n}$ aperiodic. Since M is finitely generated, $\mu(M)$ is a finitely generated torsion submonoid of $K^{n \times n}$ and hence finite by assumption on K. Then, $\mathrm{Im}(S) = \{\lambda \cdot A \cdot \gamma \mid A \in \mu(M)\}$ is also finite.

It remains to show that for each $k \in K$ that $S^{-1}(k)$ is aperiodic. As shown in the proof of Proposition 5.2, $\mu : M \to \mu(M)$ recognizes $S^{-1}(k)$, and by assumption $\mu(M)$ is an aperiodic monoid. Hence $S^{-1}(k)$ is aperiodic. □

Moreover, if the monoid M, the semiring K and a representation of S are given in an effective way, all the constituents of the description of S in the proof of Proposition 5.2 or 5.4 can be effectively computed. Thus we obtain:

Corollary 5.5. *Let K be a computable semiring, M be a computable finitely generated monoid, and $S, T \in K\langle\langle M \rangle\rangle$. Assume that either K is locally finite and S, T have effectively given representations, or K has Burnside matrix monoids and S, T have effectively given aperiodic representations. Then from this, the following are decidable:*

1. $S = T$
2. $\mathrm{supp}(S) = \emptyset$
3. $\mathrm{supp}(S) = M$.

Proof. Compute the descriptions of S and T given in Propositions 5.2 and 5.4 and compare the arising languages and the coefficients of their characteristic series. □

The following is a further immediate consequence of Propositions 5.2 and 5.4.

Corollary 5.6. *Let K be a semiring, M be a finitely generated monoid, and $L \subseteq M$.*

(a) If 1_L is recognizable and K is locally finite, then L is recognizable.
(b) If 1_L is aperiodic and K has Burnside matrix monoids, then L is aperiodic.

Corollary 5.7. *Let K be a field and M a finitely generated monoid. Then, $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle = K^{\mathrm{ap}}\langle\langle M\rangle\rangle = K^{\mathrm{wap}}\langle\langle M\rangle\rangle$.*

Proof. The first equality follows from Theorem 5.3 and Proposition 5.4 and the second one from Theorem 4.2. □

Now we wish to investigate recognizable and aperiodic step functions. First we note:

Proposition 5.8. *Let K be any semiring and M be any monoid. The classes $K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$ and $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ are each closed under left and right quotients and the Hadamard product. Inverse homomorphic images of recognizable (resp. aperiodic) step functions of $K\langle\langle M\rangle\rangle$ are again recognizable (resp. aperiodic) step functions.*

Proof. First we deal with $K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$. Let $S \in K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$, $h : N \to M$ a mapping and $p \in M$. Clearly, the image of the series $h^{-1}(S) = S \circ h$ is finite and $(h^{-1}(S))^{-1}(k) = h^{-1}(S^{-1}(k))$ for any $k \in K$. Since recognizable languages are preserved under inverse homomorphic images, we deduce that if h is a morphism then $h^{-1}(S)$ is a recognizable step function. Using the fact that the left quotient $p^{-1}S$ is the inverse image of S by the mapping $h_p : M \to M$ defined by $h_p(x) = px$ and that recognizable languages are closed under left quotients, we deduce that $p^{-1}S$ is also a recognizable step function. We proceed similarly for right quotients.

Now let $S = \sum_{i=1}^{m} k_i \cdot 1_{L_i}$ and $T = \sum_{j=1}^{n} k'_j \cdot 1_{L'_j}$ with $k_i, k'_j \in K$ and recognizable languages $L_i, L'_j \subseteq M$. Then each $L_i \cap L'_j$ is recognizable in M, and $1_{L_i} \odot 1_{L'_j} = 1_{L_i \cap L'_j}$, so $S \odot T = \sum_{i,j} k_i k'_j \cdot 1_{L_i \cap L'_j} \in K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$.

For the class $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ we argue as above, using the closure of aperiodic languages under the operations inverse morphism, left and right quotients, and intersection. □

Now we will derive another version of Theorems 3.7 and 3.9 for step functions which holds also if K is non-commutative. Note that its proof does not require the (relatively complicated) arguments used for Theorems 3.7 and 3.9. Note that the Cauchy product is well-defined since the semiring is idempotent and the series considered have finite images.

Proposition 5.9. *Let M be any monoid in which the product of two aperiodic languages is again aperiodic, and let K be an idempotent semiring. Let $S_1, S_2 \in K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$. Then, $S_1 \cdot S_2 \in K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$.*

Proof. Let $S_1 = \sum_{k_1 \in I_1} k_1 \cdot 1_{S_1^{-1}(k_1)}$ and $S_2 = \sum_{k_2 \in I_2} k_2 \cdot 1_{S_2^{-1}(k_2)}$ with $I_i = \mathrm{Im}(S_i)$ finite ($i = 1, 2$) and each $S_1^{-1}(k_1), S_2^{-1}(k_2)$ aperiodic in M. For all $a, b \in K$ and $A, B \subseteq M$ we have $(a \cdot 1_A) \cdot (b \cdot 1_B) = ab \cdot (1_A \cdot 1_B)$ and since K is idempotent we also have $1_A \cdot 1_B = 1_{A \cdot B}$. Hence,

$$S_1 \cdot S_2 = \sum_{k_1 \in I_1, k_2 \in I_2} k_1 \cdot k_2 \cdot 1_{S_1^{-1}(k_1)} \cdot 1_{S_2^{-1}(k_2)} = \sum_{k_1 \in I_1, k_2 \in I_2} k_1 \cdot k_2 \cdot 1_{S_1^{-1}(k_1) \cdot S_2^{-1}(k_2)},$$

and each $S_1^{-1}(k_1) \cdot S_2^{-1}(k_2)$ is aperiodic in M. $\square$

With an analogous proof we also obtain:

Proposition 5.10. *Let M be any monoid in which the product of two recognizable languages is again recognizable, and let K be a locally finite idempotent semiring. Let $S_1, S_2 \in K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$. Then, $S_1 \cdot S_2 \in K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle$.*

Next we show for recognizable step functions the equivalence of aperiodicity and weak aperiodicity.

Proposition 5.11. *Let K be any semiring and M be any monoid. Then, $K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle \cap K^{\mathrm{wap}}\langle\langle M\rangle\rangle = K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$.*

Proof. One inclusion is clear. Conversely, let $S \in K^{\mathrm{rec-step}}\langle\langle M\rangle\rangle \cap K^{\mathrm{wap}}\langle\langle M\rangle\rangle$. We have $S = \sum_{k \in \mathrm{Im}(S)} k \cdot 1_{S^{-1}(k)}$ with $\mathrm{Im}(S)$ finite and each $S^{-1}(k) \subseteq M$ recognizable. We show that the languages $S^{-1}(k)$ are also aperiodic. Let $m \geq 0$ be such that $(S, uv^m w) = (S, uv^{m+1}w)$ for all $u, v, w \in M$. Then, for all $k \in K$ we have $uv^m w \in S^{-1}(k)$ if and only if $uv^{m+1}w \in S^{-1}(k)$ and we deduce that the syntactic monoid of $S^{-1}(k)$ is aperiodic. Therefore, the languages $S^{-1}(k)$ are all aperiodic. $\square$

As an immediate consequence of Proposition 5.2 and Proposition 5.11, if K is a locally finite semiring, any weakly aperiodic series S is aperiodic.

6 Star-free series

In a monoid M, the collection $\mathrm{SF}(M)$ of star-free languages is defined as the smallest system of languages in M containing all finite languages and being closed under the operations union, complement and product. We define a corresponding notion for formal power series as follows. Let K be a semiring and $\overline{} : K \to K$ be any mapping such that $\overline{0} = 1$ and $\overline{1} = 0$. Then, we call $\overline{}$ a complement function on K and $(K, \overline{})$ a semiring with complement function. For $S \in K\langle\langle M \rangle\rangle$, we define $\overline{S} \in K\langle\langle M \rangle\rangle$, the complement of S, by $(\overline{S}, w) = \overline{(S, w)}$ for $w \in M$. Oberve that for characteristic series, we have $\overline{1_L} = 1_{\overline{L}}$ where $\overline{L}$ denotes the complement of L in M.

Definition 6.1. *Let M be a monoid and K a semiring with complement. Assume either that each element of M has only finitely many factorizations or that K is idempotent. The collection $K^{\mathrm{sf}}\langle\langle M \rangle\rangle$ of all star-free series in $K\langle\langle M \rangle\rangle$ is the smallest collection of formal power series containing all polynomials and being closed under the operations sum, product and complement.*

In [8] we have defined star-free series using a complement restricted to characteristic series. All the results presented in this section also hold for this alternative definition.

Note that if K is idempotent, by structural induction all star-free series in $K\langle\langle M \rangle\rangle$ have finite image and the product operation is well-defined on $K^{\mathrm{sf}}\langle\langle M \rangle\rangle$. Note that if S is star-free and $k \in K$, then the series $k \cdot S$ is star-free, since k is a polynomial.

Proposition 6.2. *Let M be any monoid, let K be an idempotent semiring with complement and let $L \subseteq M$ be star-free. Then 1_L is a star-free series.*

Proof. By structural induction on L. If L is finite, 1_L is a polynomial. If $L = L_1 \cup L_2$, we have $1_L = 1_{L_1} + 1_{L_2}$ since $(K, +)$ is idempotent. For the same reason, $L = L_1 \cdot L_2$ implies $1_L = 1_{L_1} \cdot 1_{L_2}$. Finally, $1_{\overline{L}} = \overline{1_L}$ by definition. □

Schützenberger [27] showed that in the free monoid Σ^*, aperiodic languages are star-free. This was generalized by Guaiana [12] to arbitrary finitely generated monoids:

Lemma 6.3 ([12, Thm. 5.1.4]). *Let M be any finitely generated monoid. Then each aperiodic language $L \subseteq M$ is star-free.*

As a consequence of Proposition 6.2 and Lemma 6.3 we obtain

Proposition 6.4. *Let M be a finitely generated monoid, and let K be an idempotent semiring with complement. Then, $K^{\mathrm{ap-step}}\langle\langle M \rangle\rangle \subseteq K^{\mathrm{sf}}\langle\langle M \rangle\rangle$.*

Proof. Let $S \in K\langle\langle M \rangle\rangle$ be an aperiodic step function. Then $S = \sum_{k \in \mathrm{Im}(S)} k \cdot 1_{S^{-1}(k)}$ with $\mathrm{Im}(S)$ finite and each $S^{-1}(k)$ aperiodic in M. By Lemma 6.3, $S^{-1}(k)$ is star-free in M. Now apply Proposition 6.2 to obtain that S is star-free. □

Now we can prove the main result of this section.

Theorem 6.5. *Let M be a finitely generated monoid. The following are equivalent:*

(1) $\mathrm{SF}(M) = \mathrm{AP}(M)$
(2) $K^{\mathrm{sf}}\langle\langle M\rangle\rangle = K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ *for some idempotent semiring* K *with complement.*
(3) $K^{\mathrm{sf}}\langle\langle M\rangle\rangle = K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ *for any idempotent semiring* K *with complement.*

Proof. (1) $\Rightarrow$ (3) : Let K be an idempotent semiring with complement. By Proposition 6.4 it remains to show that each star-free series in $K\langle\langle M\rangle\rangle$ is an aperiodic step function. We proceed by induction. Each singleton in M is star-free, hence aperiodic by (1). Thus $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ contains all polynomials. Clearly, $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ is closed under the sum operation and by Proposition 5.9 also under Cauchy product. Now let $S = \sum_{i=1}^{n} k_i 1_{L_i}$ be an aperiodic step function. We may assume that the languages $(L_i)_{1\le i\le n}$ form a partition of M. Then, $\overline{S} = \sum_{i=1}^{n} \overline{k_i} 1_{L_i}$ is also an aperiodic step function.

(3) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (1) : Choose K as in (2). Let $L \subseteq M$ be star-free. By Proposition 6.2, 1_L is star-free and hence an aperiodic step function by (2). Thus, L is aperiodic. The converse is just Lemma 6.3. □

It follows directly from Theorem 6.5 that if M is a finitely generated monoid with $\mathrm{SF}(M) = \mathrm{AP}(M)$ and K is an idempotent semiring with complement then $K^{\mathrm{sf}}\langle\langle M\rangle\rangle = K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ does not depend on the complement function of K and, by the proof of Propositions 6.4 and 6.2, it coincides with the alternative class of star-free series considered in [8] where complement is restricted to characteristic series.

As a consequence of Theorem 6.5 and Proposition 5.8 we obtain an analogue of Proposition 5.4 for star-free series, the converse of Proposition 6.2, and further closure properties of star-free series.

Corollary 6.6. *Let* K *be an idempotent semiring with complement. Let* M *be any finitely generated monoid satisfying* $\mathrm{SF}(M) = \mathrm{AP}(M)$*.*

(a) If $S \in K\langle\langle M\rangle\rangle$ *is star-free, then for any* $k \in K$ *the set* $S^{-1}(k)$ *is star-free in* M*. In particular,* $\mathrm{Supp}(S)$ *is star-free.*
(b) If $L \subseteq K$*, then* L *is star-free iff* 1_L *is star-free.*
(c) $K^{\mathrm{sf}}\langle\langle M\rangle\rangle$ *is closed under left and right quotients and the Hadamard product.*
(d) Let N *be a finitely generated monoid,* $h : N \to M$ *be a morphism and* $S \in K^{\mathrm{sf}}\langle\langle M\rangle\rangle$*. Then* $h^{-1}(S) \in K^{\mathrm{sf}}\langle\langle N\rangle\rangle$*.*

The following theorem derives from Proposition 5.4 and Theorem 6.5. It holds in particular for the semirings $\mathcal{M}$, $\mathcal{P}$, $\mathcal{R}at$ (Theorem 5.3) and $\mathcal{R}_{\max,\min}$, or any distributive lattice with 0 and 1.

Theorem 6.7. *Let* K *be an idempotent semiring with complement and with Burnside matrix monoids, and let* M *be any finitely generated monoid satisfying* $\mathrm{SF}(M) = \mathrm{AP}(M)$*. Then,* $K^{\mathrm{ap}}\langle\langle M\rangle\rangle = K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle = K^{\mathrm{sf}}\langle\langle M\rangle\rangle$*.*

The condition $\mathrm{SF}(M) = \mathrm{AP}(M)$ is satisfied for all trace monoids by Guaiana, Restivo and Salemi [13]. It also holds, e.g., for a class of particular concurrency monoids (intersecting with trace monoids only in the class of free monoids), see [6].

7 Conclusion

We have investigated three classes of recognizable series with aperiodicity properties: the class $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle$ of aperiodic step functions, the class $K^{\mathrm{ap}}\langle\langle M\rangle\rangle$ of aperiodic series, and the class $K^{\mathrm{wap}}\langle\langle M\rangle\rangle$ of weakly aperiodic series. We have shown that these three classes have similar closure properties under inverse homomorphic images, left and right quotients, sum, Cauchy product and Hadamard product. Clearly, $K^{\mathrm{ap-step}}\langle\langle M\rangle\rangle \subseteq K^{\mathrm{ap}}\langle\langle M\rangle\rangle \subseteq K^{\mathrm{wap}}\langle\langle M\rangle\rangle$ and under suitable hypotheses we have equalities which are summarized in the following table together with their relationships with the star free series (where BMM means semiring with Burnside matrix monoids).

$K \setminus M$	any	finitely generated	fin. gen. and $\mathrm{AP}(M) = \mathrm{SF}(M)$
BMM		ap-step = ap 5.4	
loc. finite		ap-step = ap = wap 5.2 and 5.11	
field	ap = wap 4.2	ap-step = ap = wap 5.7	
idempotent		ap-step $\subseteq$ sf 6.4	ap-step = sf 6.5
idempotent and BMM		ap-step = ap $\subseteq$ sf 5.4 and 6.4	ap-step = ap = sf 6.7
idempotent and loc. fin.		ap-step = ap = wap $\subseteq$ sf 5.2, 5.11 and 6.4	ap-step = ap = wap = sf 5.2, 5.11 and 6.5

Recently, we obtained further relationships between aperiodic series and series definable in weighted first-order logic [9].

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