Ambiguity Hierarchies of Rational Series

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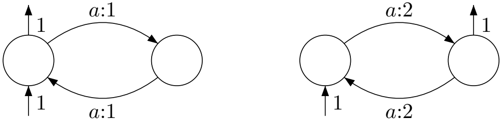

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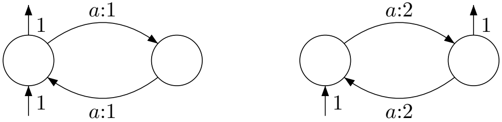

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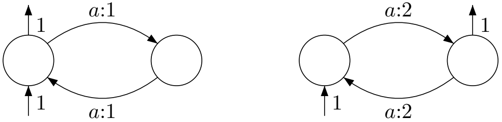

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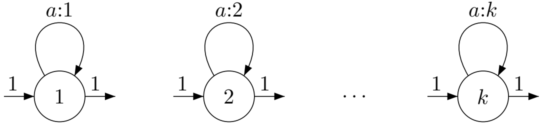

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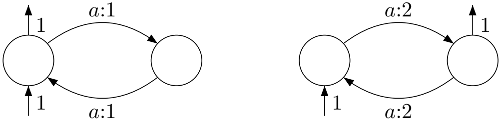

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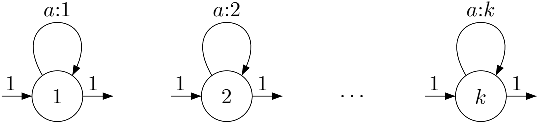

- Finitely ambiguous if it is k-ambiguous for some k.

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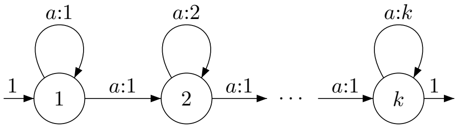

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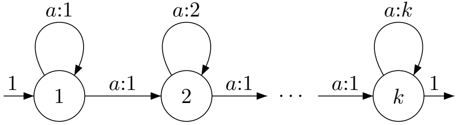

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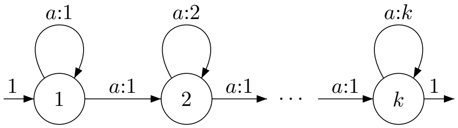

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- The same concepts also studied for weighted tree automata.

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- Unambiguous series over fields are exactly the Pólya series (J. P. Bell and D. Smertnig, 2021).

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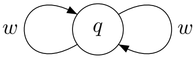

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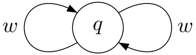

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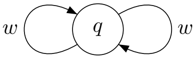

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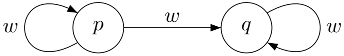

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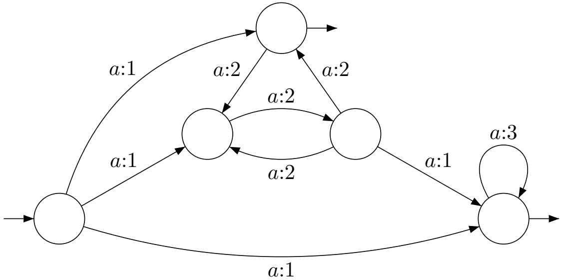

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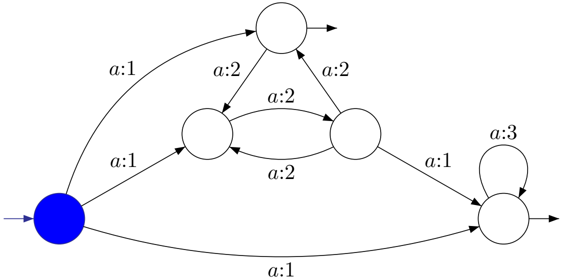

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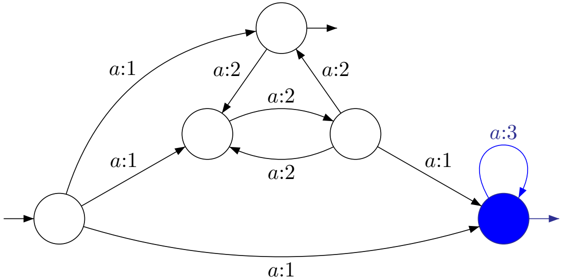

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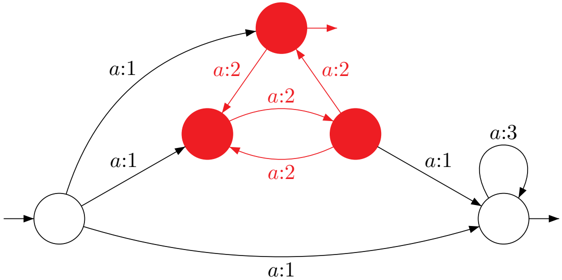

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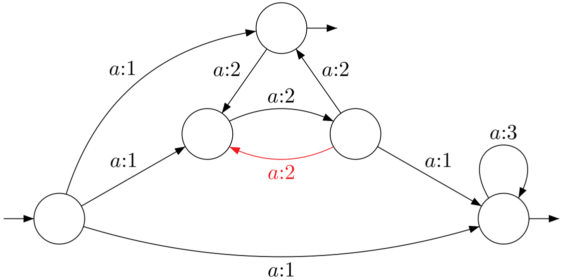

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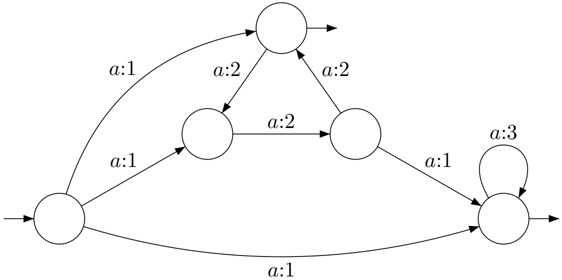

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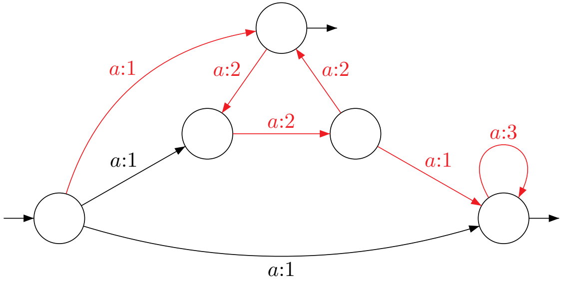

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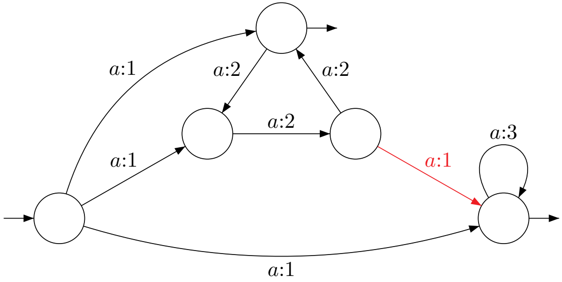

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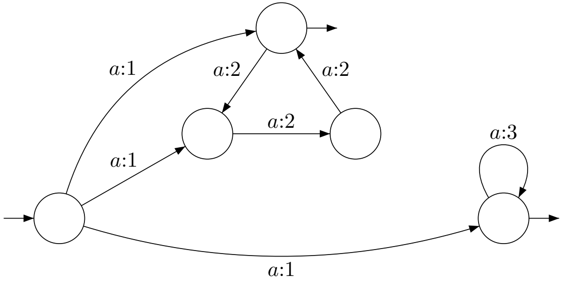

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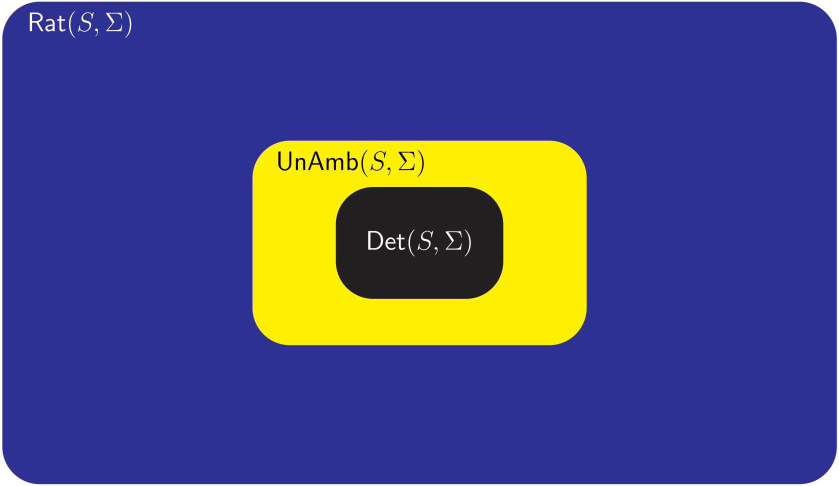

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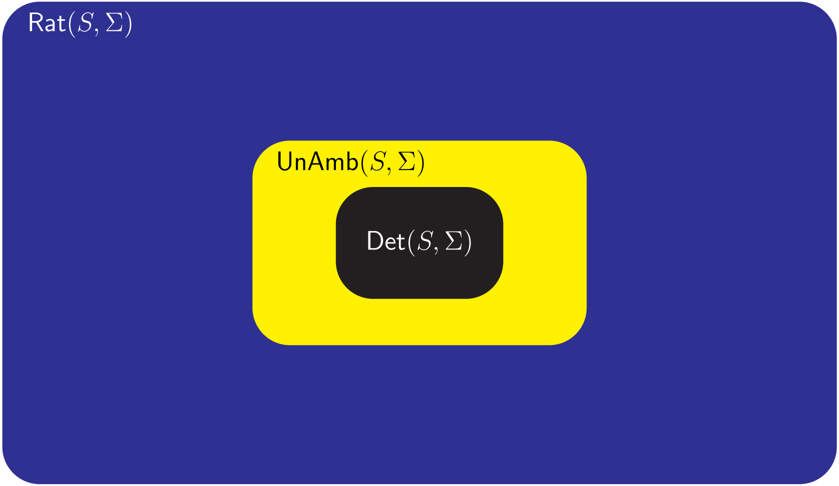

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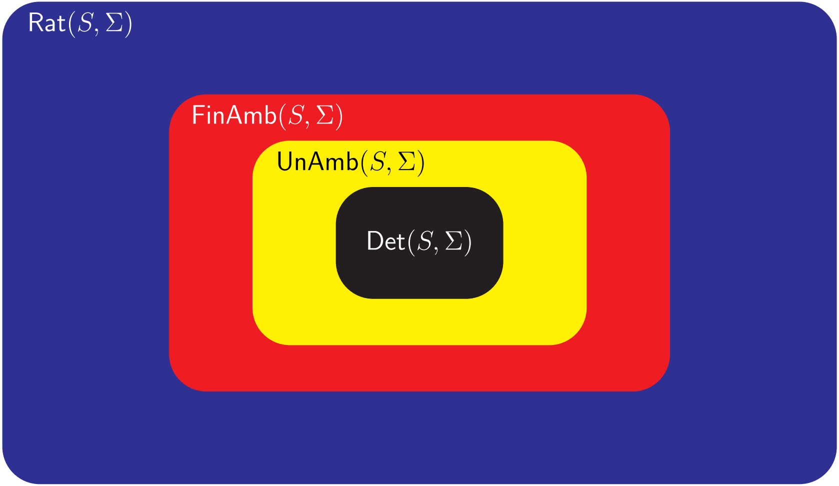

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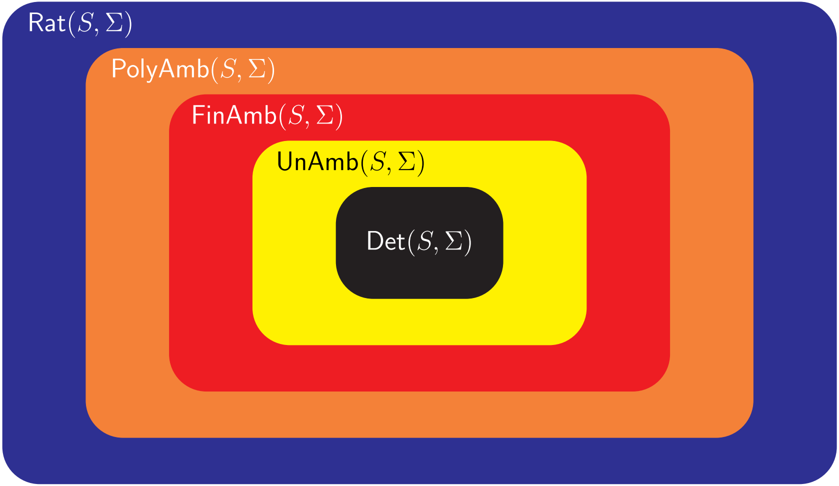
- PolyAmb(S, Σ) the set of all series realised by polynomially ambiguous weighted automata over S and Σ.



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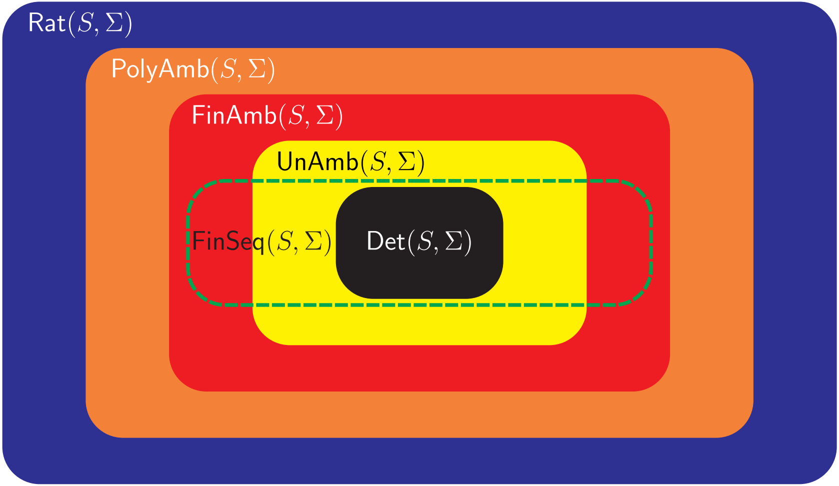

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- The finite ambiguity hierarchy:

$$\text{k-Amb}(S,\Sigma) \subseteq (\mathrm{k}+1)\text{-Amb}(S,\Sigma) \qquad \text{for all } k \geq 1$$

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- A trivial case: $\mathsf{Det}(S,\Sigma) = \mathsf{Rat}(S,\Sigma)$ for all alphabets Σ when the semiring S is locally finite.

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$$\begin{aligned}\mathsf{Det}(\mathbb{F},\Sigma) \subseteq \mathsf{UnAmb}(\mathbb{F},\Sigma) \subseteq \mathsf{FinAmb}(\mathbb{F},\Sigma) \subseteq \\ \subseteq \mathsf{PolyAmb}(\mathbb{F},\Sigma) \subseteq \mathsf{Rat}(\mathbb{F},\Sigma)\end{aligned}$$

and

$$\mathsf{k\text{-}Amb}(\mathbb{F},\Sigma) \subseteq \mathsf{(k+1)\text{-}Amb}(\mathbb{F},\Sigma) \qquad \text{for } k \geq 1$$

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- (How) do the answers depend on the properties of $\mathbb{F}$?
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- All inclusions are strict already for $\Sigma = \{a\}$ when $\mathbb{F} = \mathbb{Q}$ (C. Barloy et al., 2020).

The Finite Ambiguity Hierarchy

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Theorem (K., 2022)

When $\mathbb{F}$ *is not locally finite,* $\text{k-Amb}(\mathbb{F}, \{a\}) \subsetneq (\text{k}+1)\text{-Amb}(\mathbb{F}, \{a\})$ *holds for all* $k \geq 1$*.*

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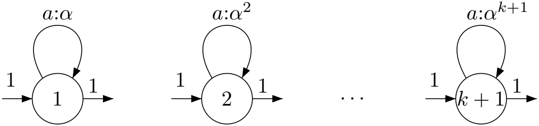

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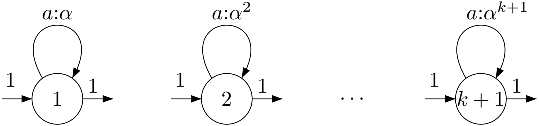

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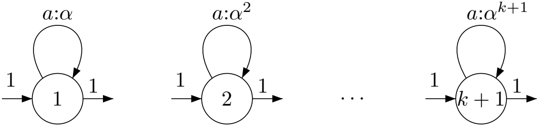

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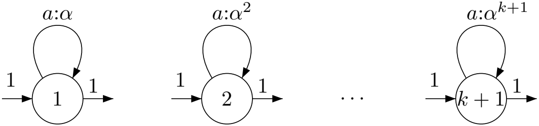

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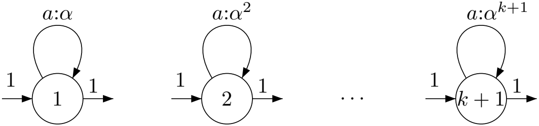

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- $\mathcal{A}$ "needs" $k+1$ cycles with a successful run through each upon every a^t with t sufficiently large.
- $\mathcal{A}$ cannot be k-ambiguous, so $r \notin \text{k-Amb}(\mathbb{F}, \{a\})$.

The Finite Ambiguity Hierarchy

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Corollary

For any Σ*, the inclusions* $\mathsf{k}\text{-}\mathsf{Amb}(\mathbb{F}, \Sigma) \subseteq (\mathsf{k}+1)\text{-}\mathsf{Amb}(\mathbb{F}, \Sigma)$ *for* $k \geq 1$ *are strict iff* $\mathbb{F}$ *is not locally finite.*

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Corollary

For any Σ*, the inclusion* $\mathsf{UnAmb}(\mathbb{F}, \Sigma) \subseteq \mathsf{FinAmb}(\mathbb{F}, \Sigma)$ *is strict iff the field* $\mathbb{F}$ *is not locally finite.*

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For $\mathbb{F}$ of characteristic 0, $r \in \mathsf{PolyAmb}(\mathbb{F}, \{a\}) \setminus \mathsf{FinAmb}(\mathbb{F}, \{a\})$.

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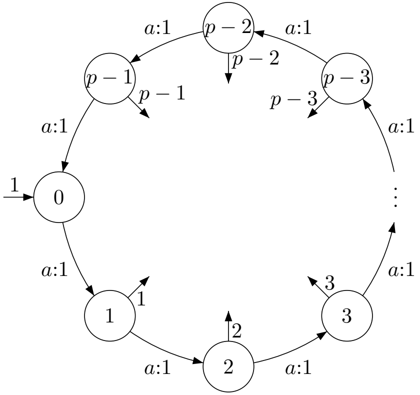

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- When $\mathbb{F}$ is of characteristic $p > 0$, lengthening a cycle may increase multiplicities of its eigenvalues.
- There is no need to go through more cycles in a single run.

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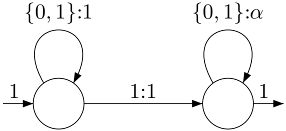

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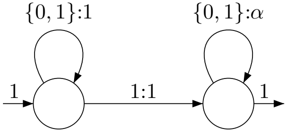

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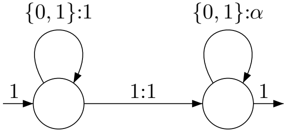

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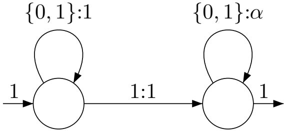

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- Consider words of the form $w_t = (10^t)^{k+1}$.

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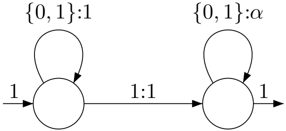

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FinAmb($\mathbb{F}, \Sigma$) vs. PolyAmb($\mathbb{F}, \Sigma$): Larger Alphabets

Theorem (K., submitted)

Let $\mathbb{F}$ be a field that is not locally finite and Σ contain at least two letters. Then FinAmb($\mathbb{F}, \Sigma$) $\subsetneq$ PolyAmb($\mathbb{F}, \Sigma$).

Proof idea:

- Let $\alpha \in \mathbb{F}$ be of infinite multiplicative order.
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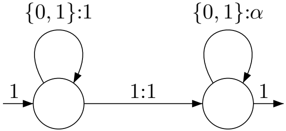

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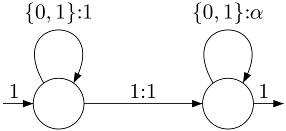

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- $\ldots$ and polynomially ambiguous, as J is upper triangular.

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- Let $\mathcal{A}$ be represented by $\mathbf{i} = (1, 0, \ldots, 0)$, A, and $\mathbf{f} \neq \mathbf{0}$.
- Suppose $\mathcal{A}$ has a polynomially ambiguous equivalent $\mathcal{B}$ represented by $\mathbf{j}$, B, $\mathbf{g}$.

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- $p(x)$ divides some of the factors: a contradiction.

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- It suffices to focus on proper subfields of $\overline{\mathbb{Q}(S)}$, over which such a polynomial can be found.

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Let $\mathbb{F}$ *be of characteristic* 0*. Then* $\mathsf{PolyAmb}(\mathbb{F}, \{a\}) \subseteq \mathsf{Rat}(\mathbb{F}, \{a\})$ *is strict iff* $\mathbb{F}$ *is not algebraically closed.*

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Open Problem

What about other than algebraically closed (and locally finite) fields of characteristic $p > 0$*?*

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Conjecture

Let Σ be an alphabet containing at least two letters. Then the inclusion $\mathsf{PolyAmb}(\mathbb{F}, \Sigma) \subseteq \mathsf{Rat}(\mathbb{F}, \Sigma)$ is strict if and only if the field $\mathbb{F}$ is not locally finite.

Finite Sequentiality

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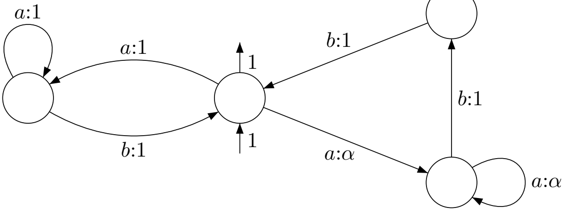

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