Cascade Decomposition of Asynchronous Zielonka Automata

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LMF, ENS Paris-Saclay, IRL ReLaX

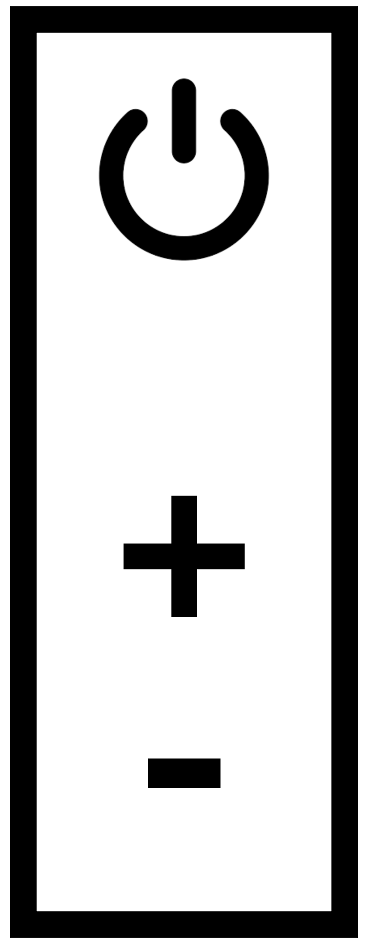
Joint Work with Bharat Adsul (IIT Bombay), Saptarshi Sarkar (IIT Bombay) and Pascal Weil (LaBRI, Univ. Bordeaux)

Based on CONCUR'20, LMCS'22, CONCUR'22 and work in progress

Outline

- Labelling functions, sequential transducers and cascade product
- Krohn-Rhodes theorem for aperiodic/regular word languages
- Model of concurrency: Mazurkiewicz traces and asynchronous Zielonka automata
- Asynchronous labelling functions, transducers and cascade product
- Propositional dynamic logic for traces
- Conclusion

Labelling function $\theta : \Sigma^* \rightarrow \Gamma^*$

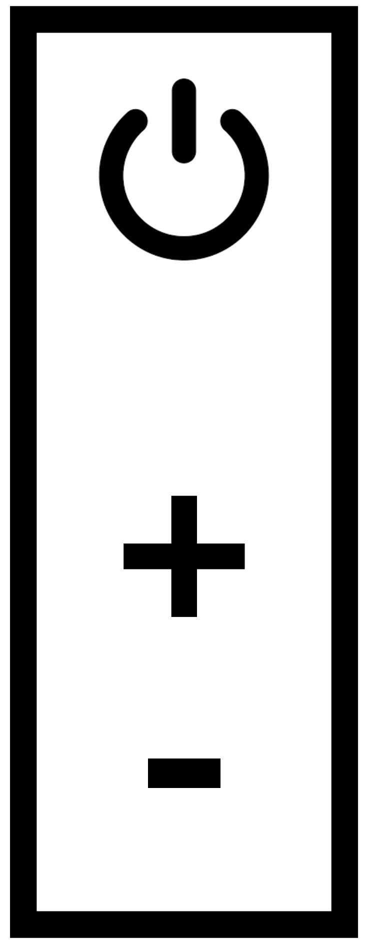


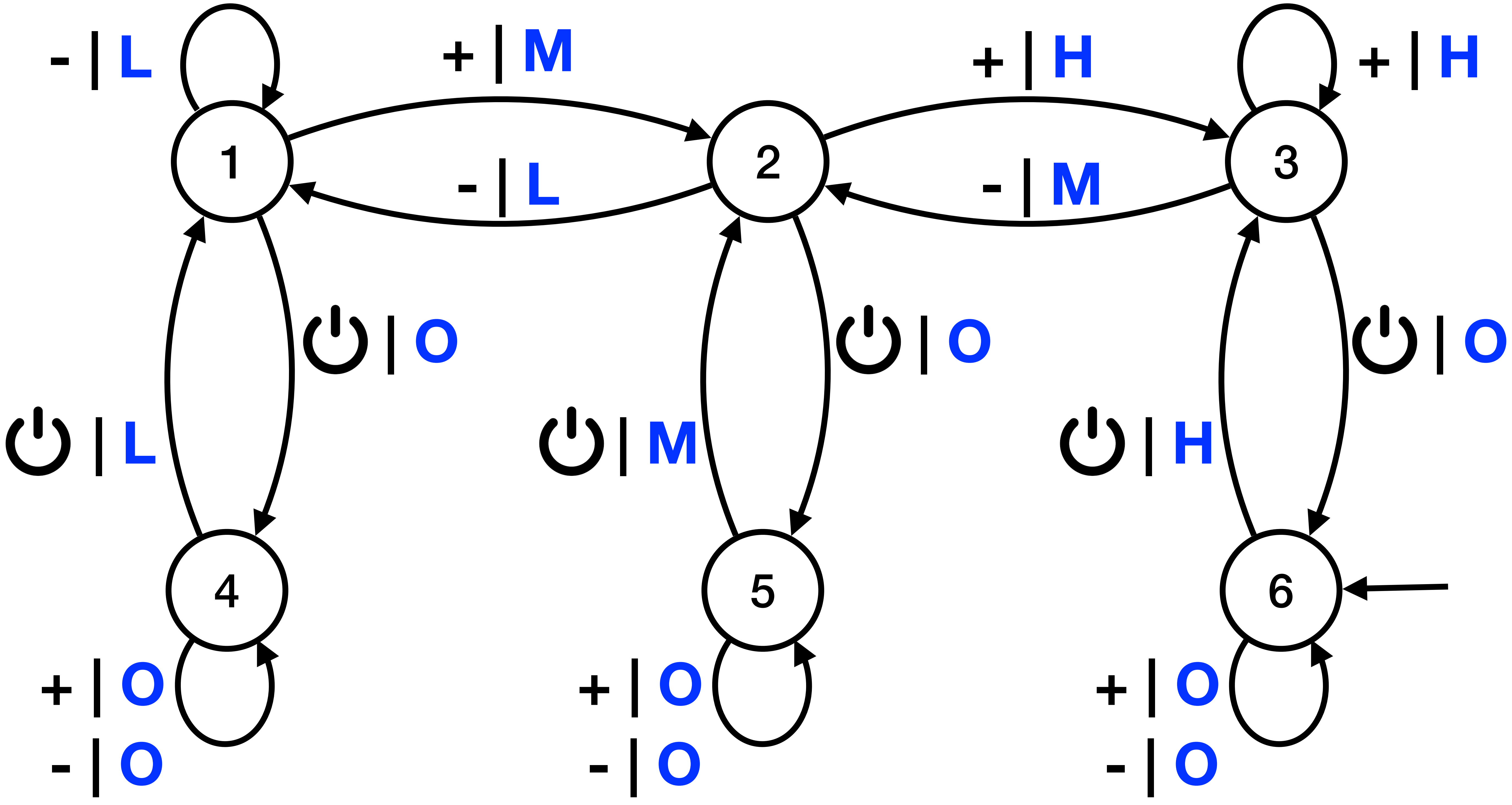
-	⏻	-	⏻	+	⏻	-	-	⏻	⏻	+	+	+	⏻
O	H	M	O	O	M	L	L	O	L	M	H	H	O

$\Sigma = \{$ ⏻ $, -, + \}$ and $\Gamma = \{\mathrm{O}, \mathrm{L}, \mathrm{M}, \mathrm{H}\}$

- ⏻ - ⏻ + ⏻ - - ⏻ ⏻ + + + ⏻

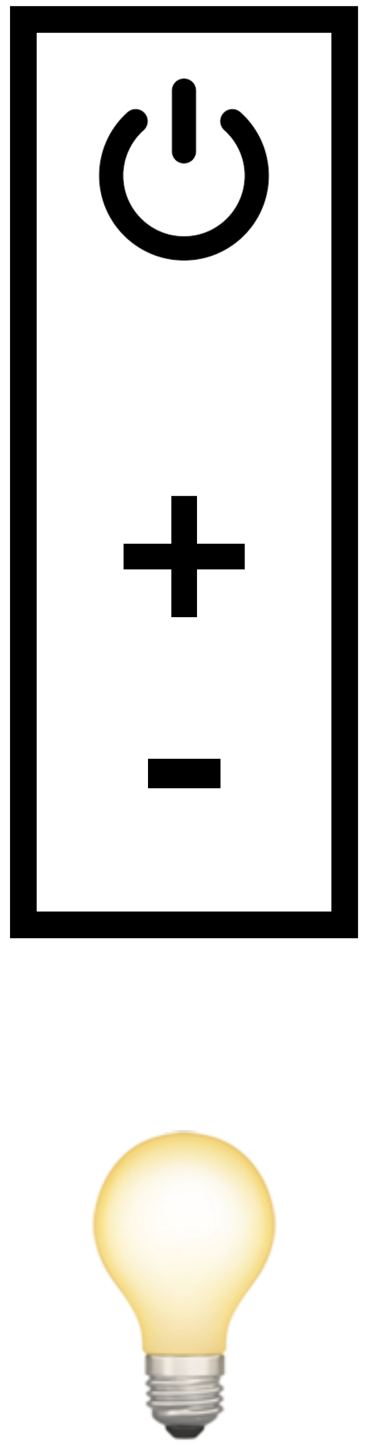
O H M O O M L L O L M H H O

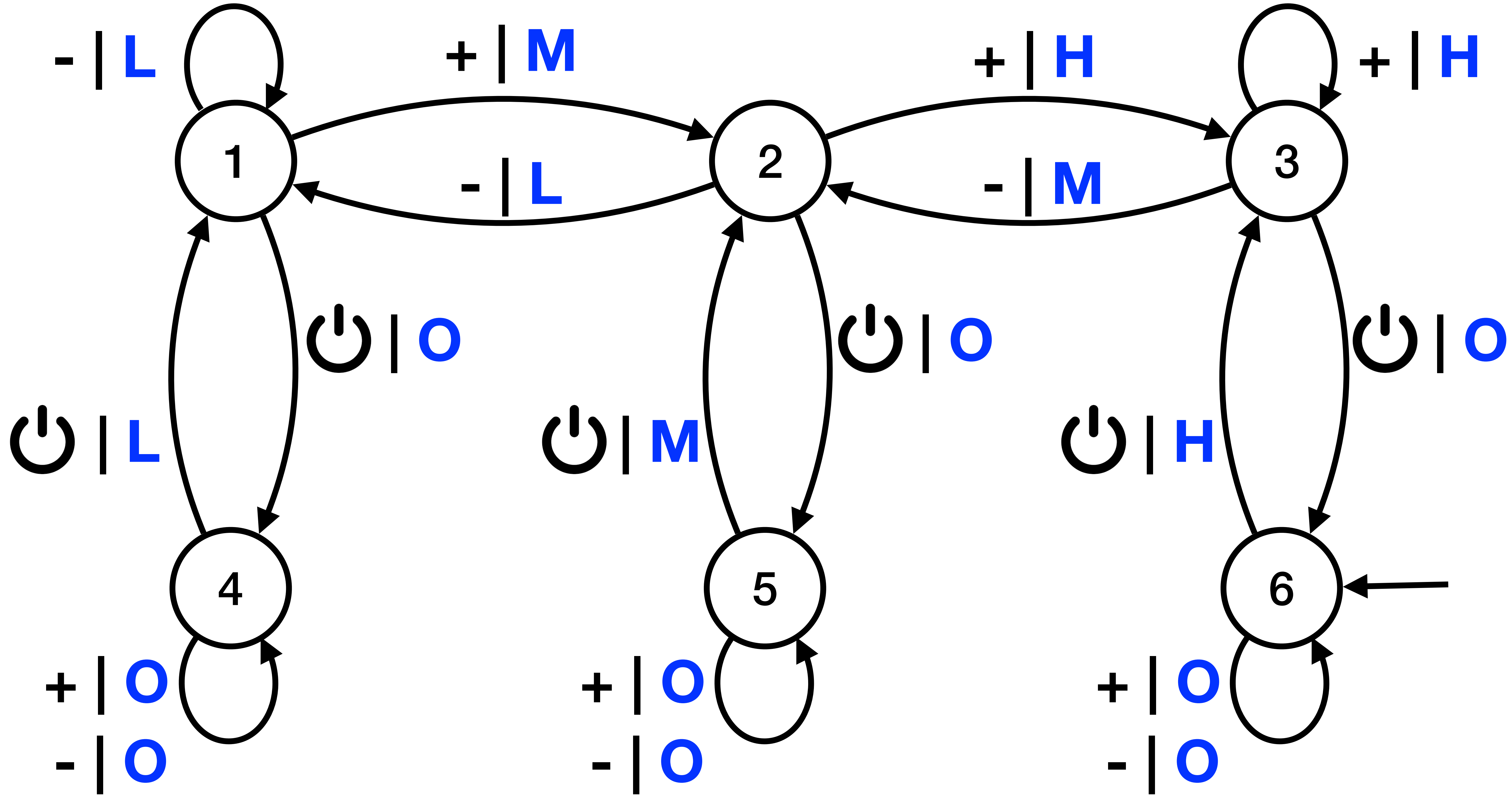




- ⏻ - ⏻ + ⏻ - - ⏻ ⏻ + + + ⏻

O H M O O M L L O L M H H O





letter-to-letter sequential transducer $\mathscr{T} = (Q, q_0, \Sigma, \delta, \Gamma, \mu)$

Composition - Cascade product

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- Composition of labelling functions

$$\Sigma^* \xrightarrow{\theta_1} \Gamma^* \xrightarrow{\theta_2} \Pi^*$$

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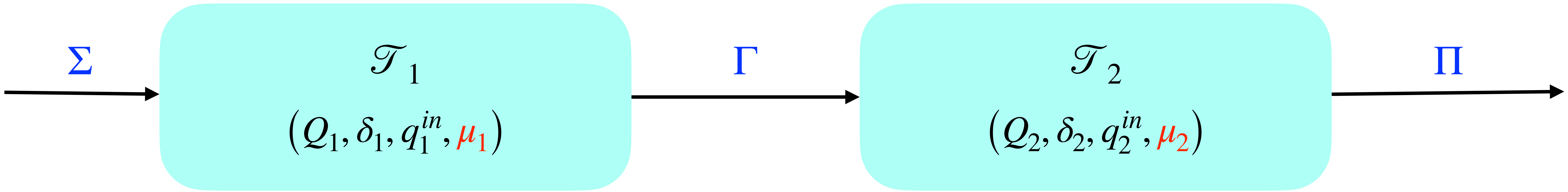
- Cascade product of (letter-to-letter) sequential transducers



Composition - Cascade product

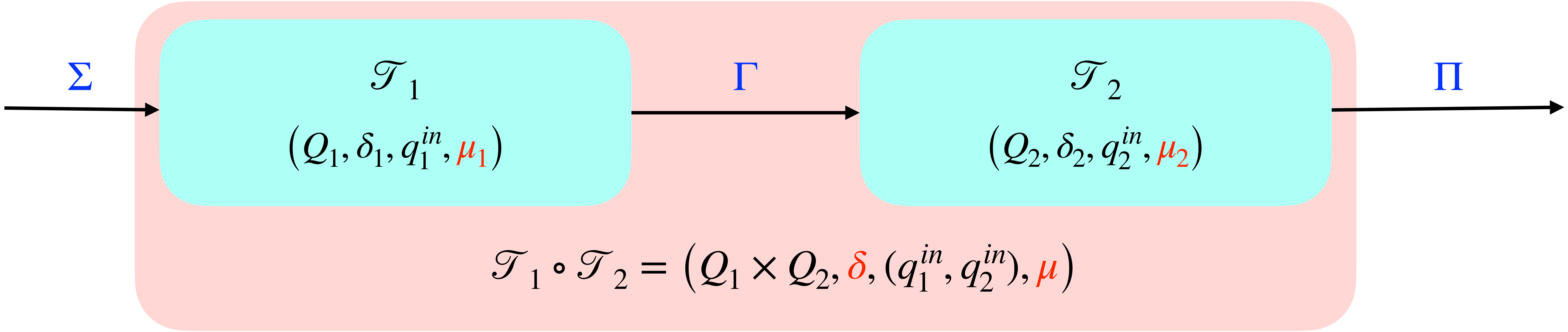
- Composition of labelling functions

$$\Sigma^* \xrightarrow{\theta_1} \Gamma^* \xrightarrow{\theta_2} \Pi^*$$

$$\delta((p,q),a) = \big(\delta_1(p,a), \delta_2(q, \mu_1(p,a))\big)$$

$$\mu((p,q),a) = \mu_2(q, \mu_1(p,a))$$

- Cascade product of (letter-to-letter) sequential transducers



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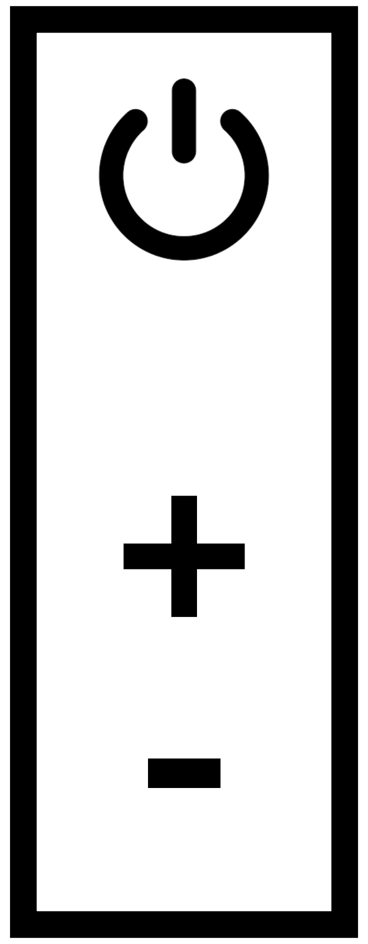

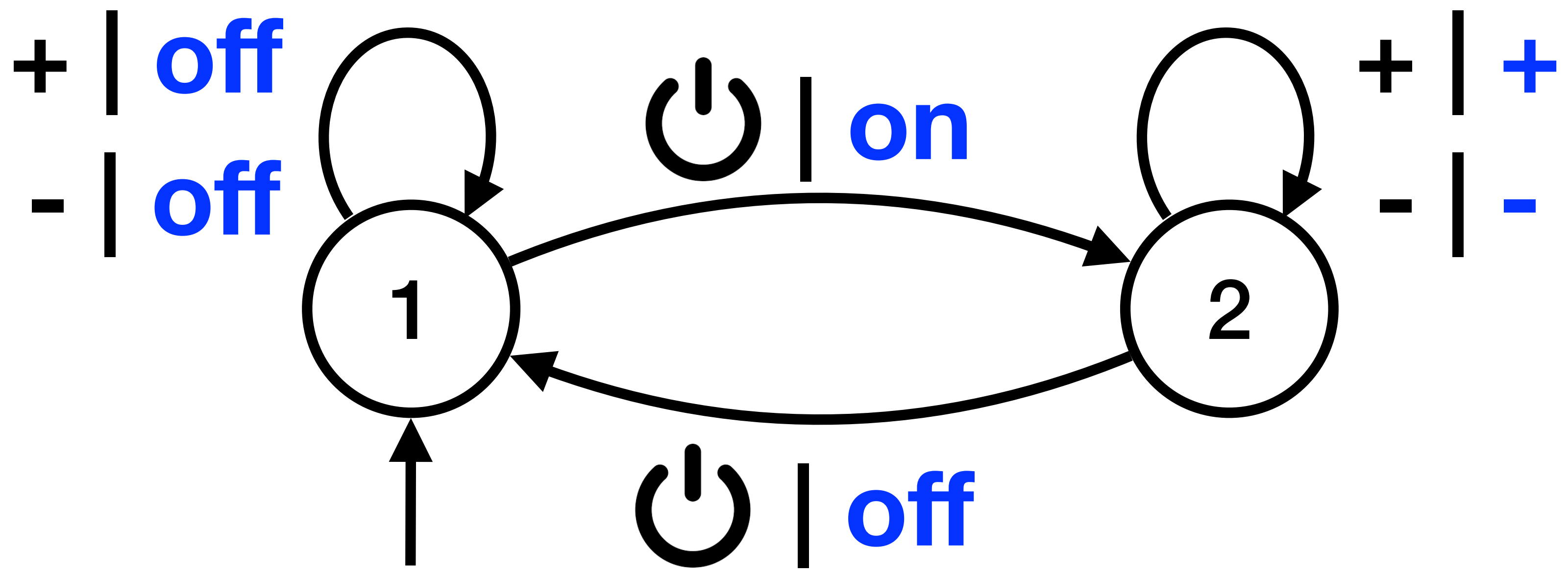

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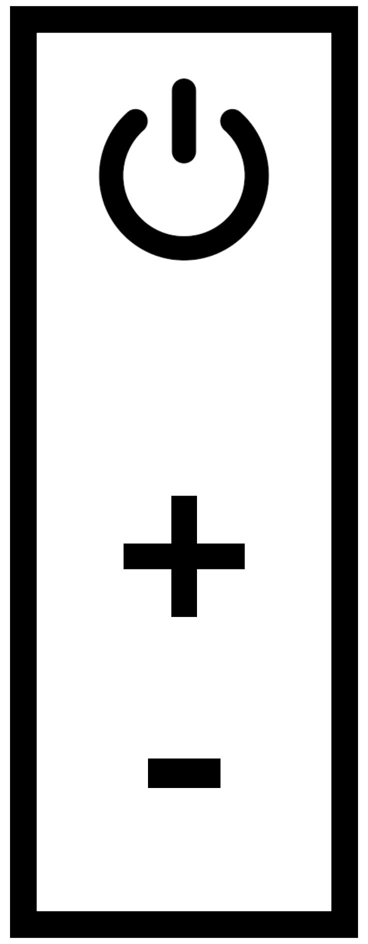

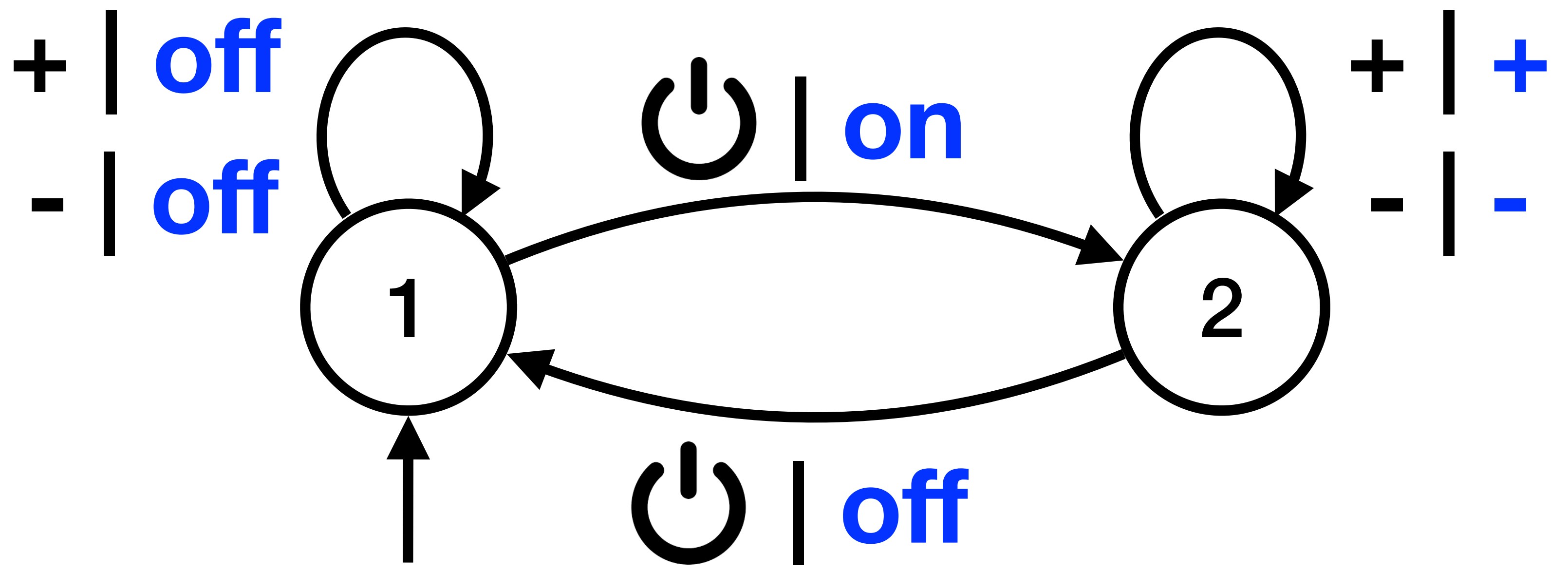

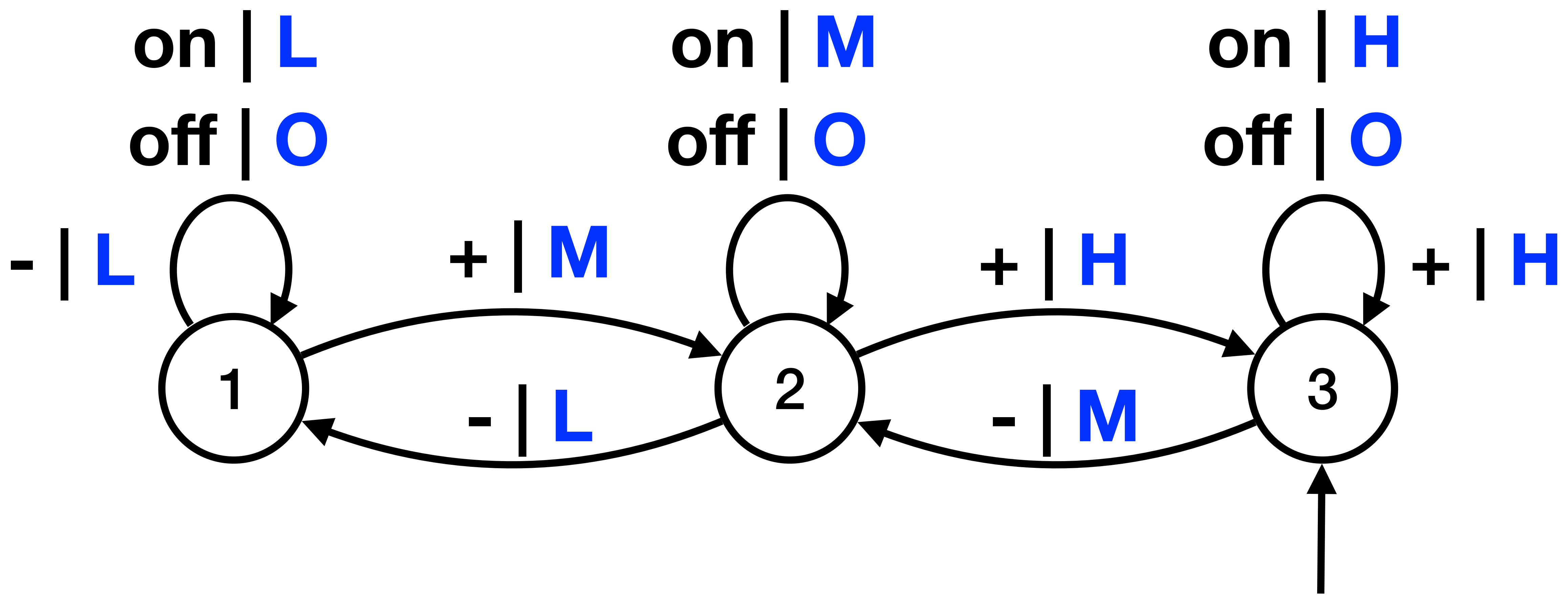

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Krohn-Rhodes

Theorem

Any (letter-to-letter) sequential transducer $\mathcal{T}$ can be realised by a cascade product of reset or permutation transducers:

$$\mathcal{T} \equiv \mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_n$$

Reset or Permutation is a property of the underlying input automaton:

Krohn-Rhodes

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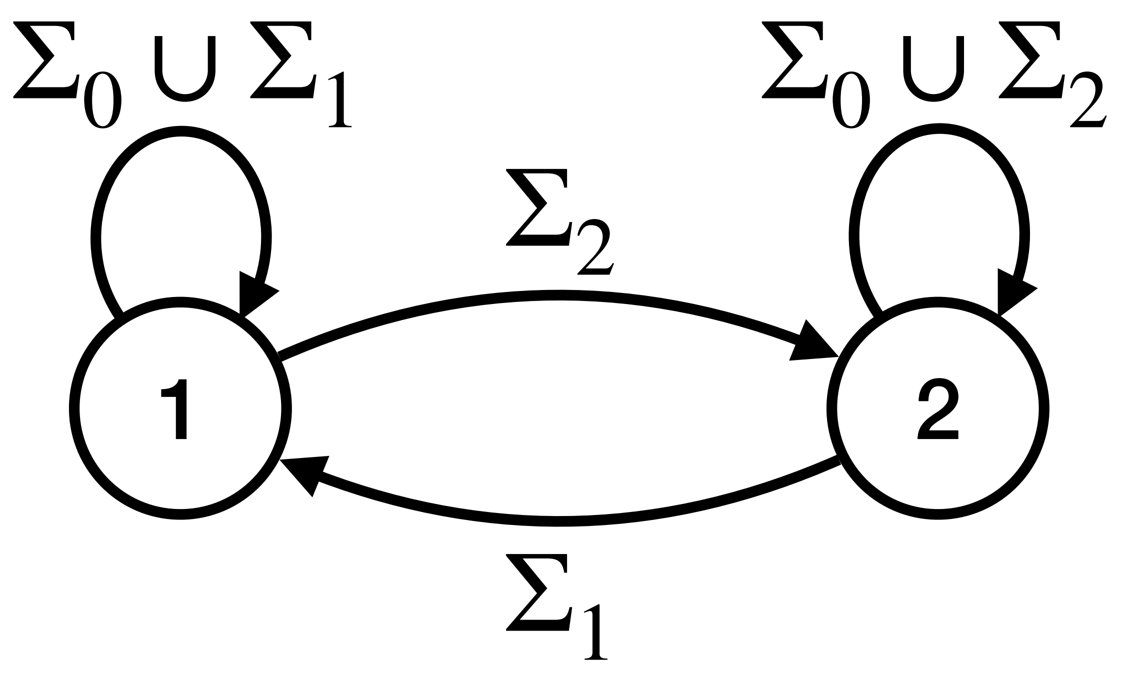
Reset - $\mathcal{U}_2$



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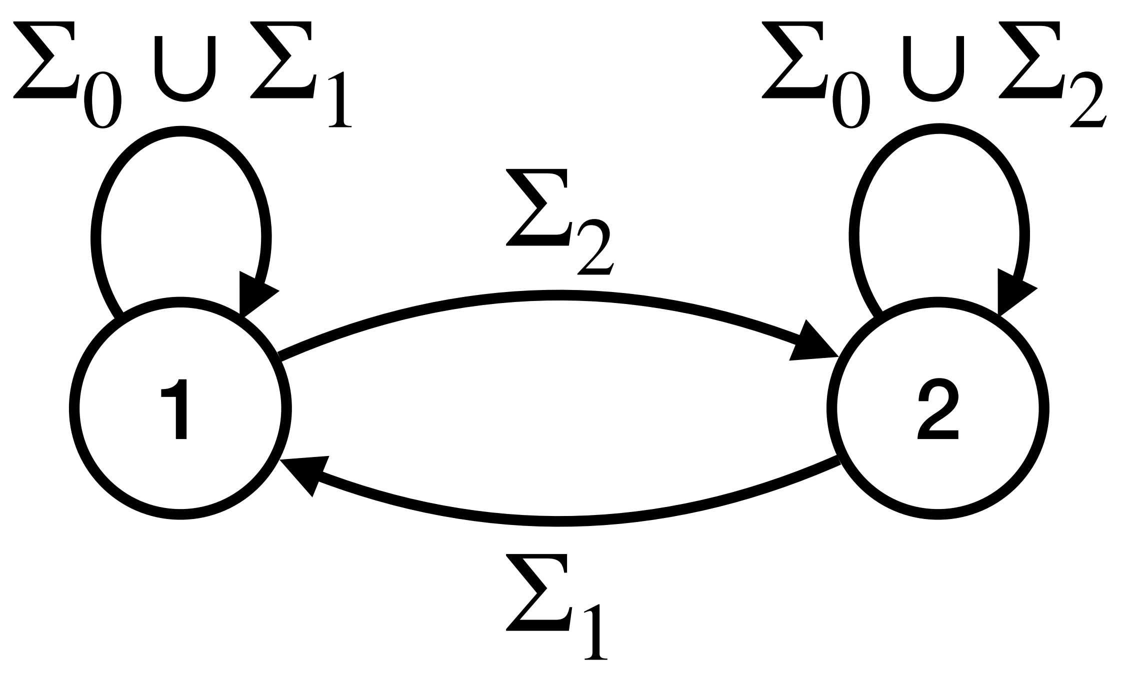
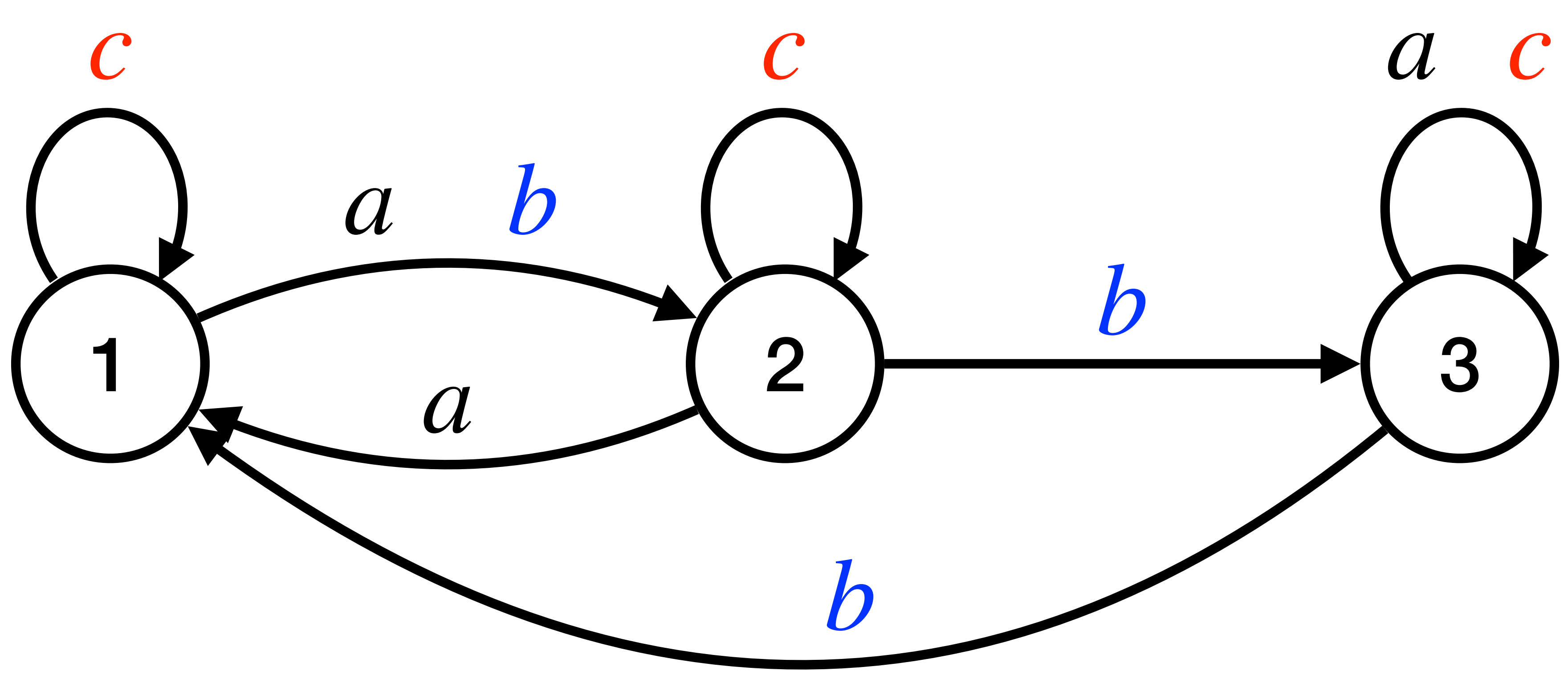
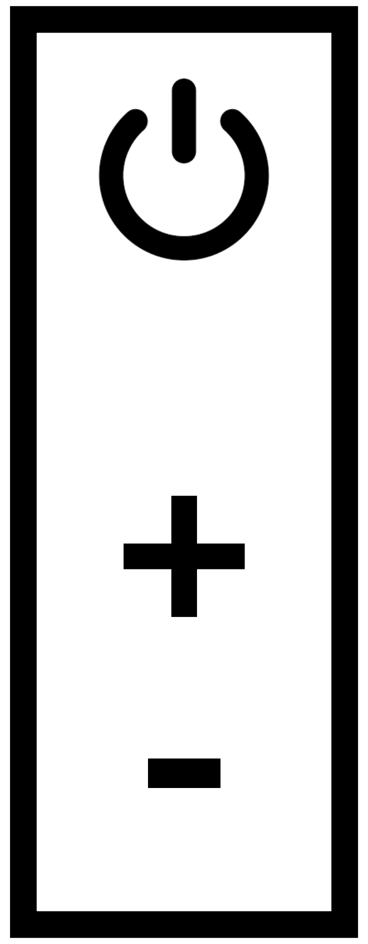


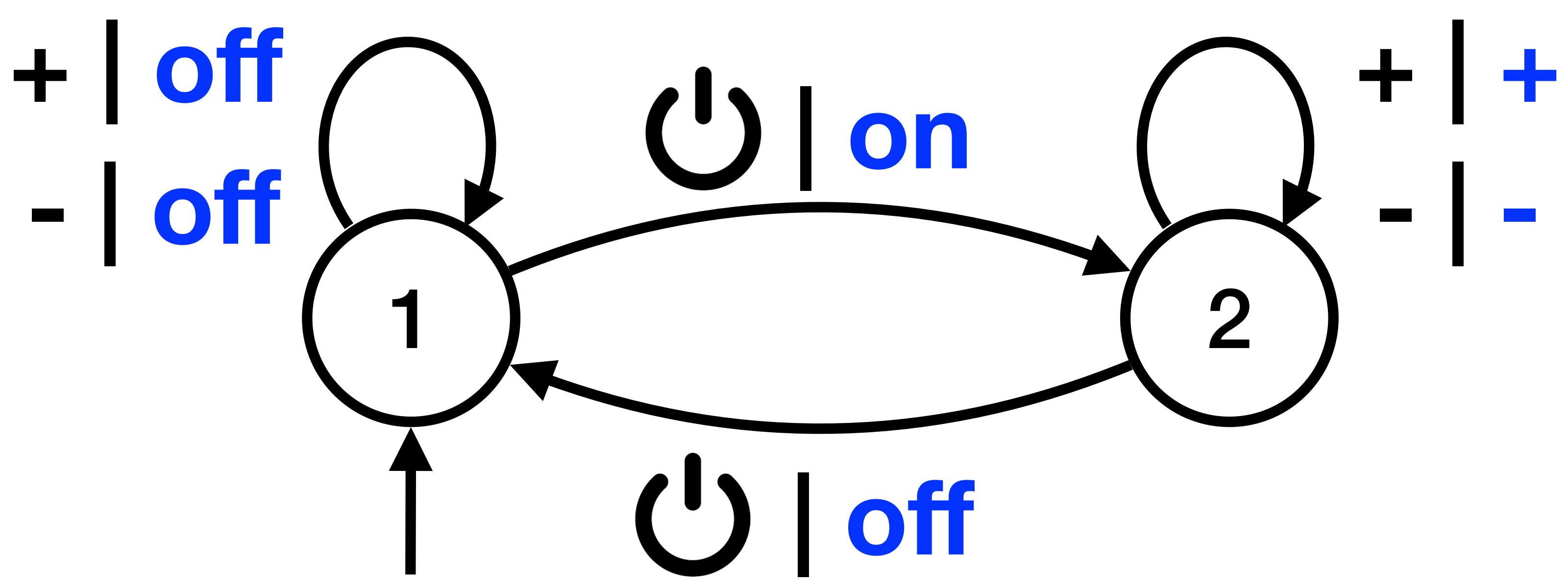
Permutation



- ⏻ - ⏻ + ⏻ - - ⏻ ⏻ + + + ⏻

O H M O O M L L O L M H H O

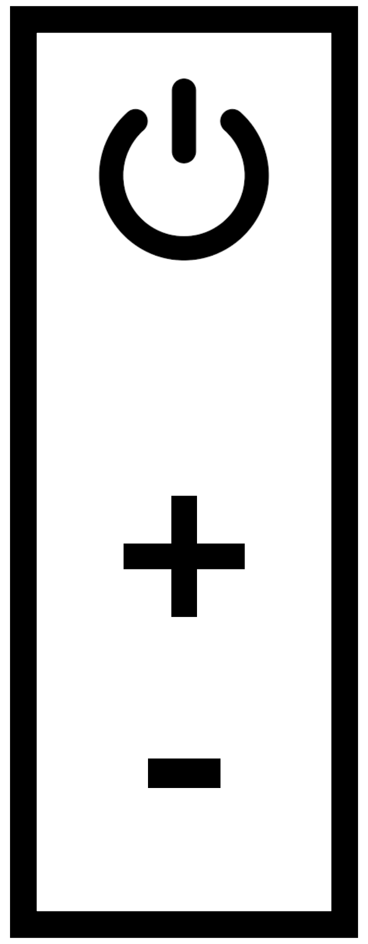


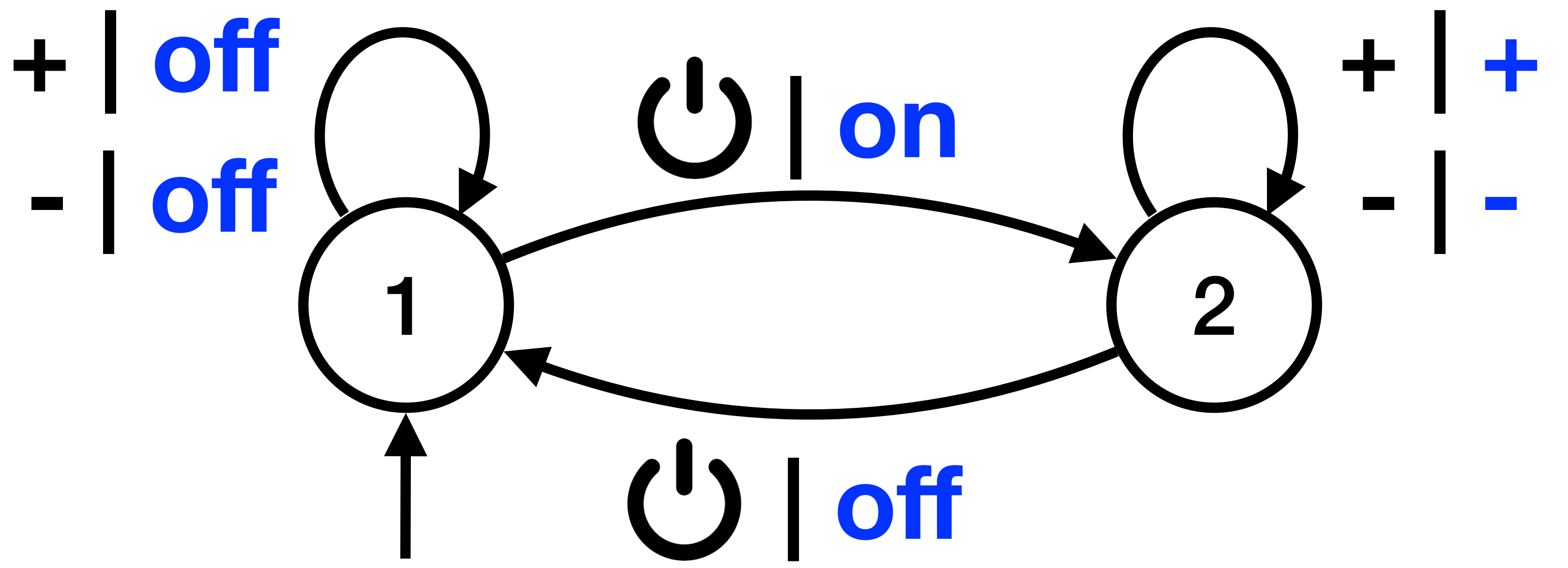


Permutation

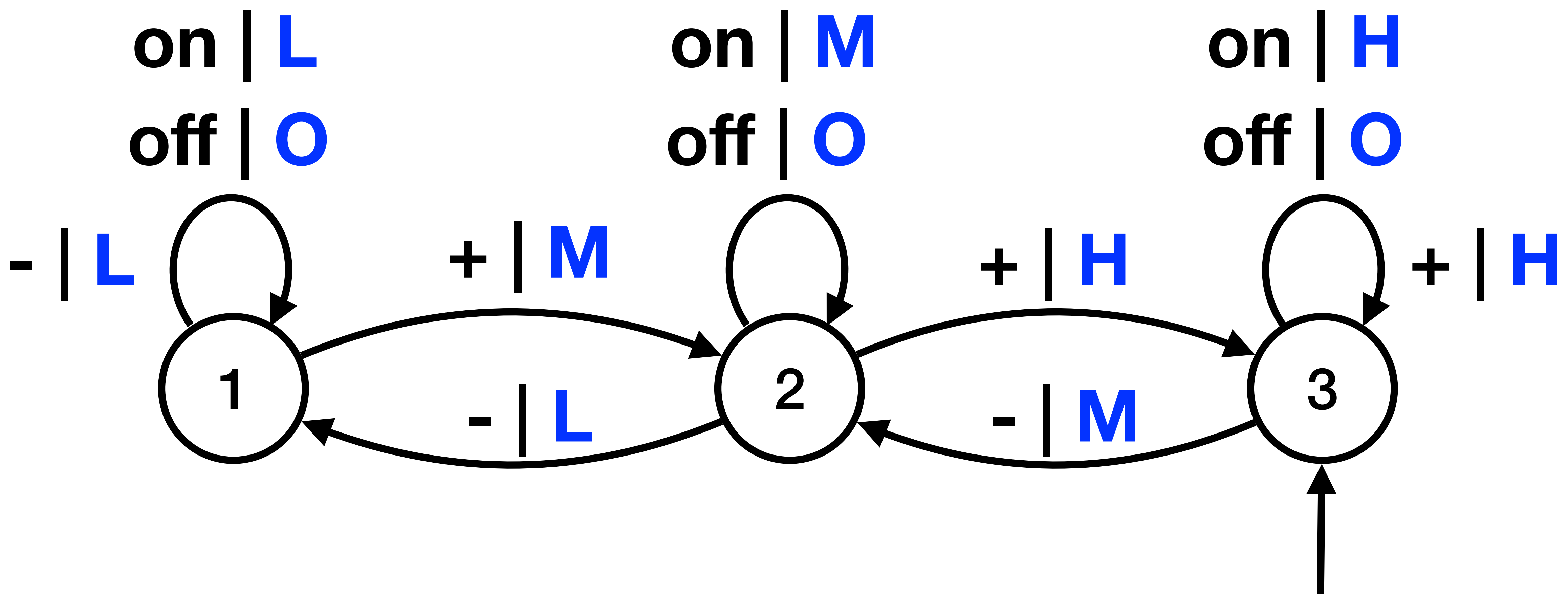
- ⏻ - ⏻ + ⏻ - - ⏻ ⏻ + + + ⏻

O H M O O M L L O L M H H O





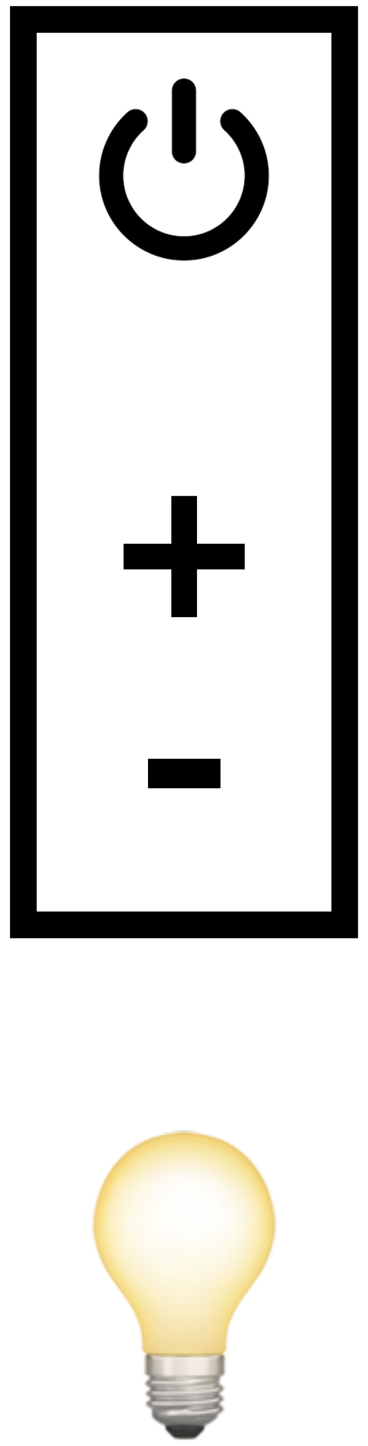
Permutation



Neither Reset nor Permutation

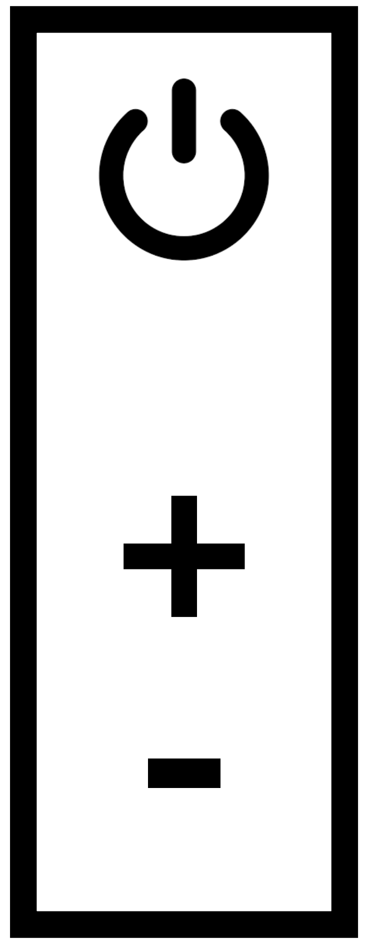
- ⏻ - ⏻ + ⏻ - + ⏻ ⏻ + - - ⏻

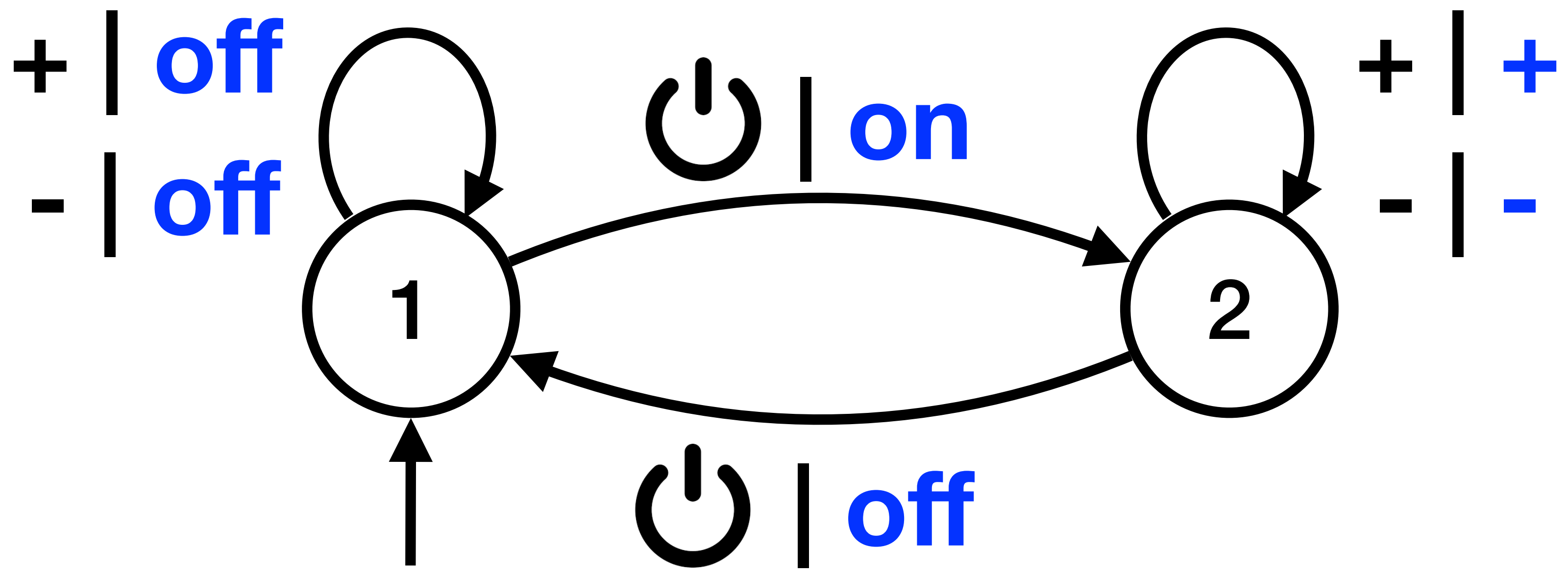
O H L O O L L H O H H L L O



- ⏻ - ⏻ + ⏻ - + ⏻ ⏻ + - - ⏻

O H L O O L L H O H H L L O

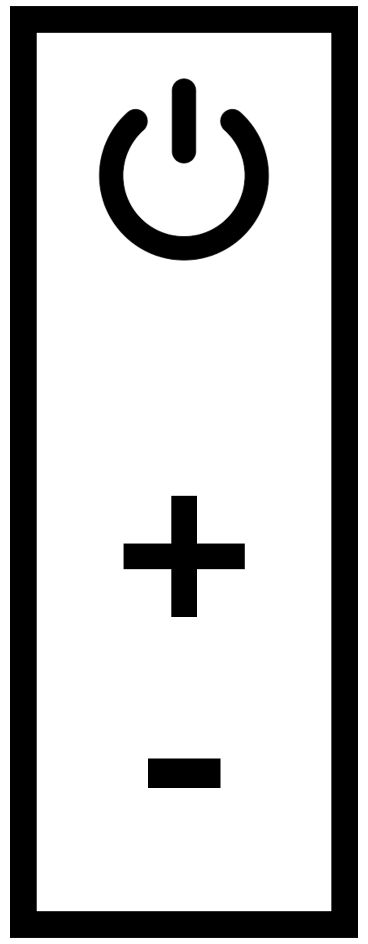


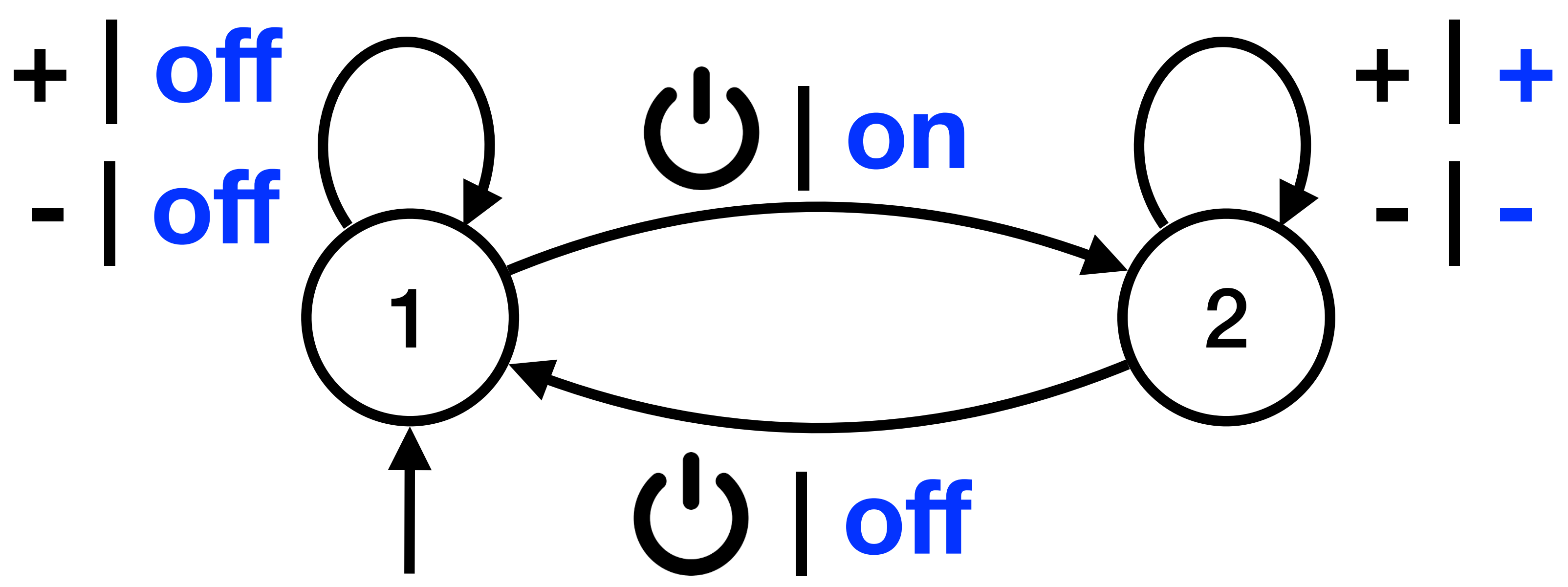


Permutation

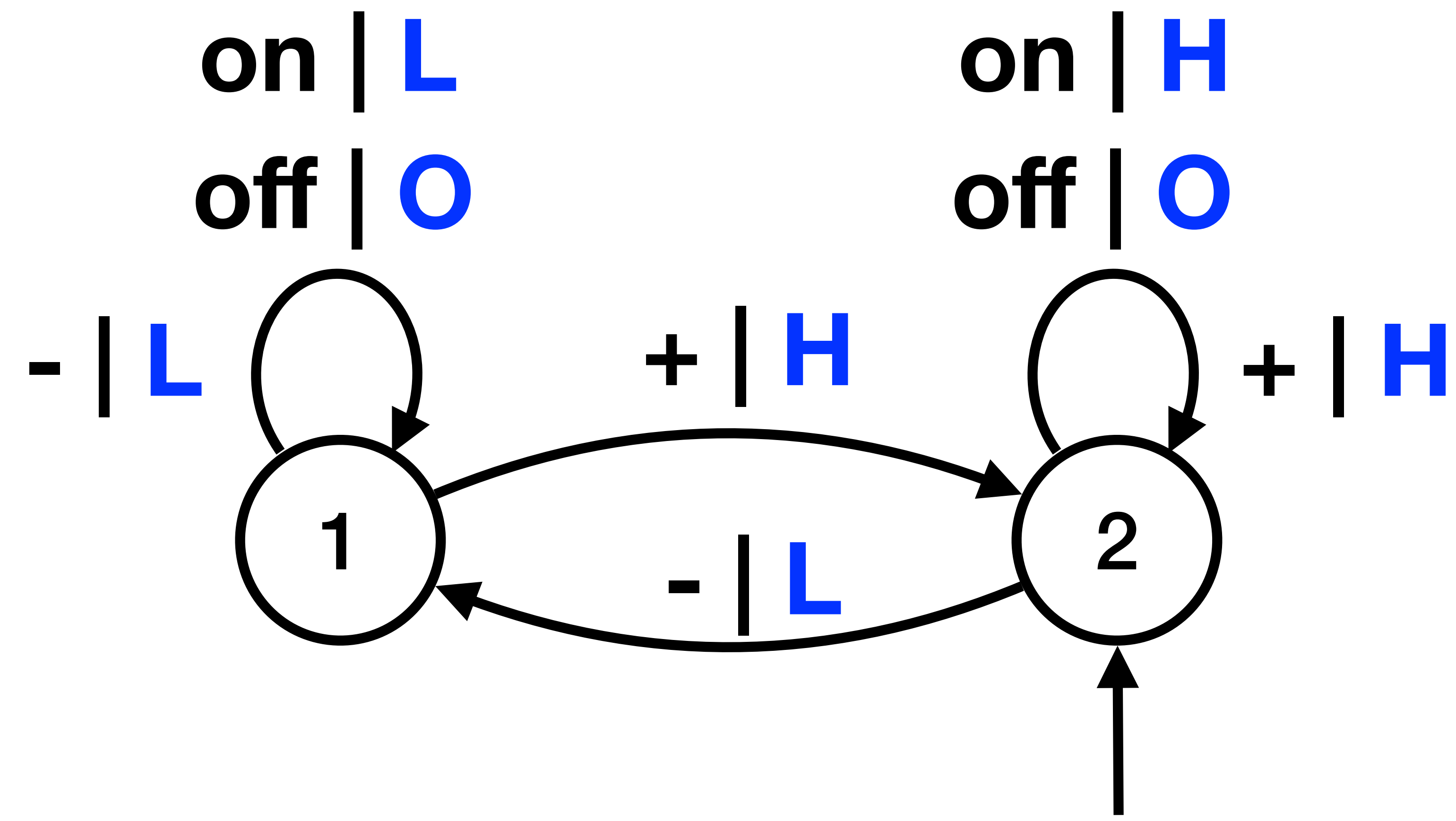
- ⏻ - ⏻ + ⏻ - + ⏻ ⏻ + - - ⏻

O H L O O L L H O H H L L O



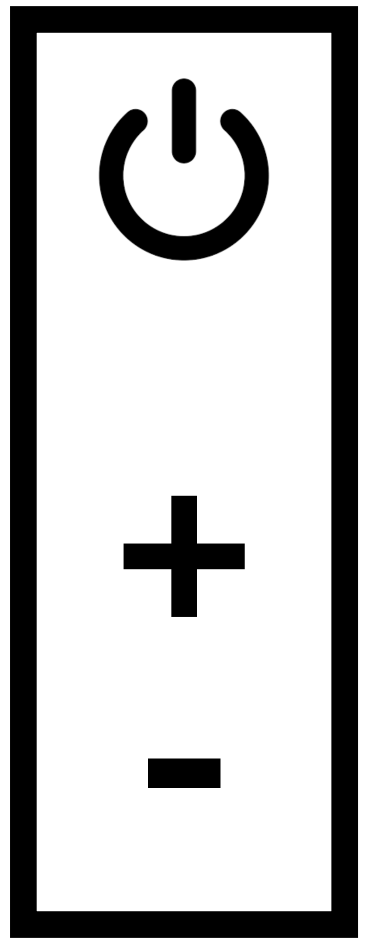


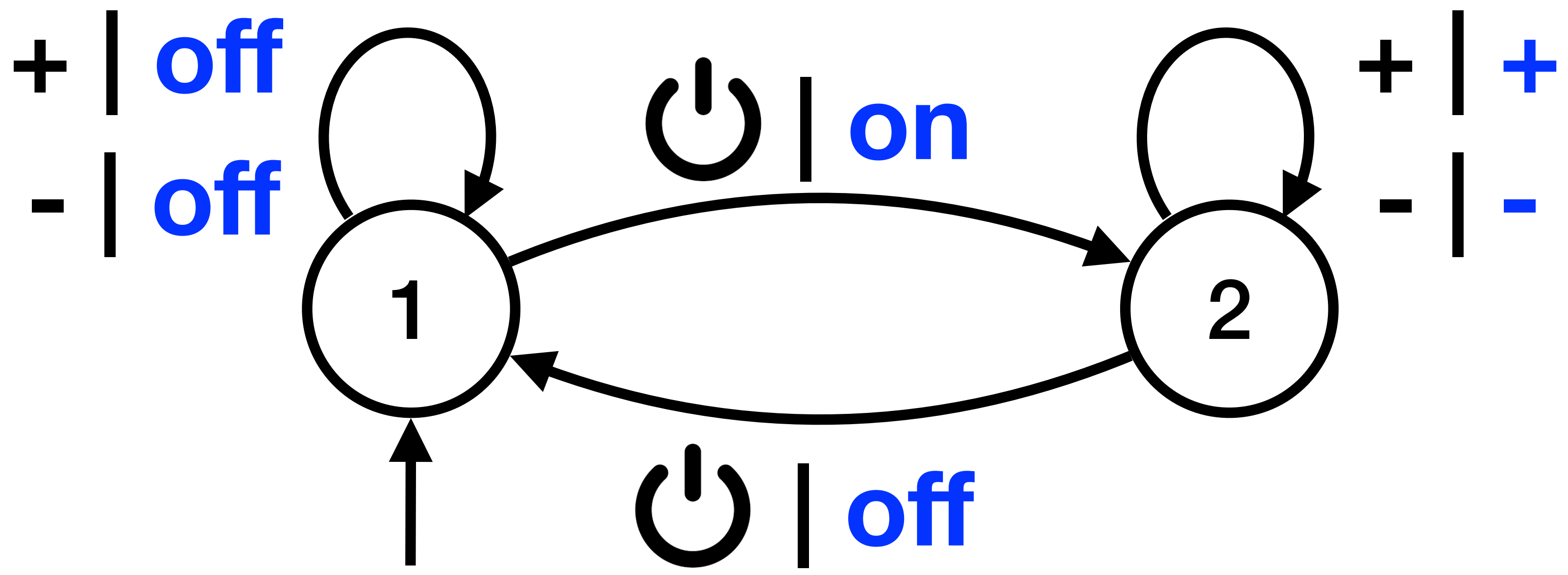
Permutation



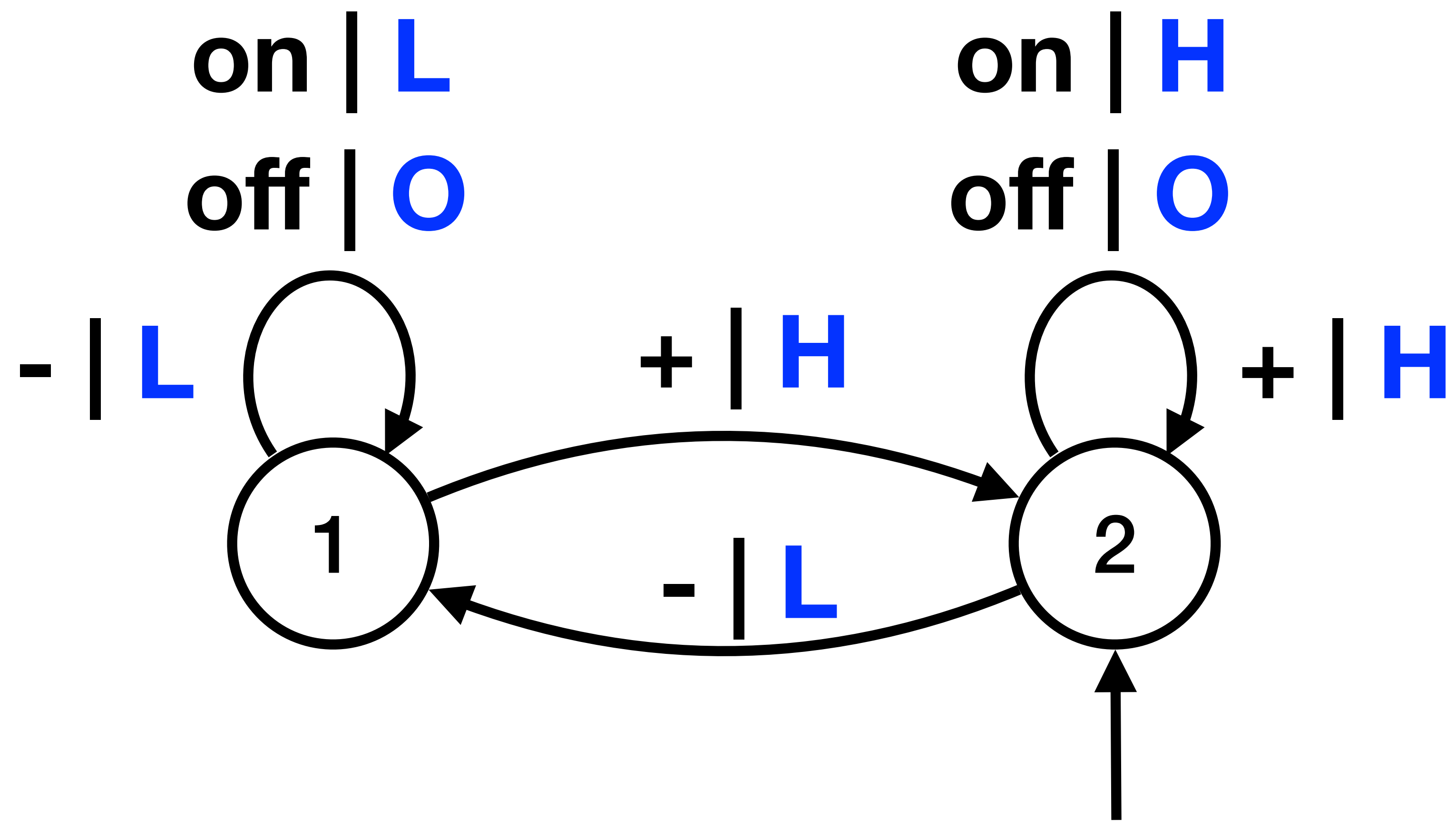
- ⏻ - ⏻ + ⏻ - + ⏻ ⏻ + - - ⏻

O H L O O L L H O H H L L O





Permutation

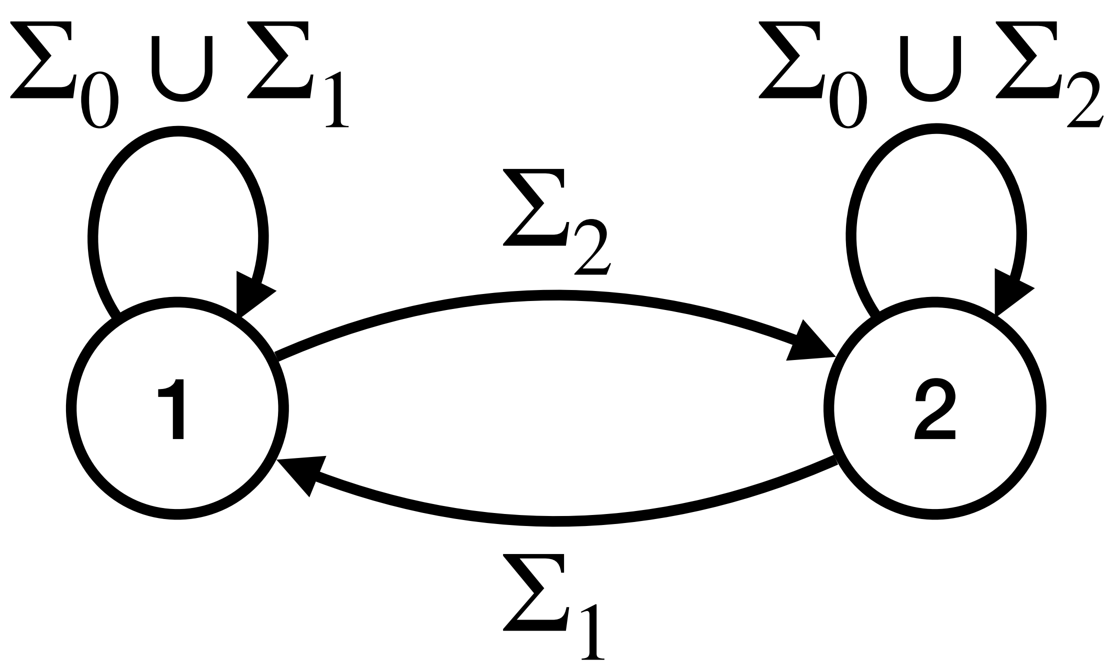


Reset

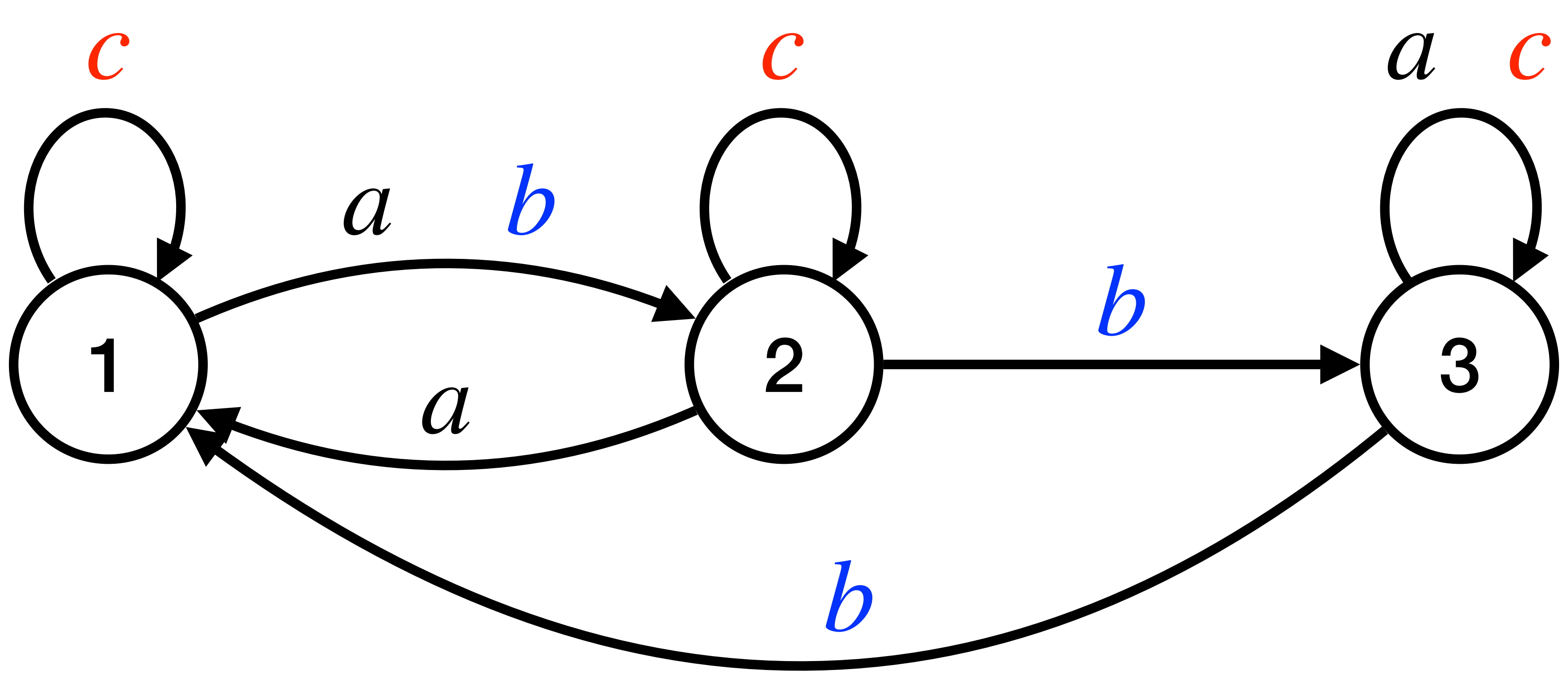
Krohn-Rhodes

Reset or Permutation automata:

Reset - $\mathscr{U}_2$



Permutation



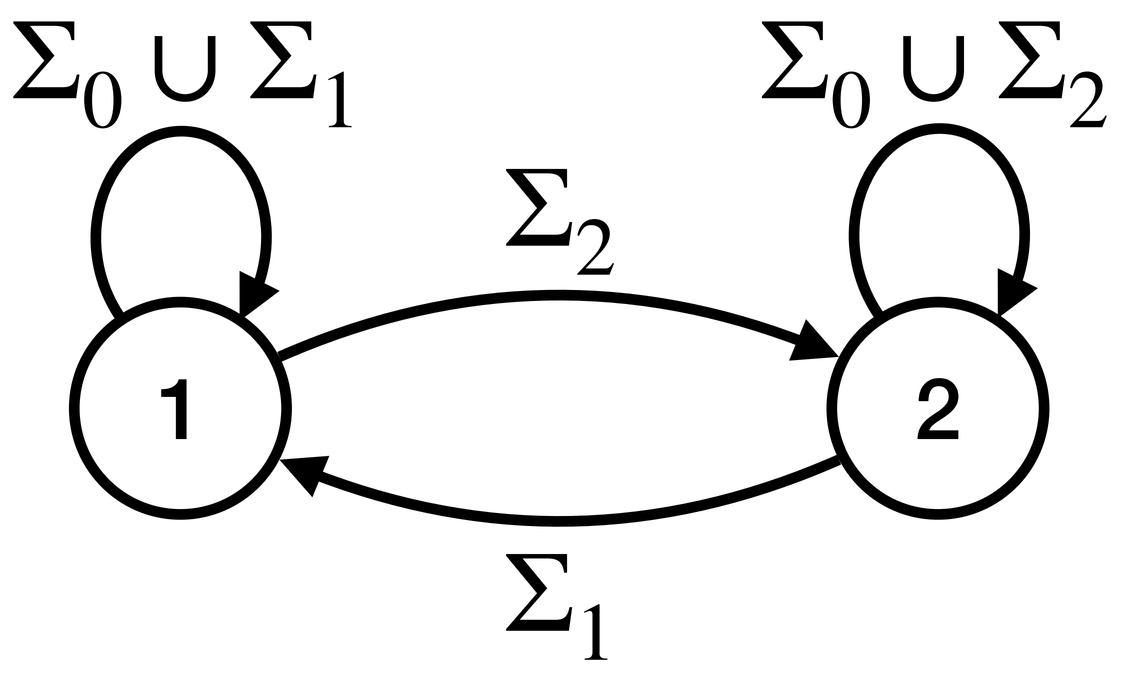
Krohn-Rhodes

Theorem

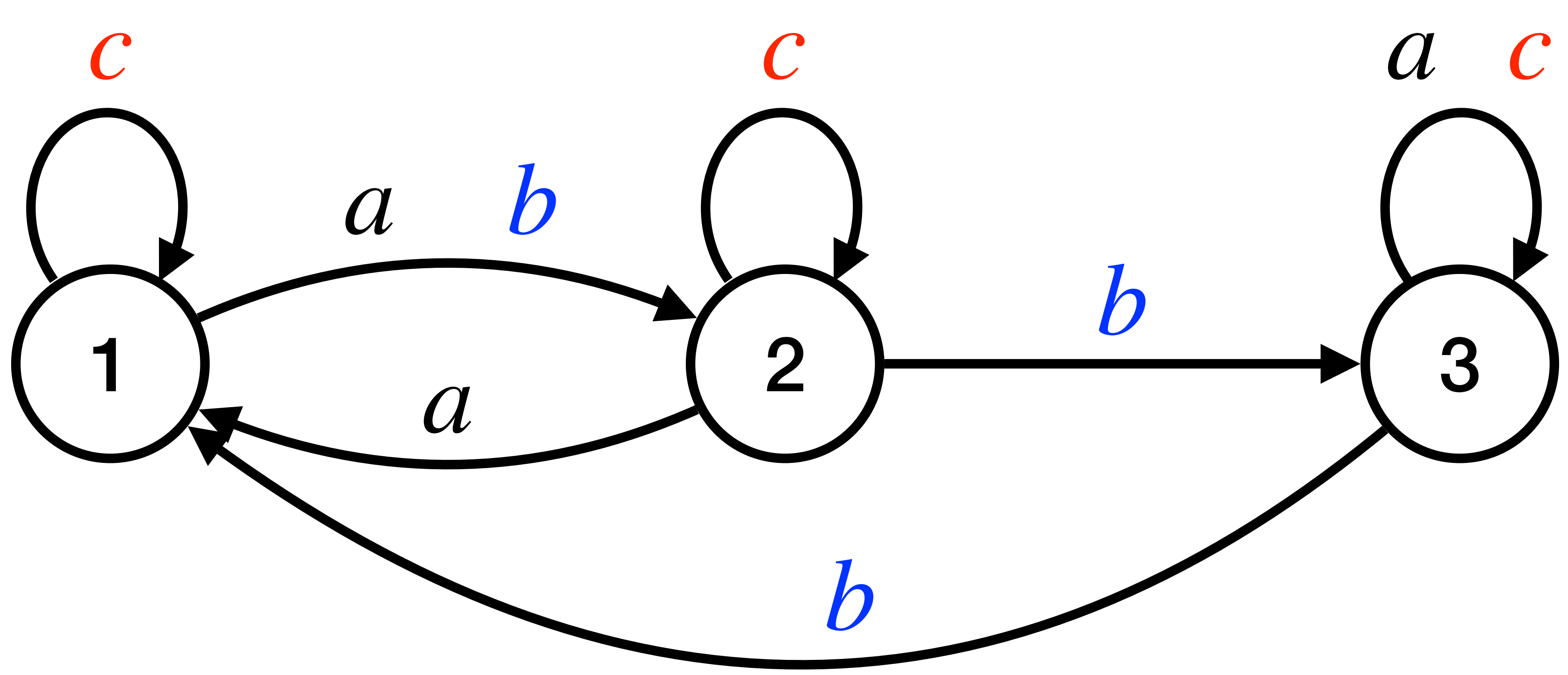
- Any regular language can be accepted by a cascade product of reset or permutation automata.

Reset or Permutation automata:

Reset - $\mathscr{U}_2$



Permutation



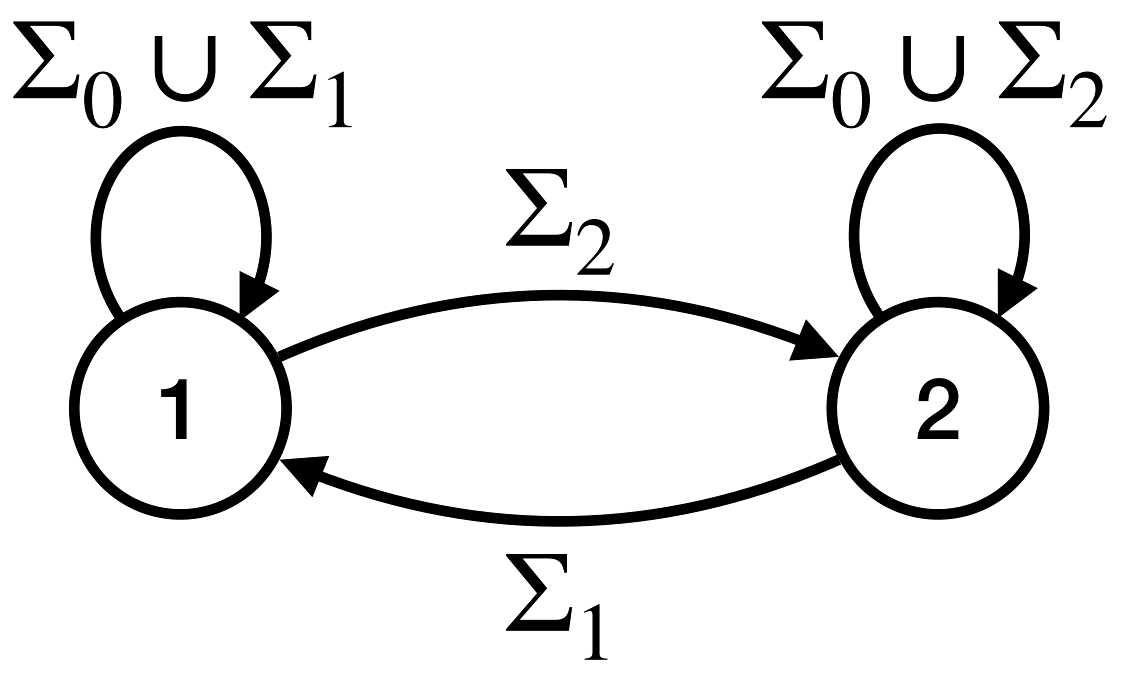
Krohn-Rhodes

Theorem

- Any regular language can be accepted by a cascade product of reset or permutation automata.
- Any aperiodic language can be accepted by a cascade product of reset automata.

Reset or Permutation automata:

Reset - $\mathscr{U}_2$



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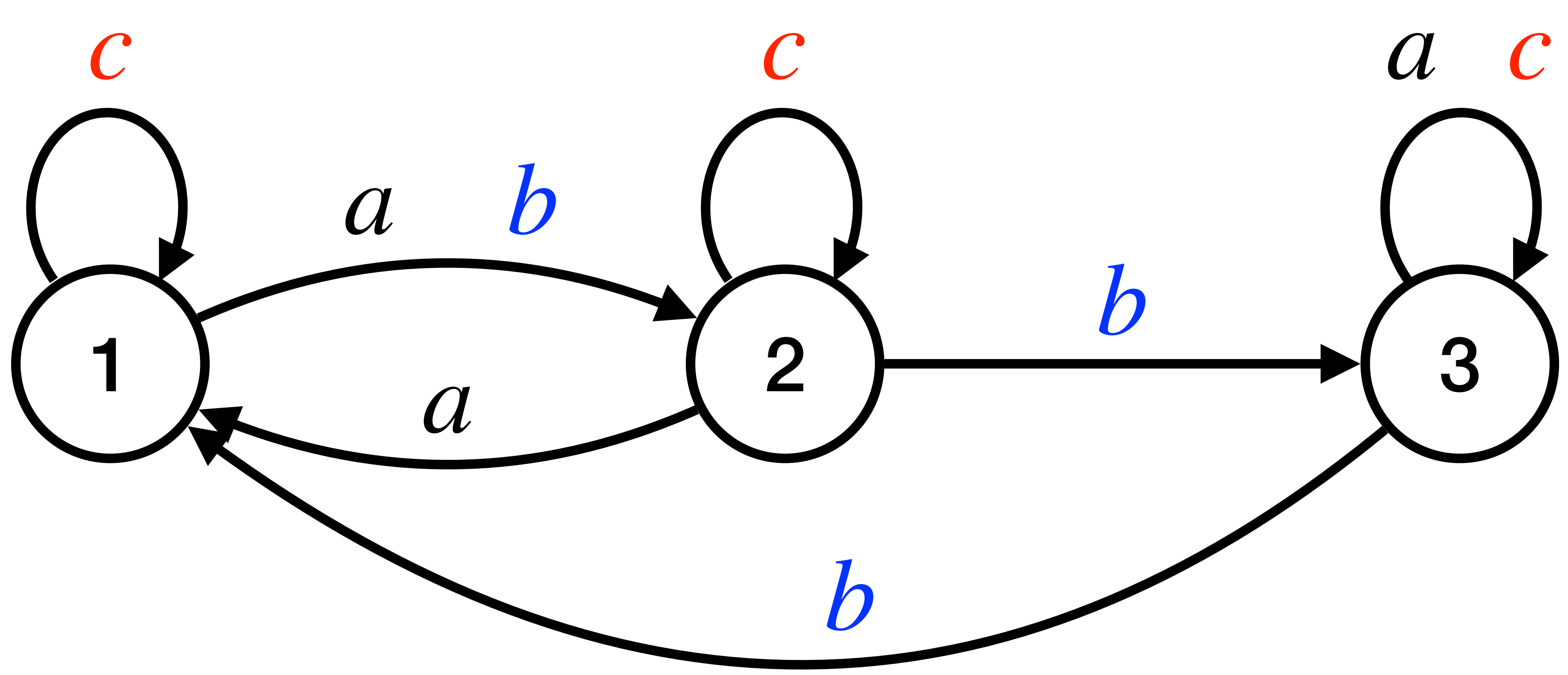
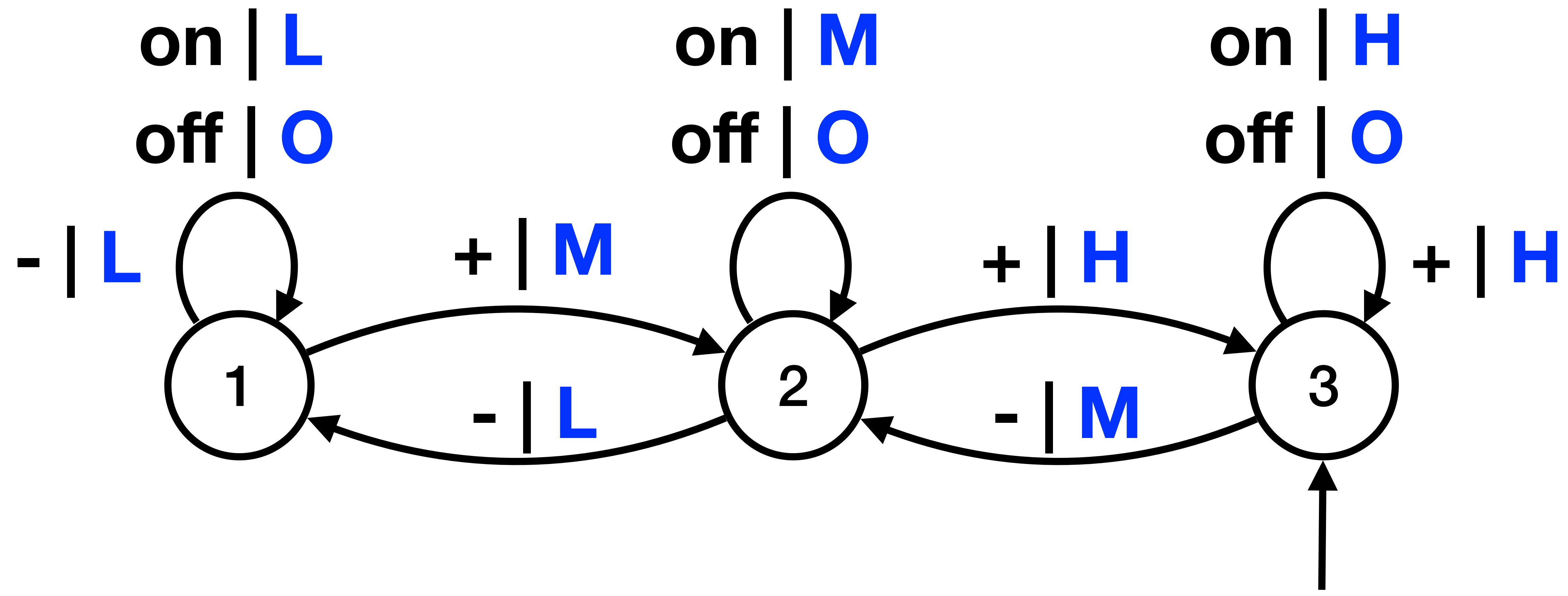


KR proof for aperiodic



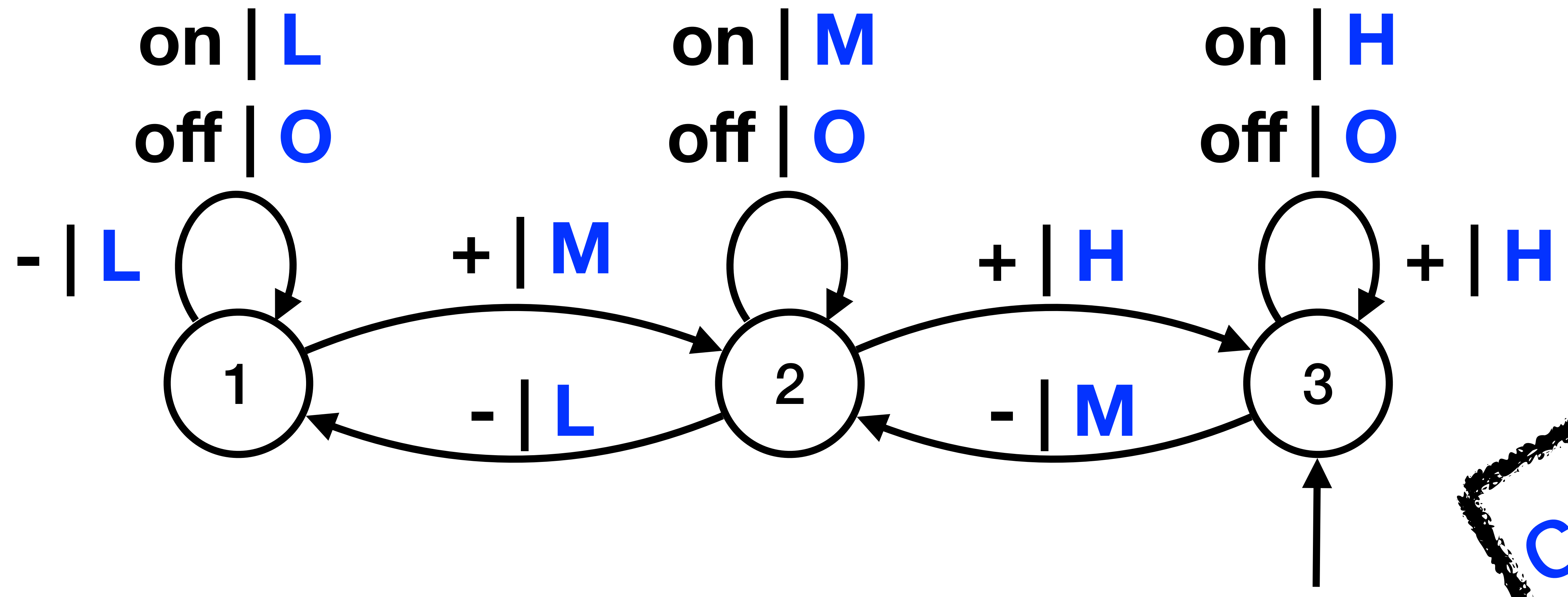
- If we ignore on and off

State_1 = $\{ + , - \}^* - - (+ -)^*$

- With $A = \{\text{on}, \text{off}, + , - \}$ and $B = \{\text{on}, \text{off}\}$

State_1 = $A^* - B^* - B^* (+ B^* - B^*)^*$

KR proof for aperiodic



cascade product of reset transducers

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KR proof for aperiodic

Theorem (Kamp)

Aperiodic = Past Temporal Logic

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State_1 = $\{ + , - \}^* - - (+ -)^*$

KR proof for aperiodic

Theorem (Kamp)

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$$- \wedge \Big((Y -) \; \vee \; (+ \to Y -) \; SS \; (- \wedge Y -) \Big)$$

KR proof for aperiodic

Theorem (Kamp)

Aperiodic = Past Temporal Logic

Each PastLTL formula φ defines a boolean labelling function

$$\theta_\varphi : \Sigma^* \to \{0,1\}$$

Each position is labelled with the truth value of φ at this position.

KR proof for aperiodic

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Example $\varphi = \mathsf{Y}\, a$

a b b b a a b a b b a
0 1 0 0 0 1 1 0 1 0 0

KR proof for aperiodic

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0 1 0 0 0 1 1 0 1 0 0

Example $\varphi = a \;\mathsf{SS}\; b$

a b b a a c c b a a c
0 0 1 1 1 1 0 0 1 1 1

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Given a PastLTL formula φ, we will implement θ_φ with a transducer $\mathcal{T}_\varphi$ constructed inductively as a cascade product of reset transducers.

KR proof for aperiodic

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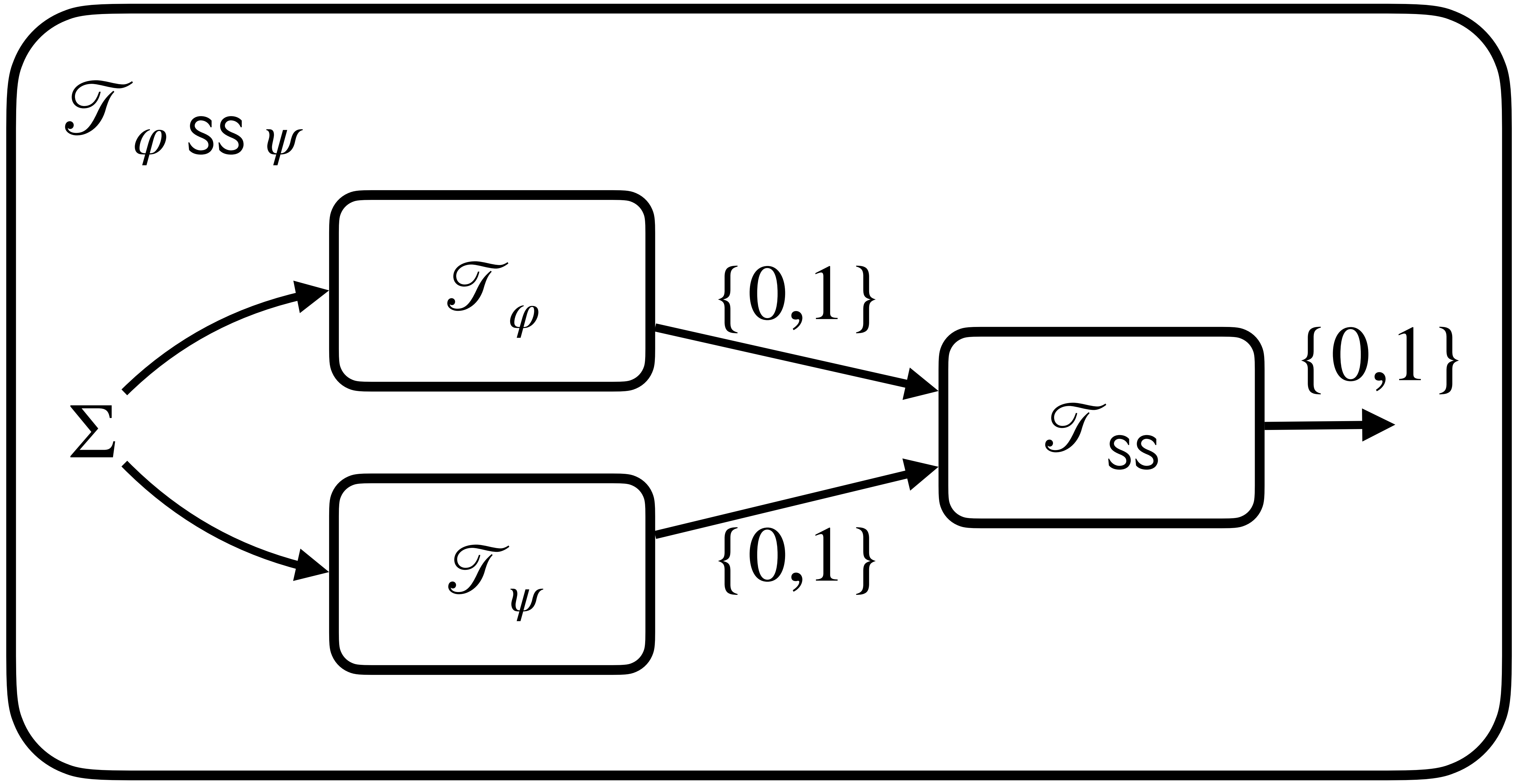
PastLTL: boolean connectives and strict-since

$$\mathsf{Y}\,\psi = \bot\ \mathsf{SS}\ \psi \quad \text{and} \quad \varphi\ \mathsf{S}\ \psi = \psi \vee (\varphi \wedge (\varphi\ \mathsf{SS}\ \psi))$$

KR proof for aperiodic



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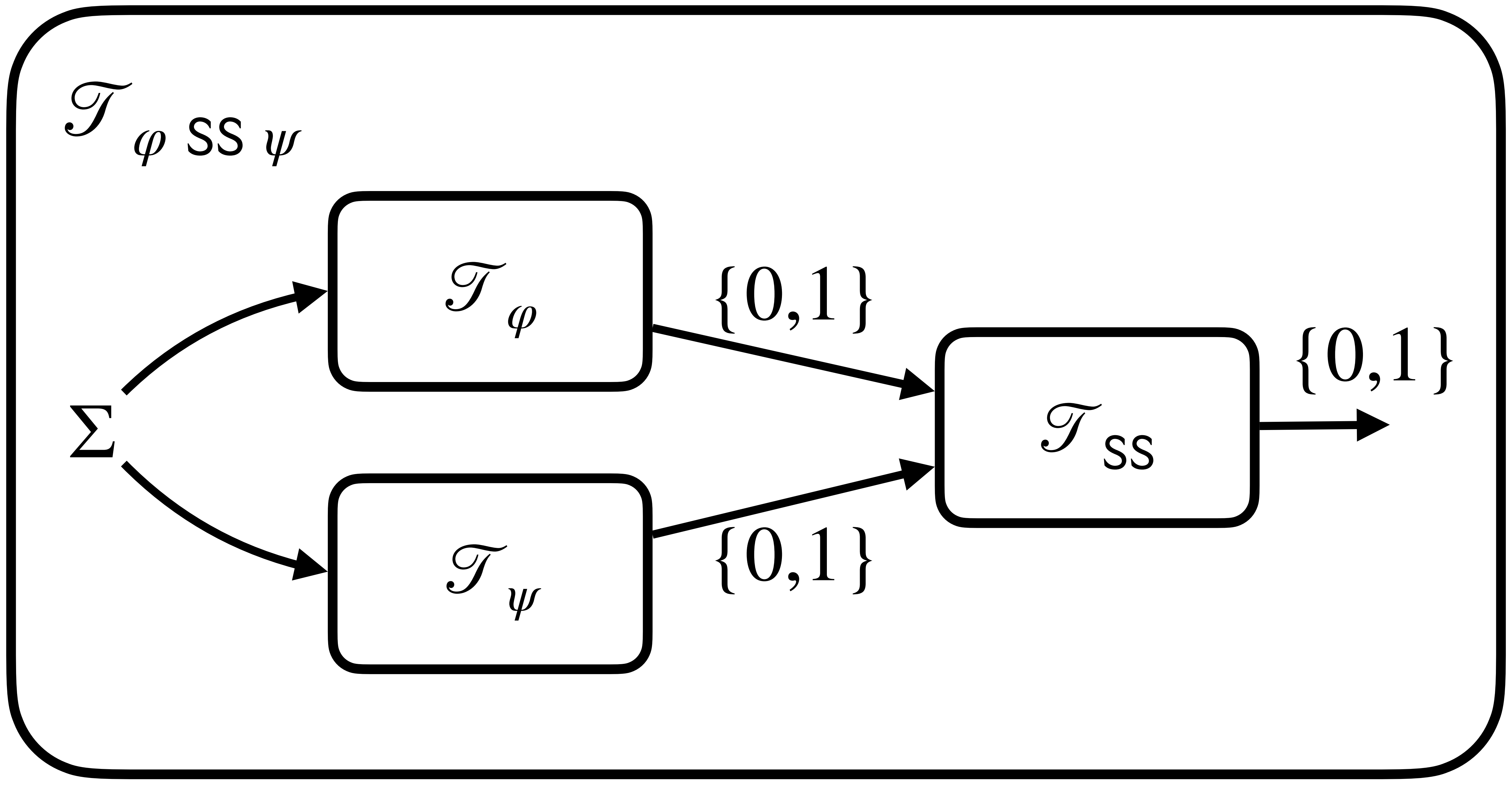

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Example $\varphi = a$ SS b

a b b a a c c b a a c
0 0 1 1 1 1 0 0 1 1 1

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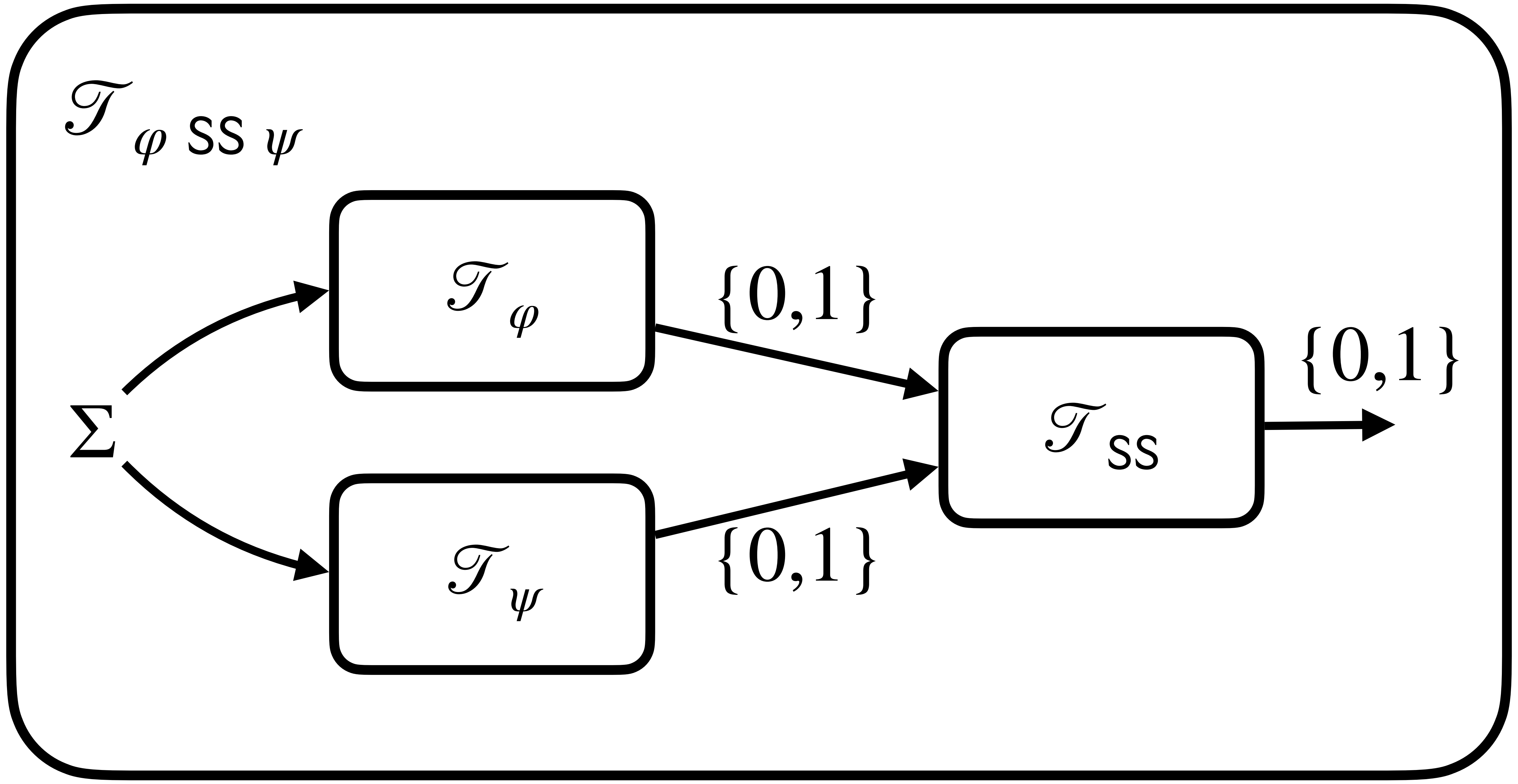

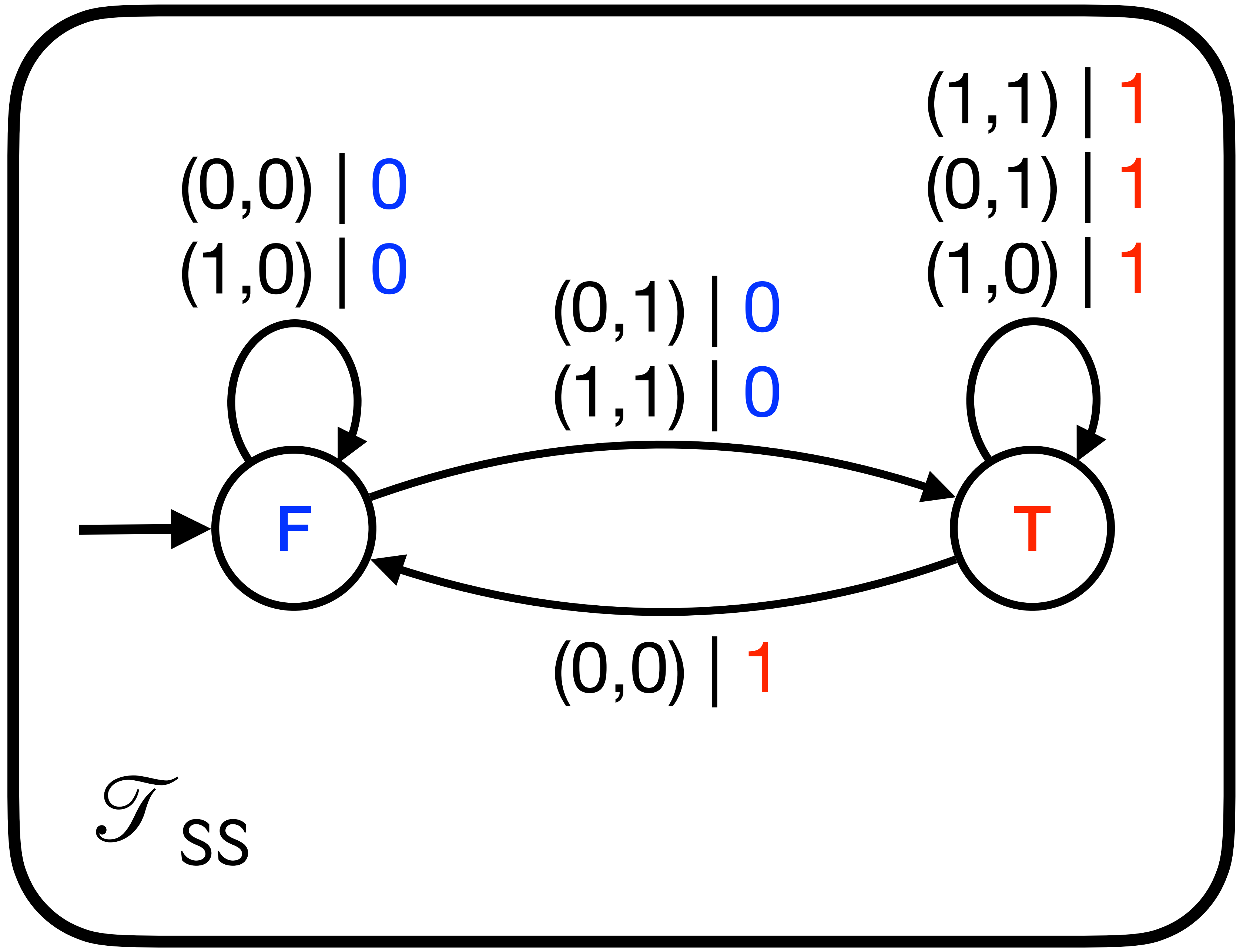

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Example $\varphi = a$ SS b

a	b	b	a	a	c	c	b	a	a	c
0	0	1	1	1	1	0	0	1	1	1

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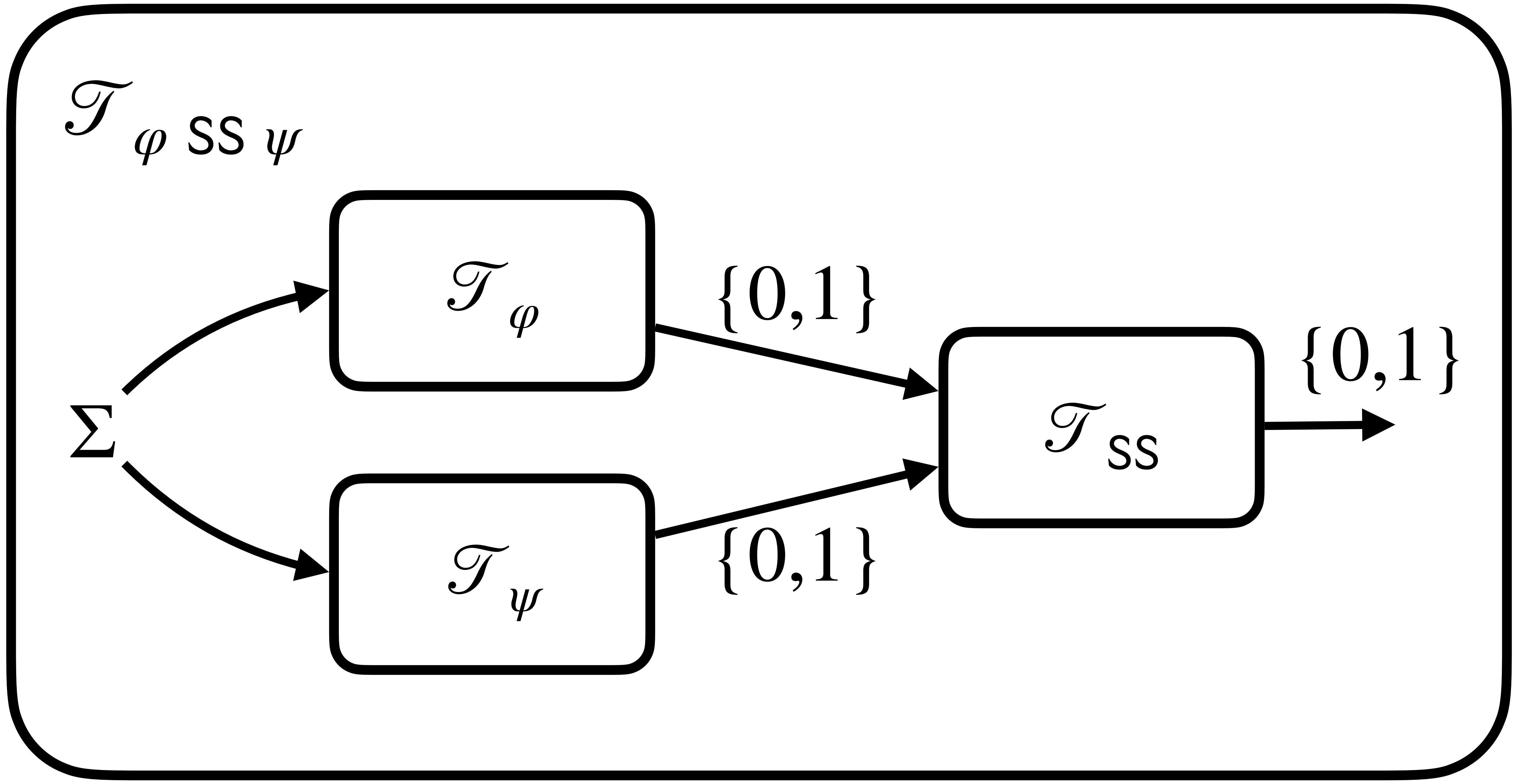

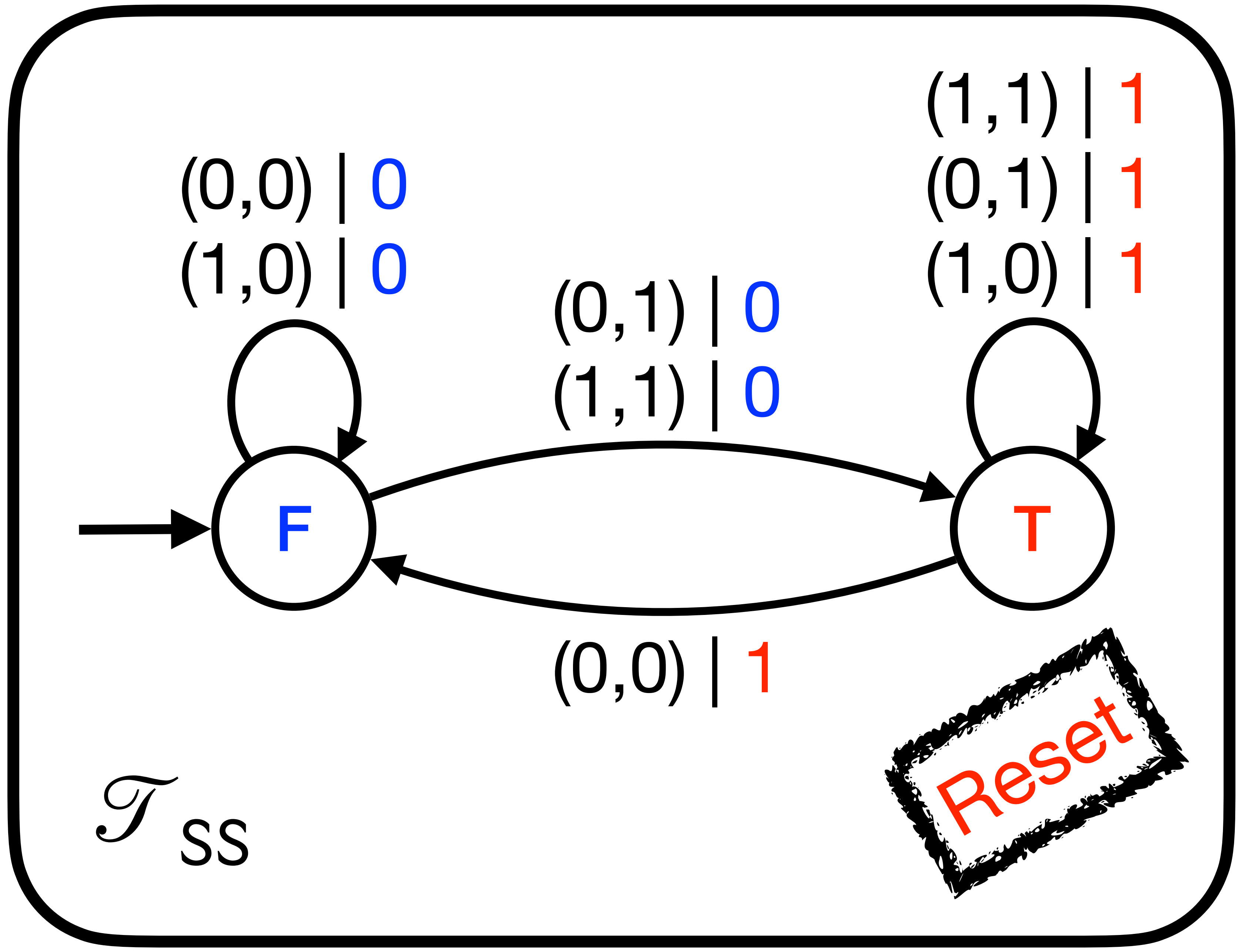

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a b b a a c c b a a c
0 0 1 1 1 1 0 0 1 1 1

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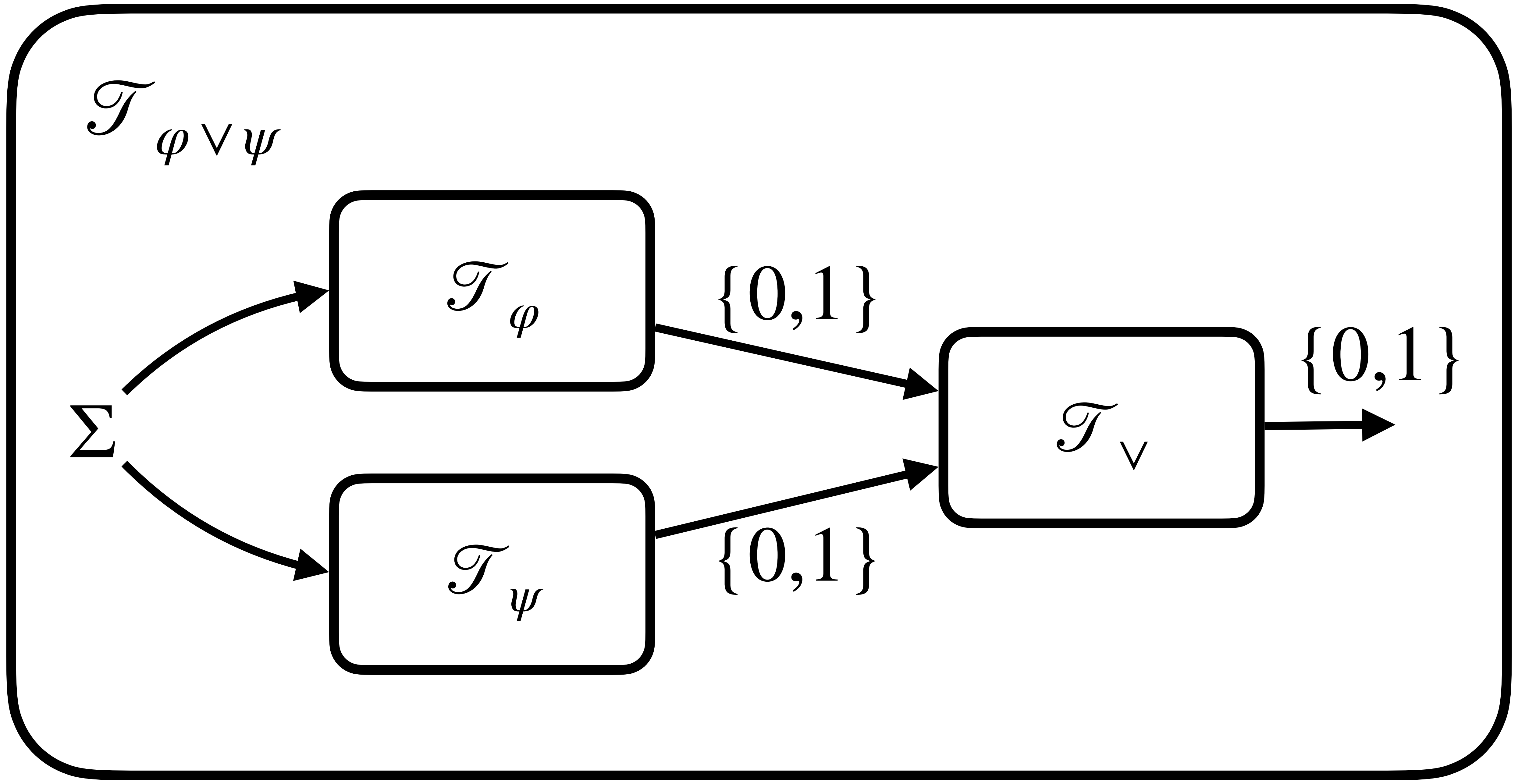

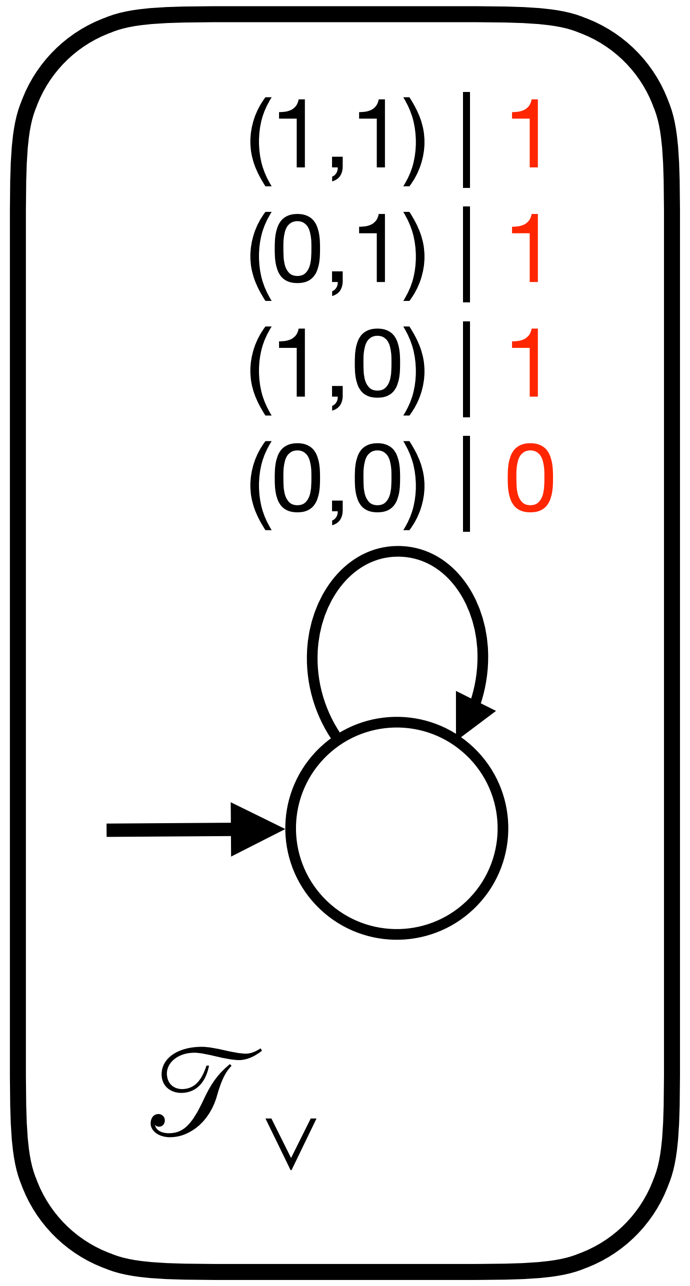

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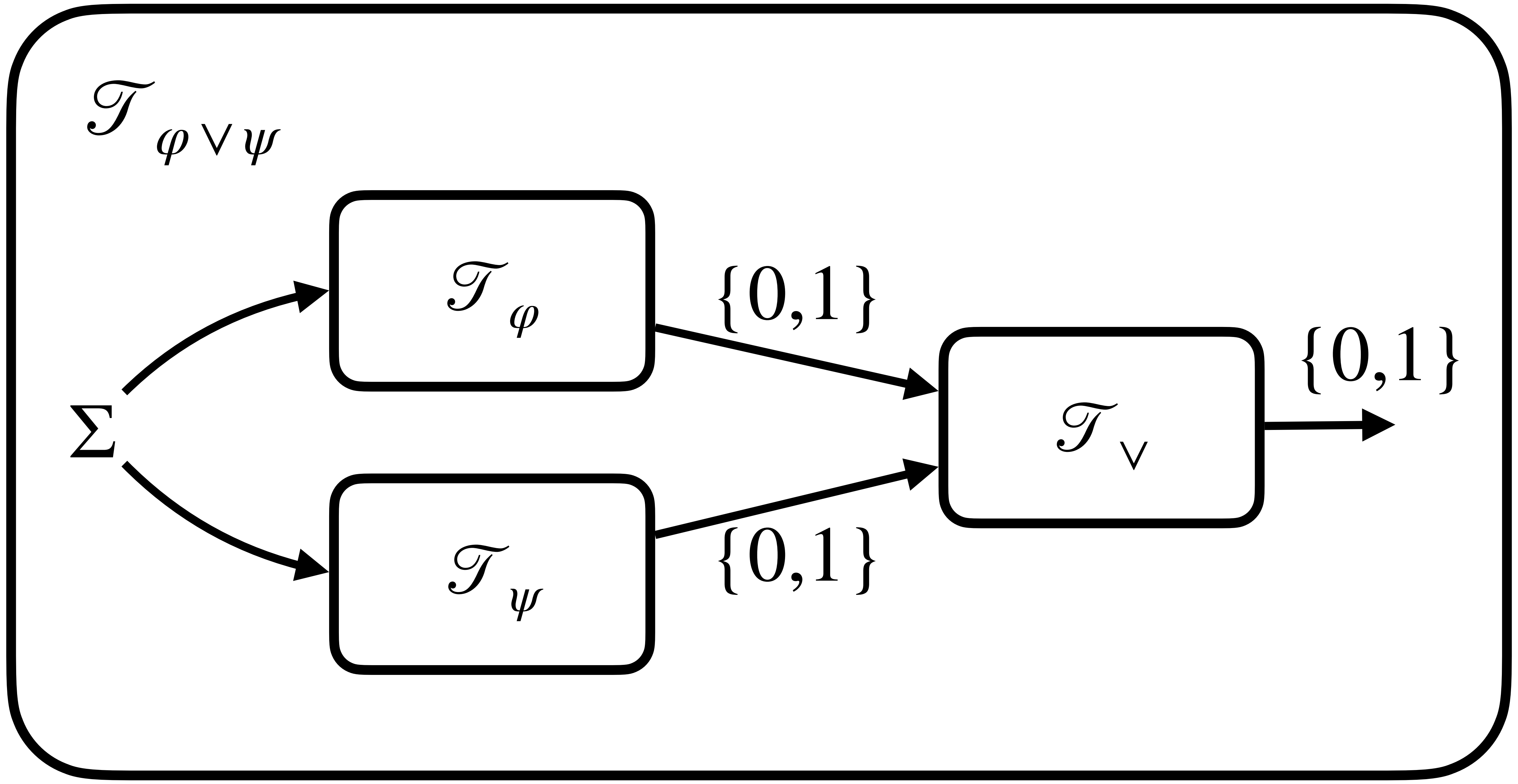
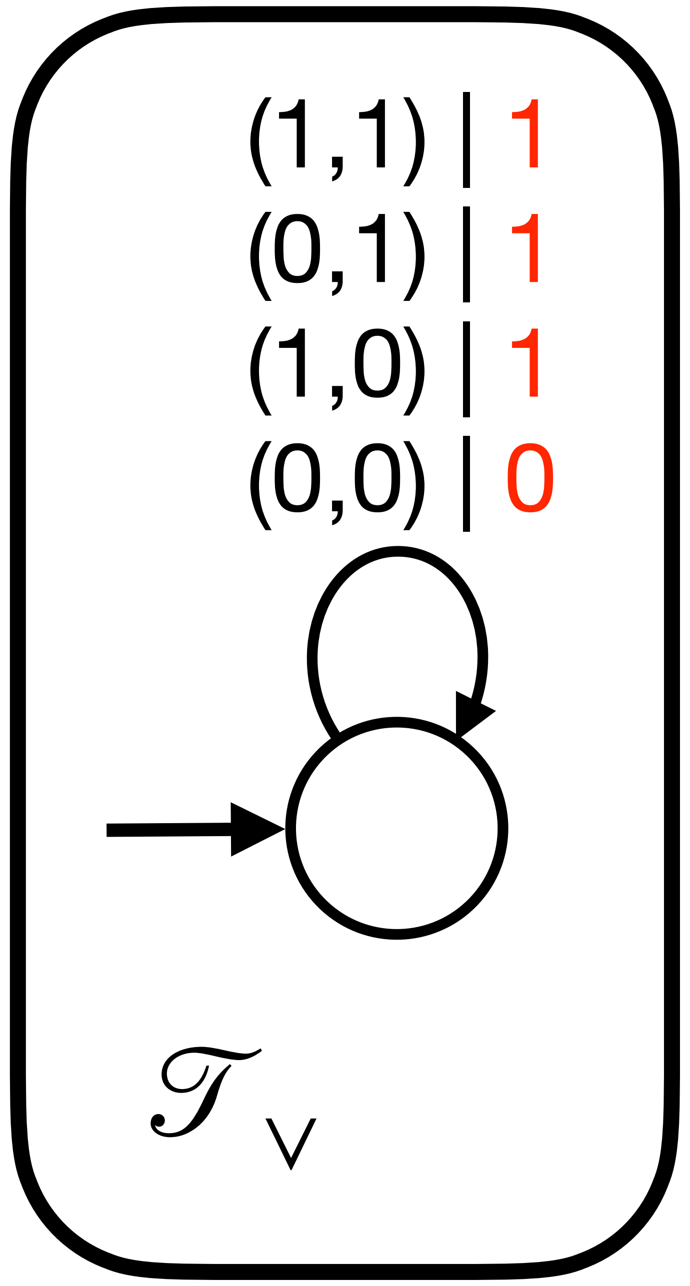
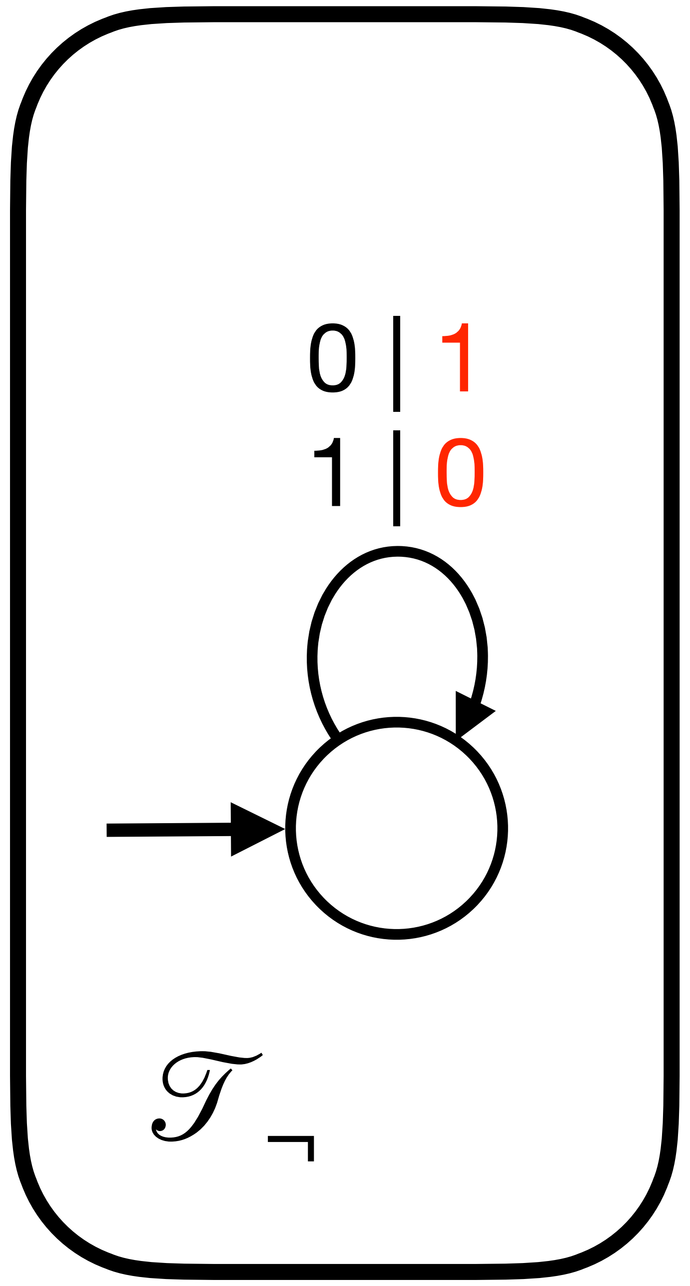
PastLTL: boolean connectives and strict-since

KR proof for aperiodic



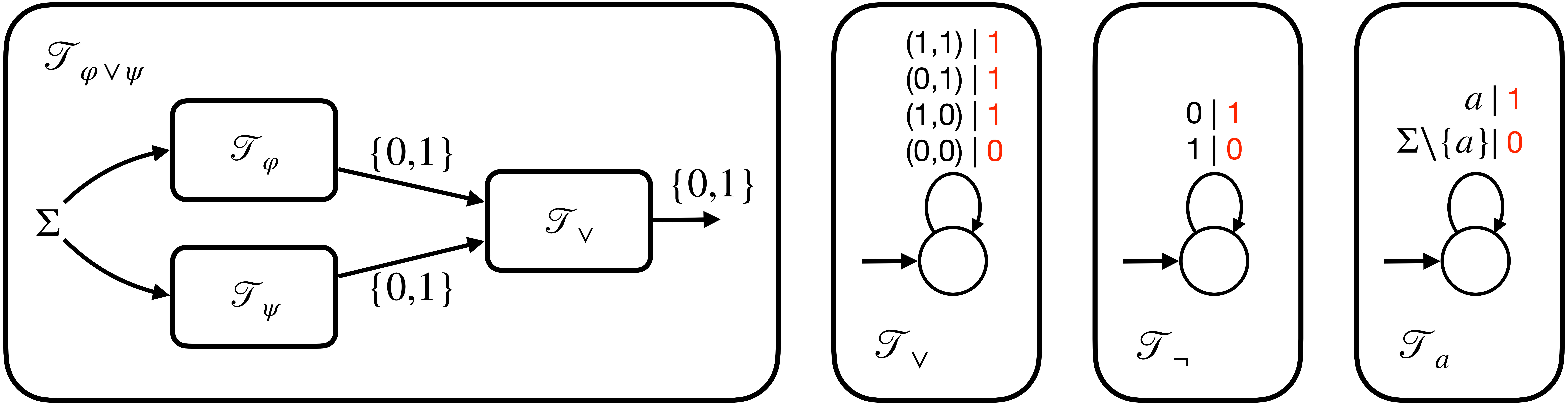




Given a PastLTL formula φ, we will implement θ_φ with a transducer $\mathscr{T}_\varphi$ constructed inductively as a cascade product of reset transducers.

PastLTL: boolean connectives and strict-since

KR proof for aperiodic



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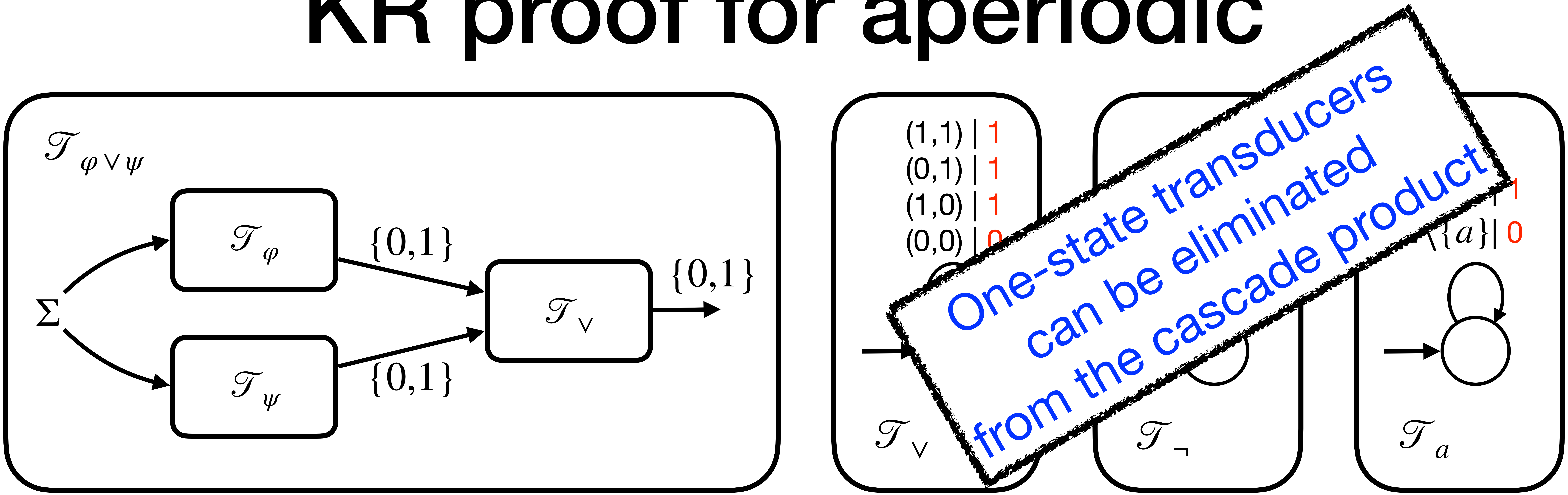

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PastLTL: boolean connectives and strict-since

Outline

- ✅ Labelling functions, sequential transducers and cascade product
- ✅ Krohn-Rhodes theorem for aperiodic/regular word languages
- Model of concurrency: Mazurkiewicz traces and asynchronous Zielonka automata
- Asynchronous labelling functions, transducers and cascade product
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Mazurkiewicz Traces

Architecture

$\mathscr{P} = \{1,2,3\}$

$\Sigma = \{a, b, c, d, e\}$

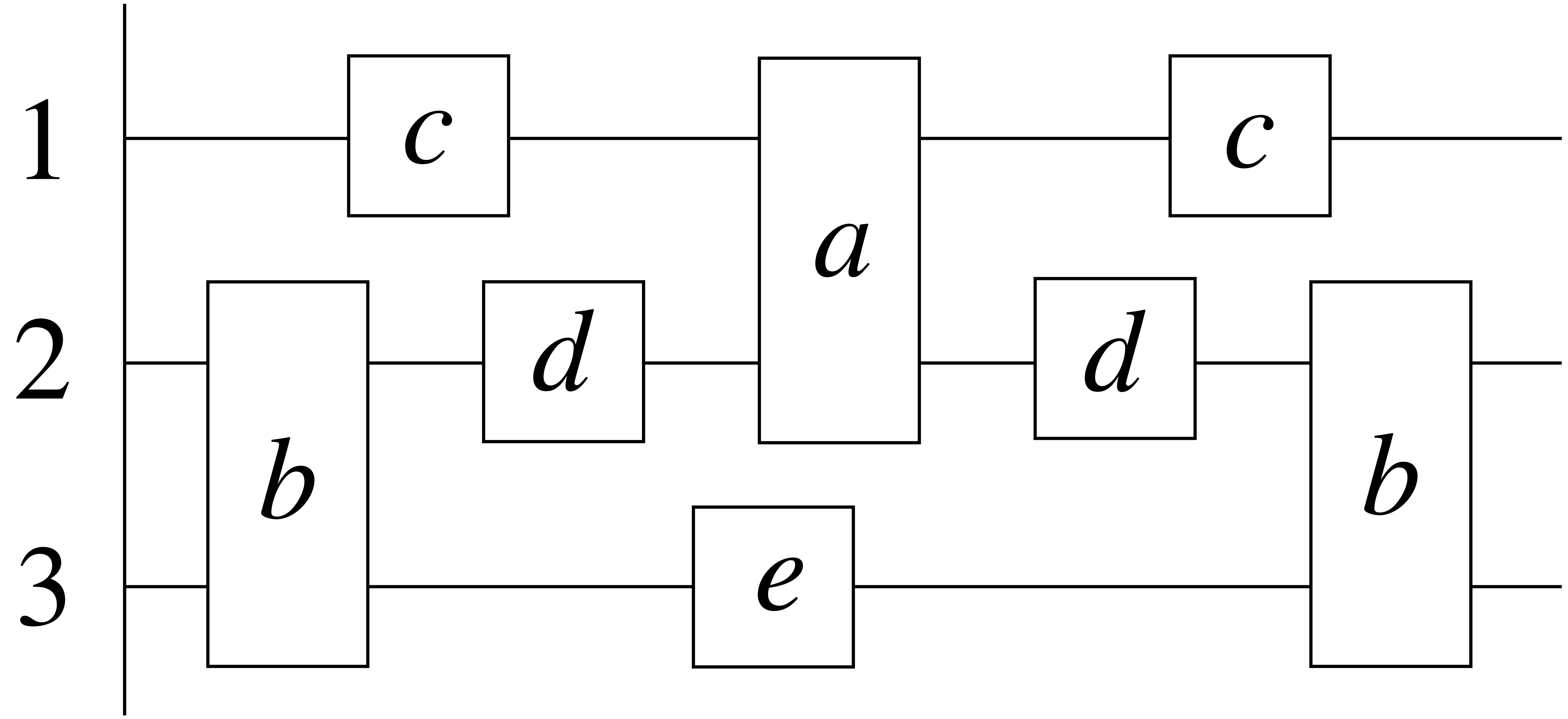
$\text{loc}(a) = \{1,2\}$

$\text{loc}(b) = \{2,3\}$

$\text{loc}(c) = \{1\}$

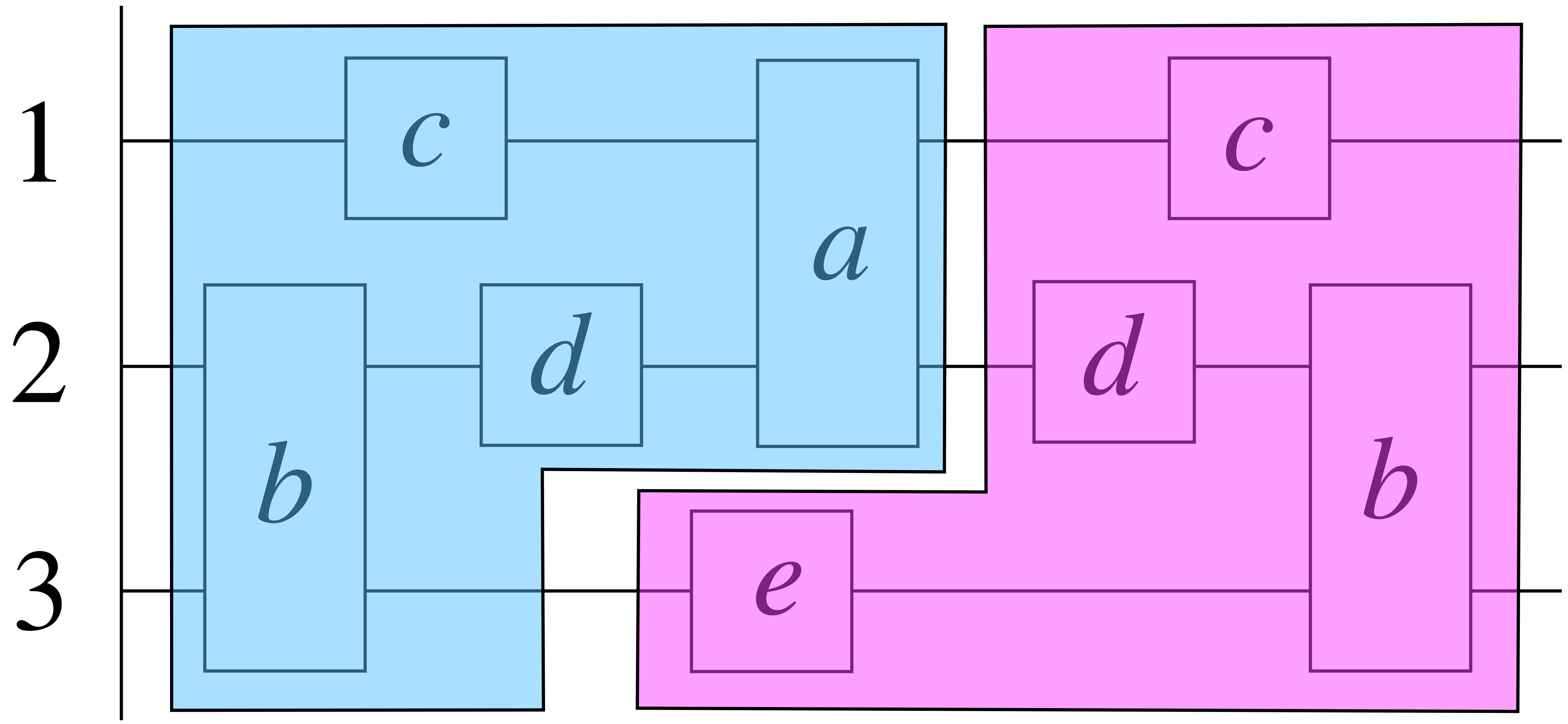
$\text{loc}(d) = \{2\}$

$\text{loc}(e) = \{3\}$



- *set of traces denoted* $Tr(\Sigma, \mathscr{P}, \text{loc})$ *or simply* $Tr(\Sigma)$
- *Trace Language:* $L \subseteq Tr(\Sigma)$

Concatenation, Independence, Commutation, Monoid



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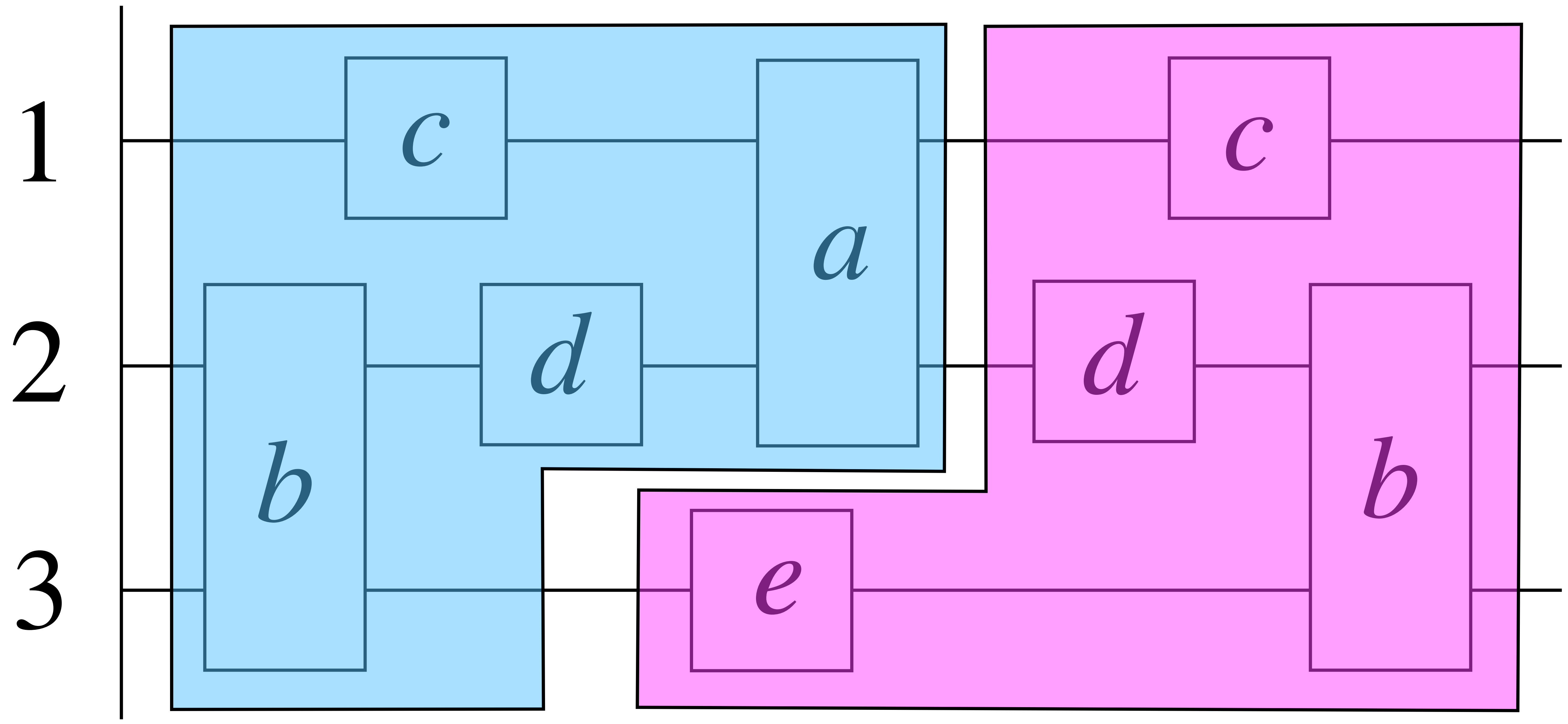

- $Tr(\Sigma)$ *with trace concatenation is a* *monoid*

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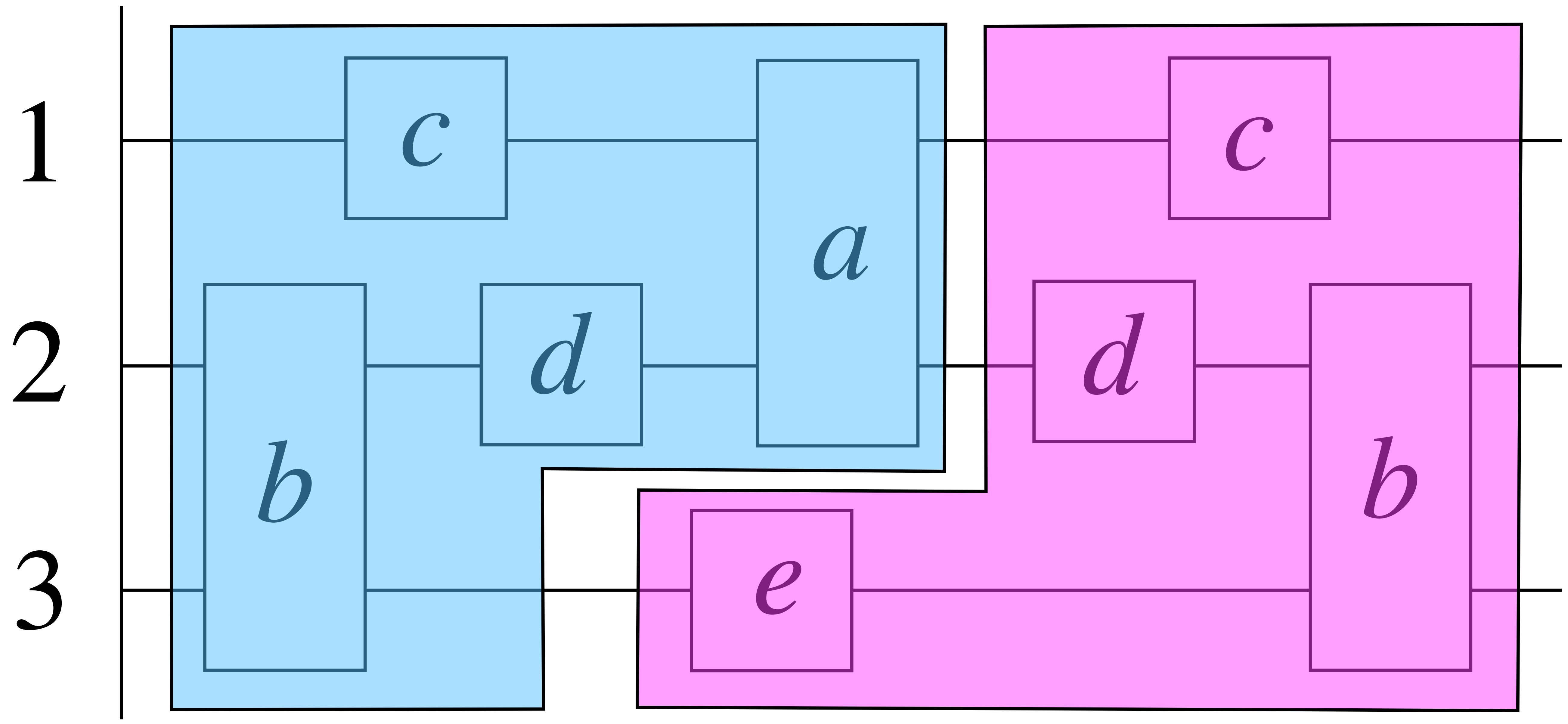

- $x \; I \; y$ iff $\mathrm{loc}(x) \cap \mathrm{loc}(y) = \varnothing$
- $a \; I \; e$
- $a \cdot e = e \cdot a$

- *$Tr(\Sigma)$ with trace concatenation is a monoid*
- *Free partially commutative monoid*

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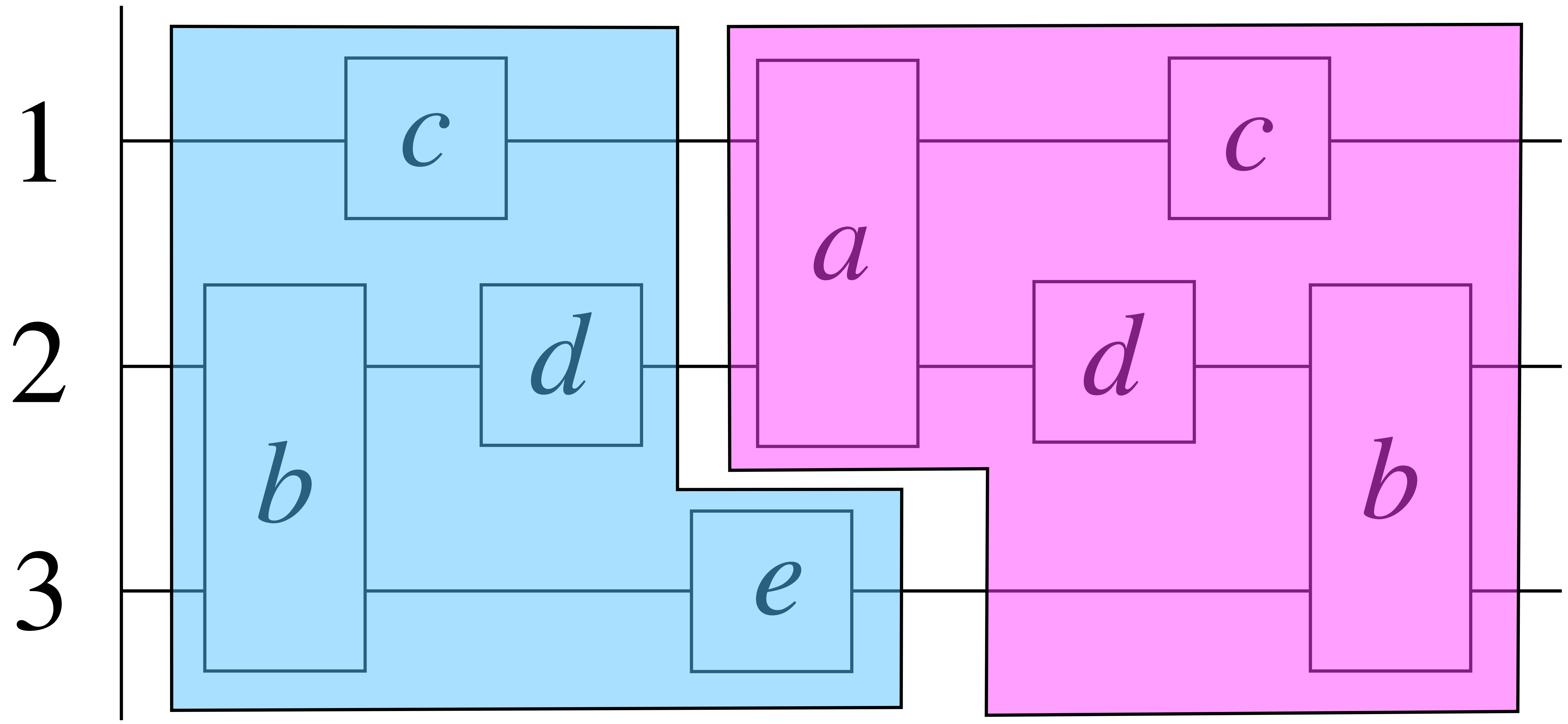

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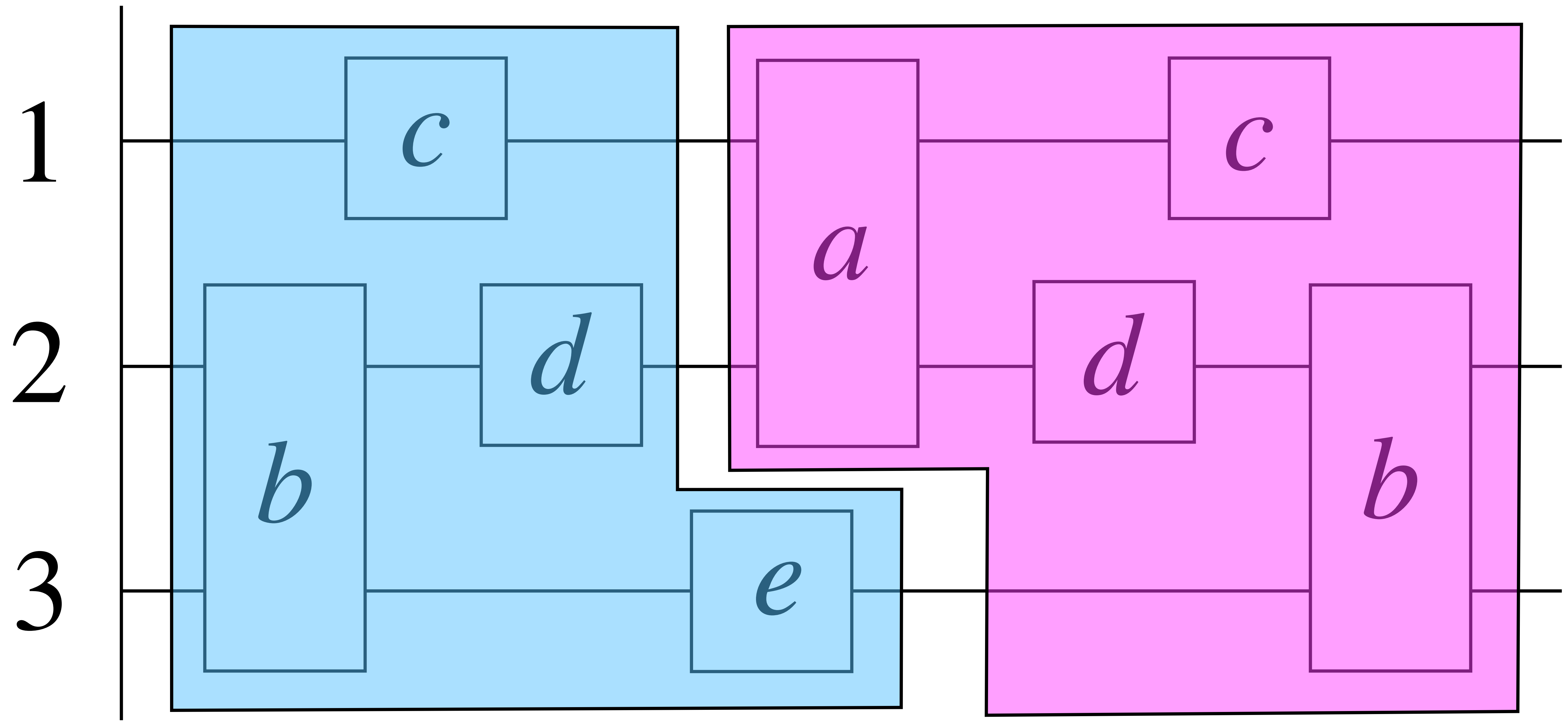

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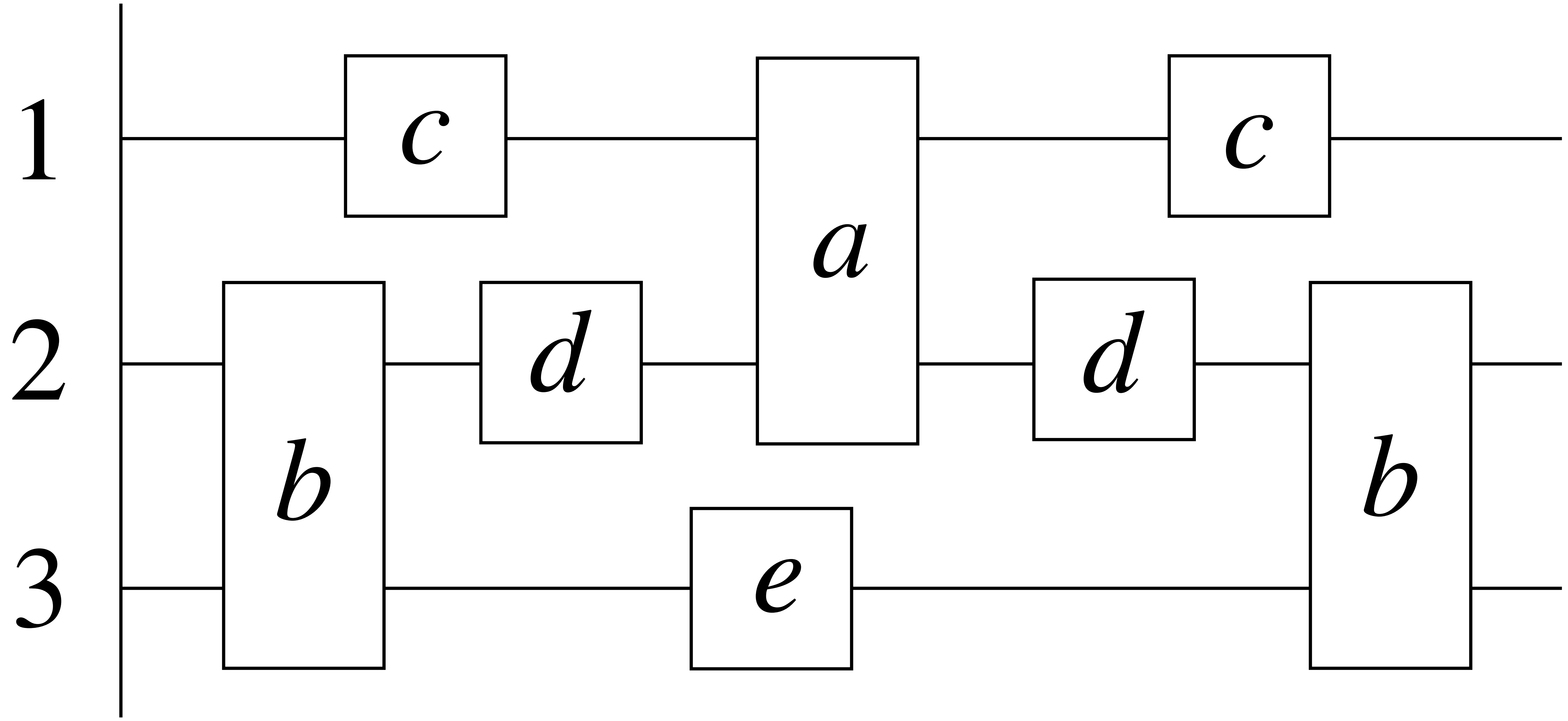
- $Tr(\Sigma)$ *with trace concatenation is a monoid*
- *Free partially commutative monoid*
- $L \subseteq Tr(\Sigma)$ *is regular if* $L = \eta^{-1}(\eta(L))$ *for some morphism* $\eta : Tr(\Sigma) \to M$ *(finite monoid)*

Asynchronous automata (Zielonka)



$\mathscr{A} = \left(\{S_i\}_{i\in\mathscr{P}}, \{\delta_a\}_{a\in\Sigma}, s^{in}, F\right)$

- S_i local states for process i
- δ_a transition function for action a
- $s^{in} = (s_1^{in}, s_2^{in}, s_3^{in})$ global initial state
- F global accepting states

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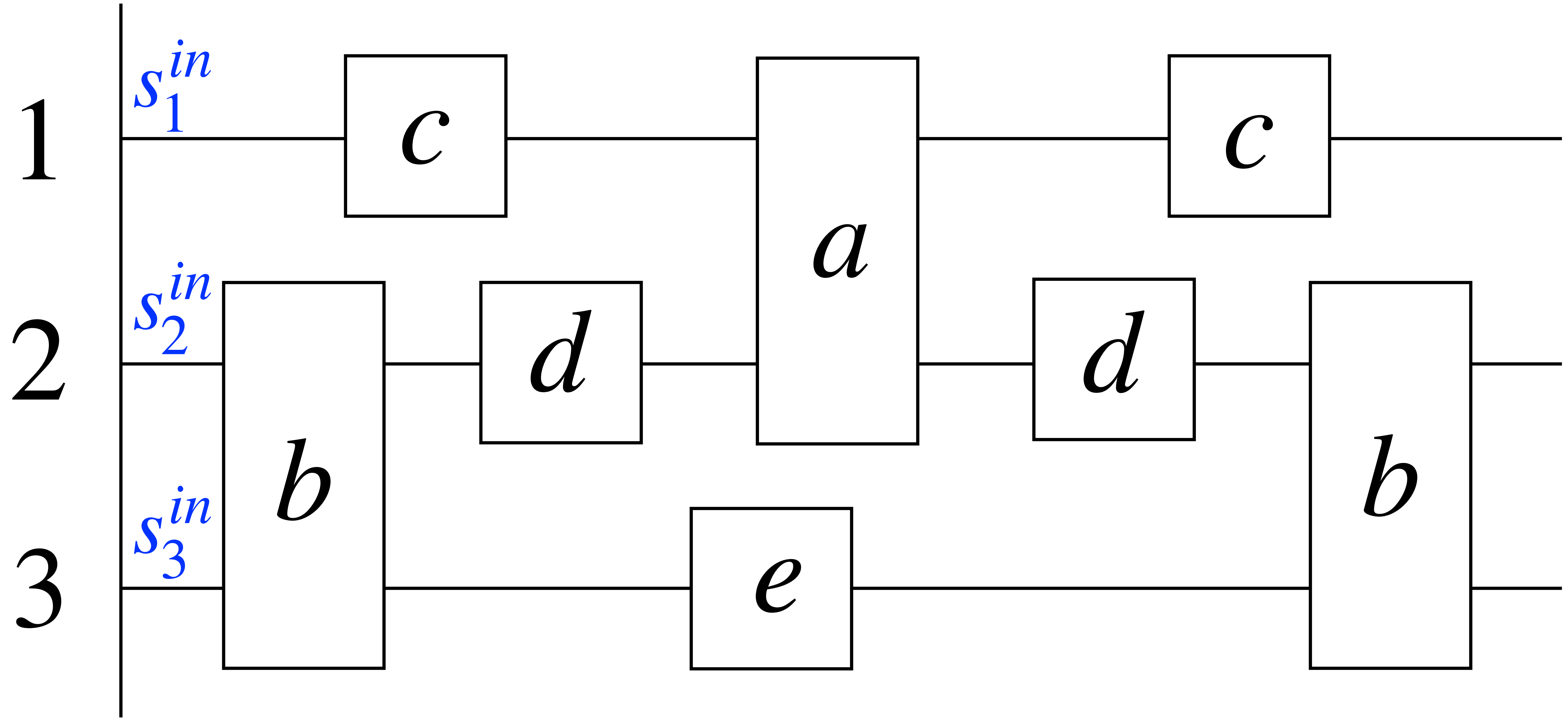

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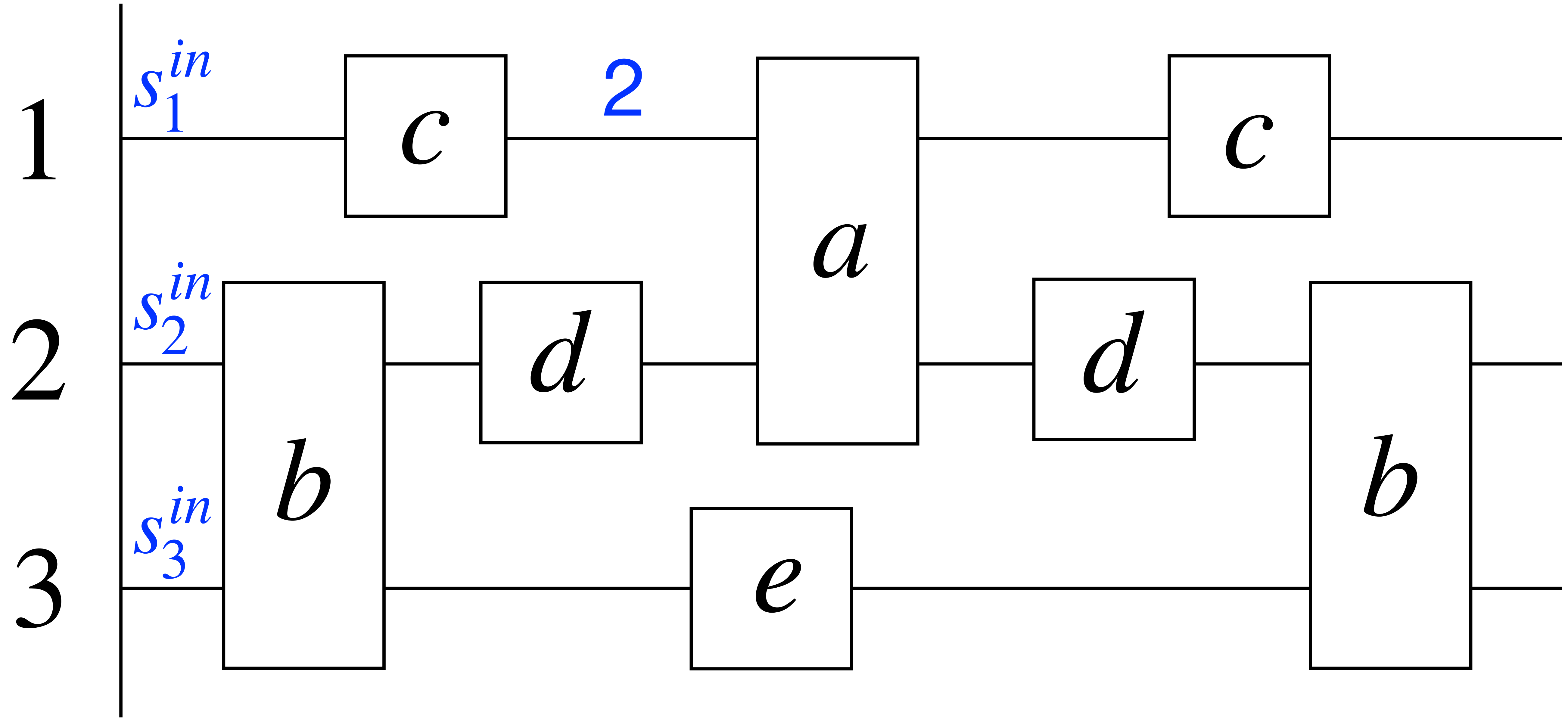

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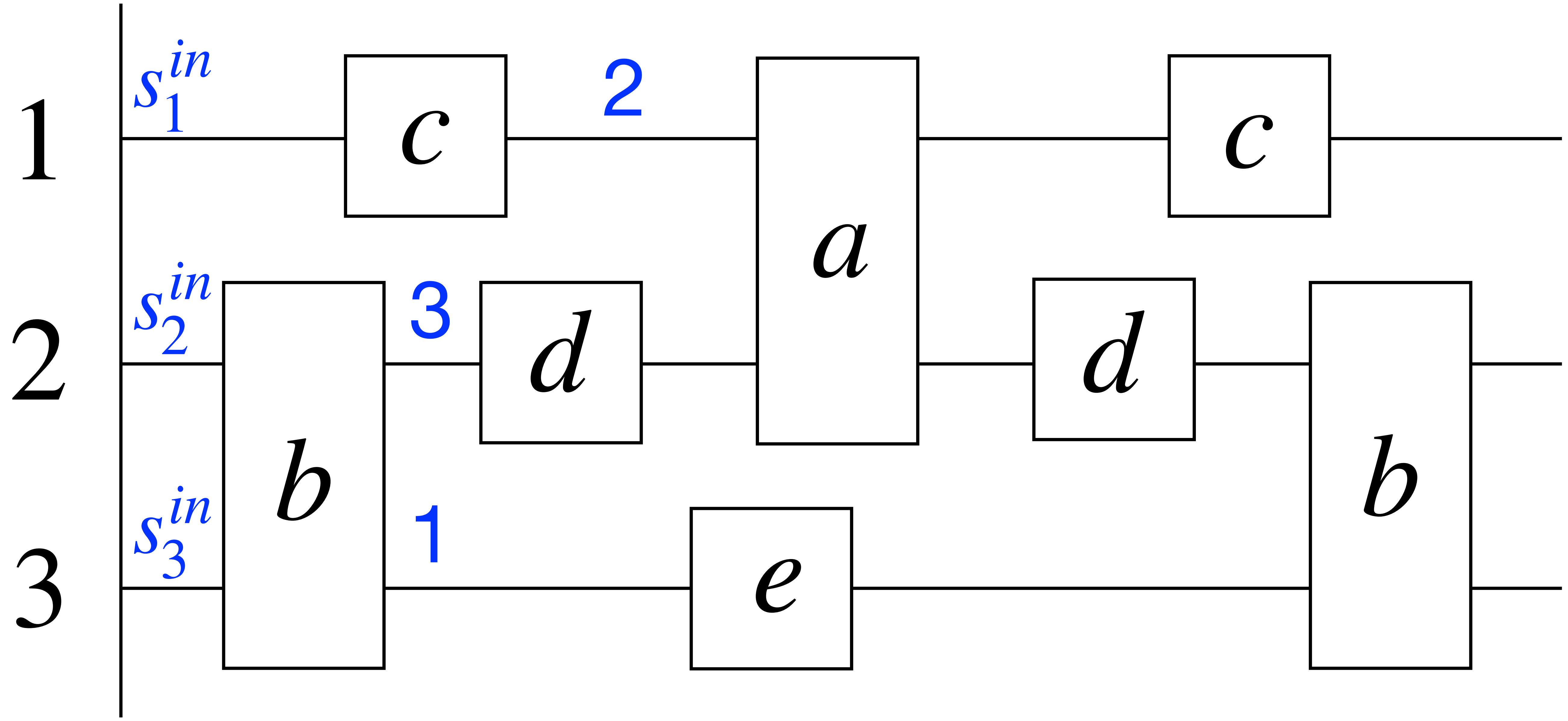

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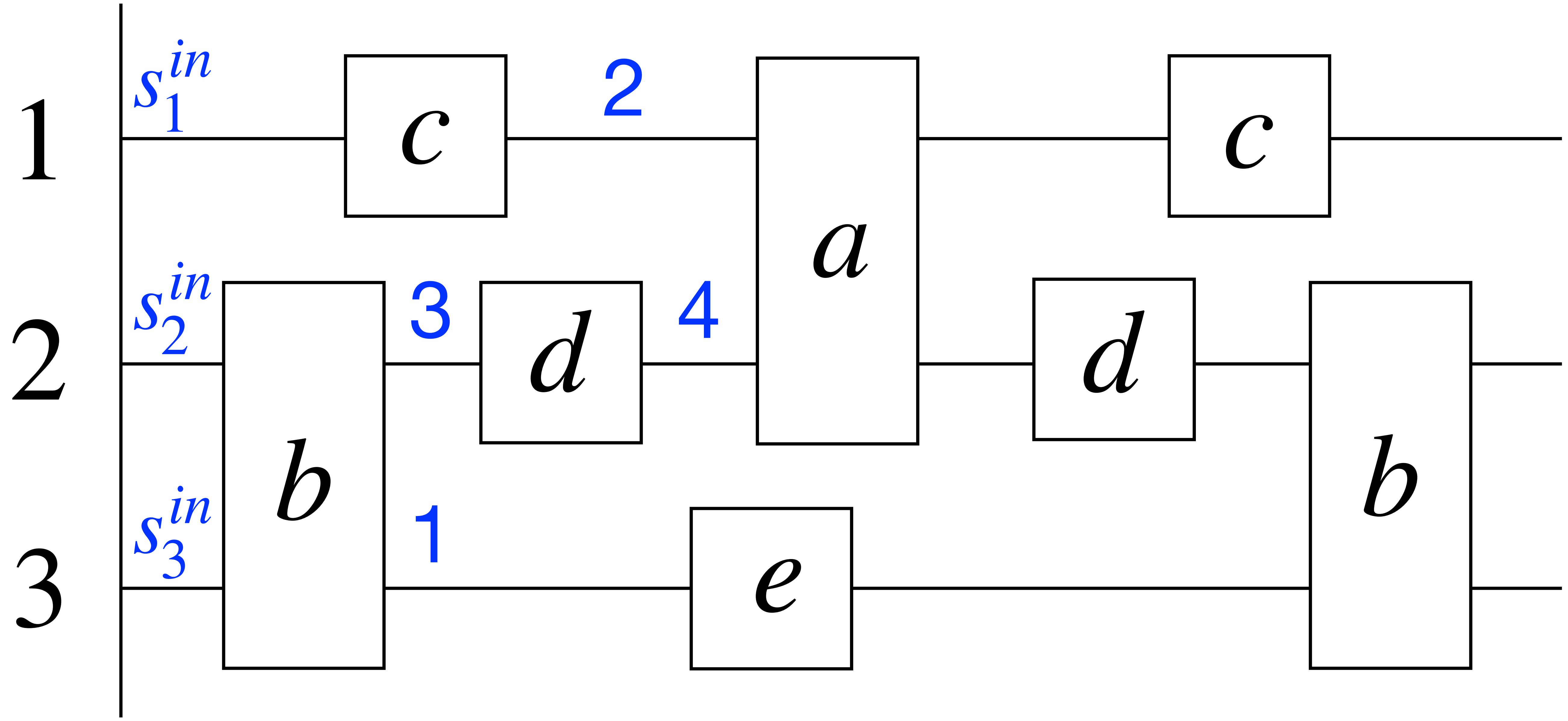

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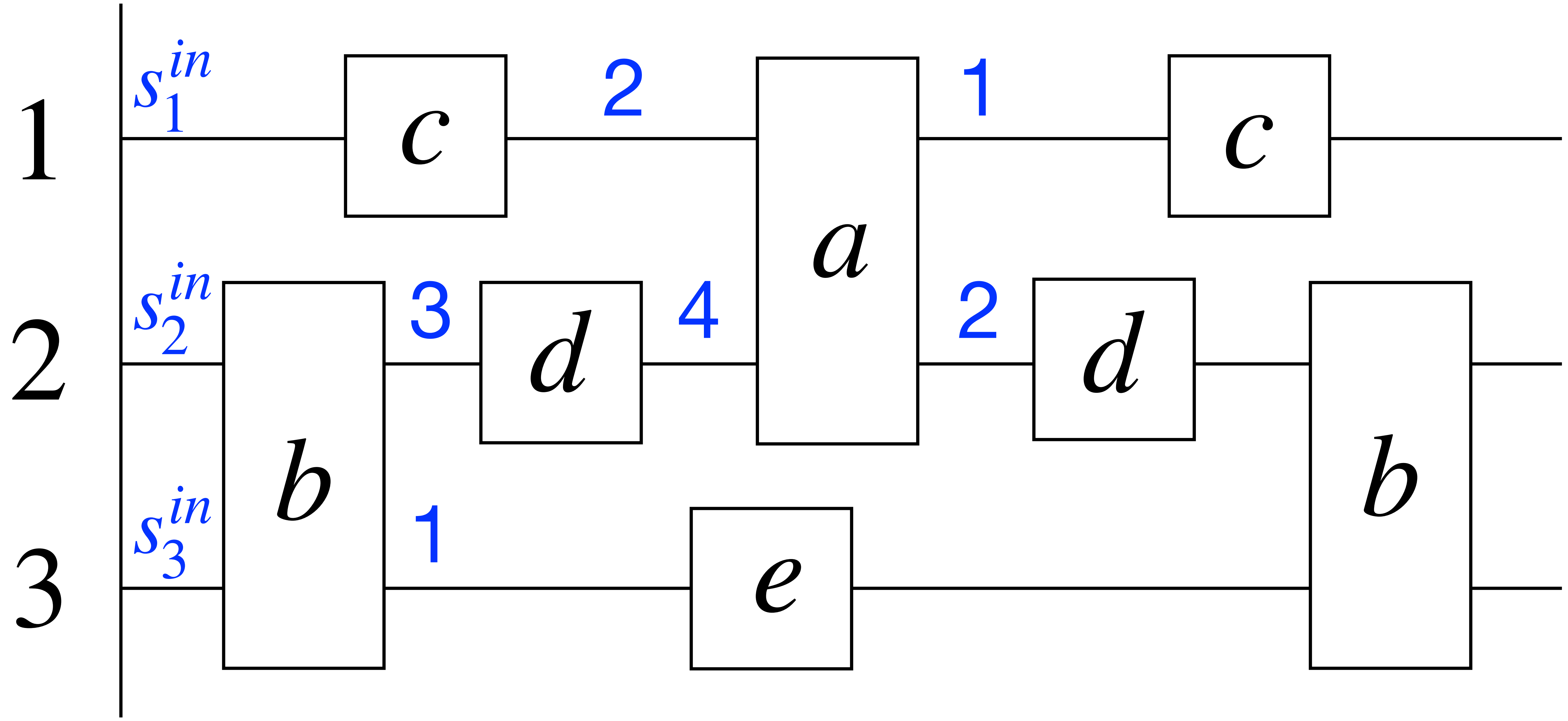

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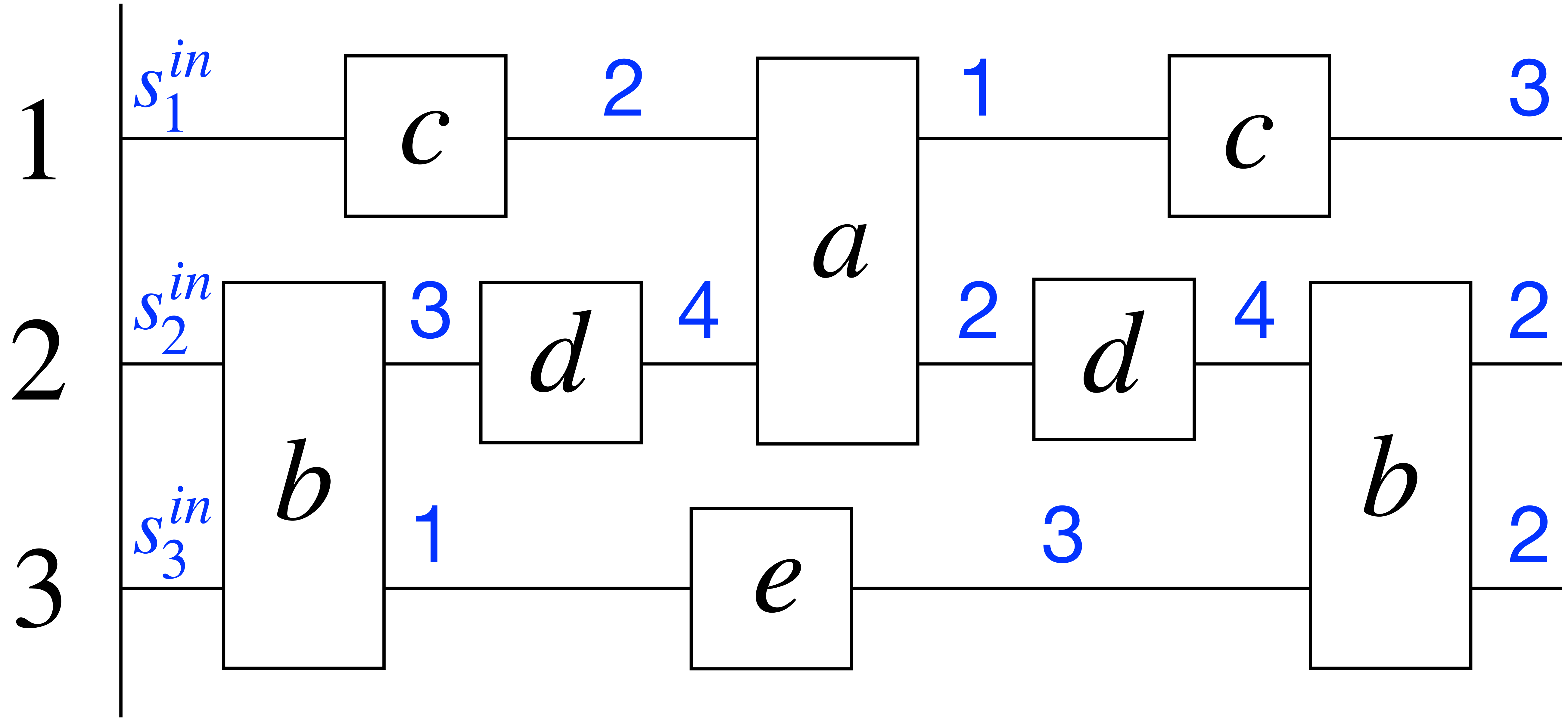

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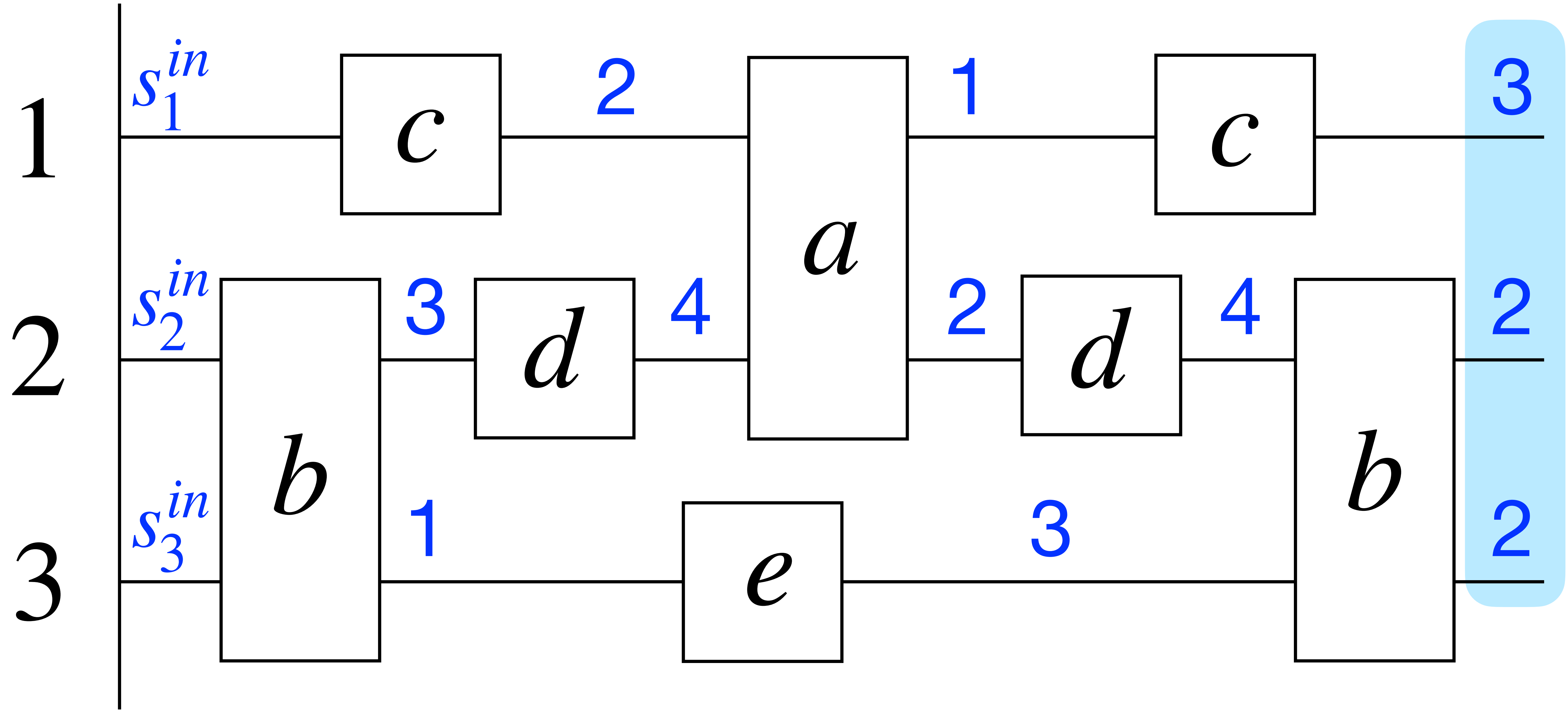

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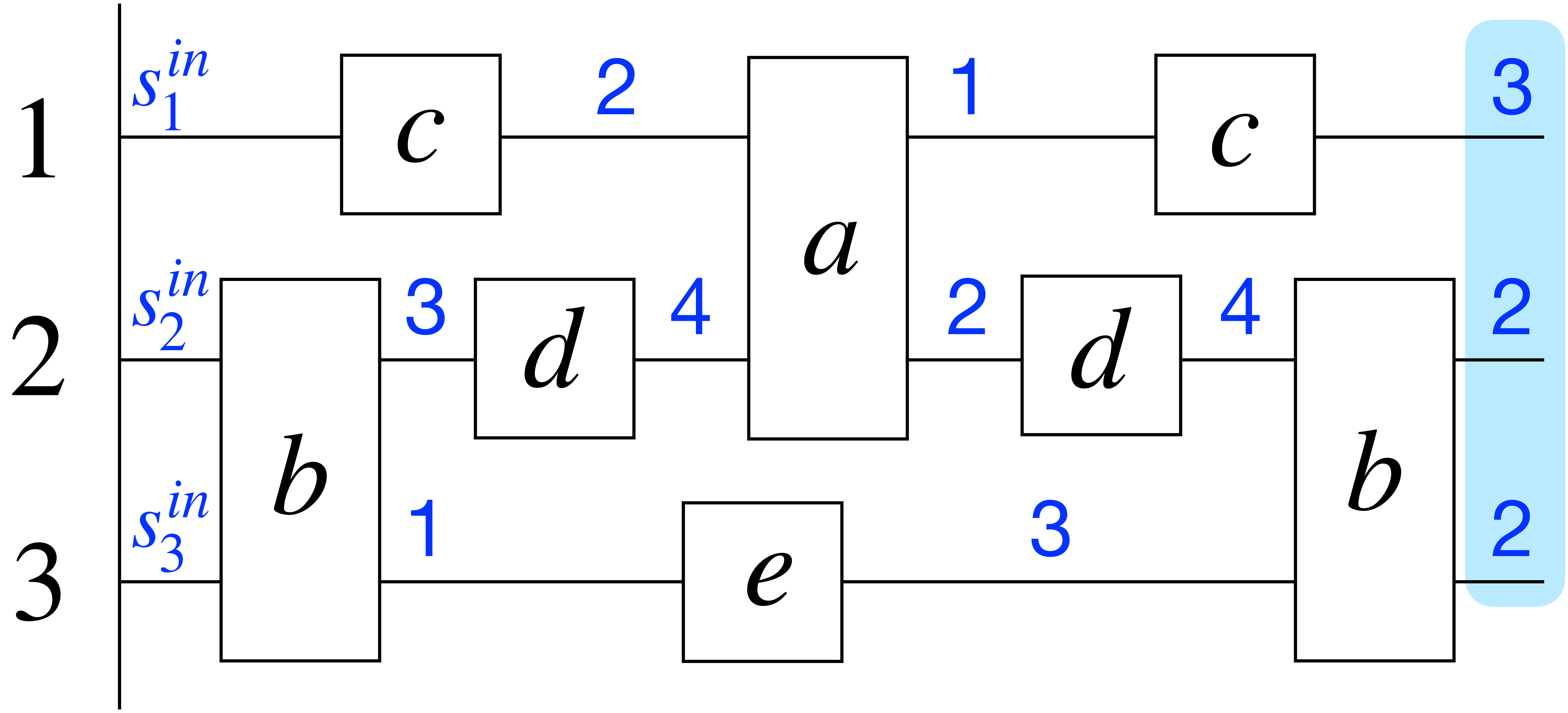

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Theorem (Zielonka, 1987)

Asynchronous Automata = Regular Trace Languages

Outline

- ✓ Labelling functions, sequential transducers and cascade product
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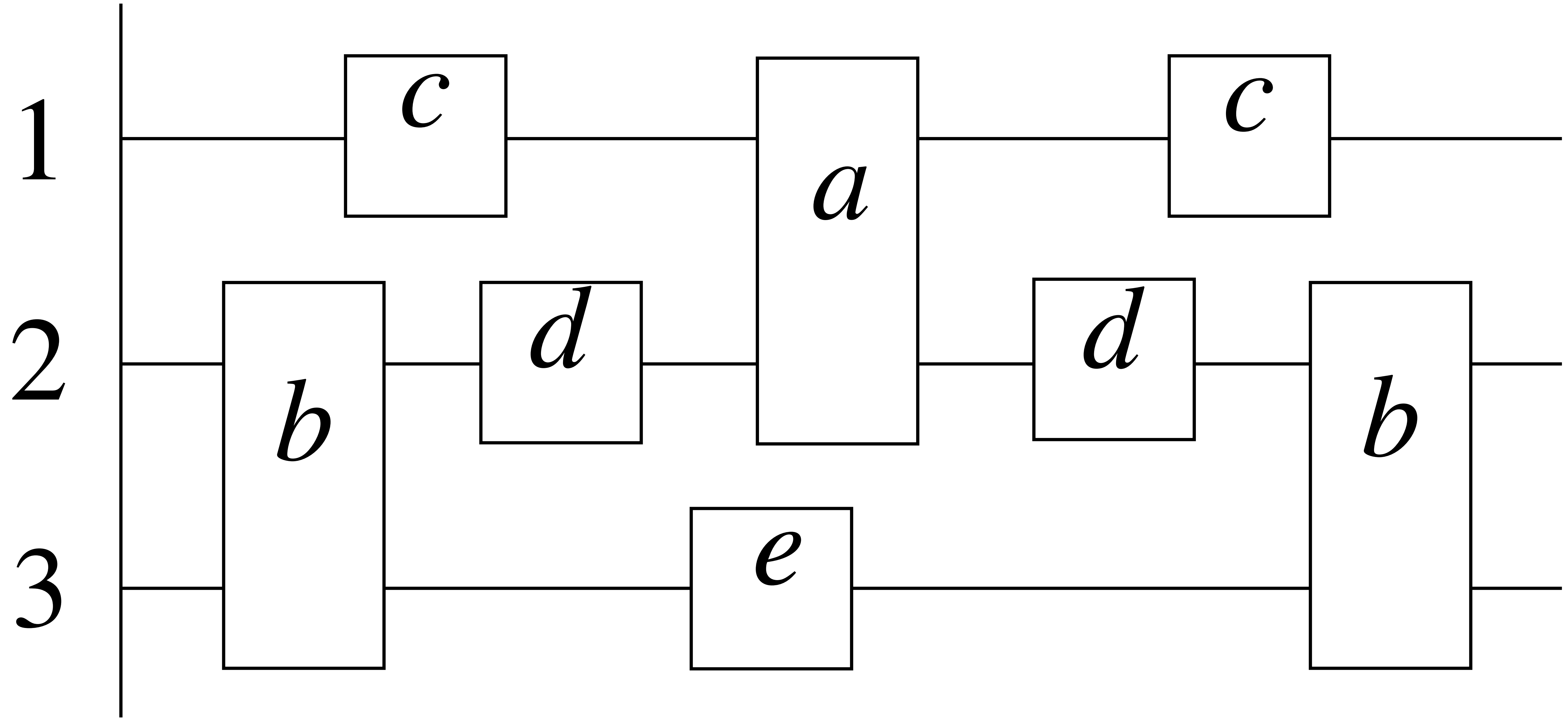
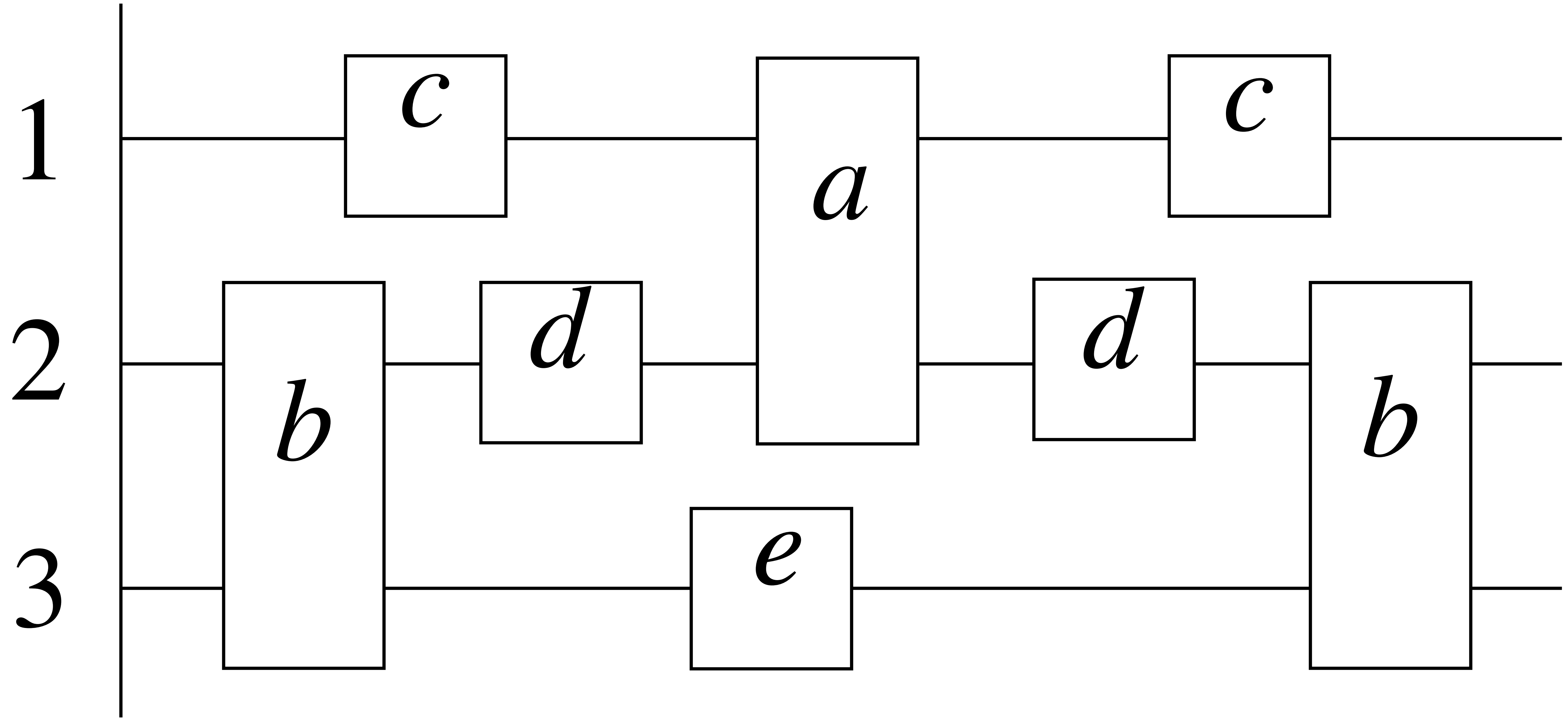
Labelling function



$\theta : Tr(\Sigma) \rightarrow Tr(\Sigma \times \Gamma)$

Labelling function



$\theta : Tr(\Sigma) \to Tr(\Sigma \times \Gamma)$

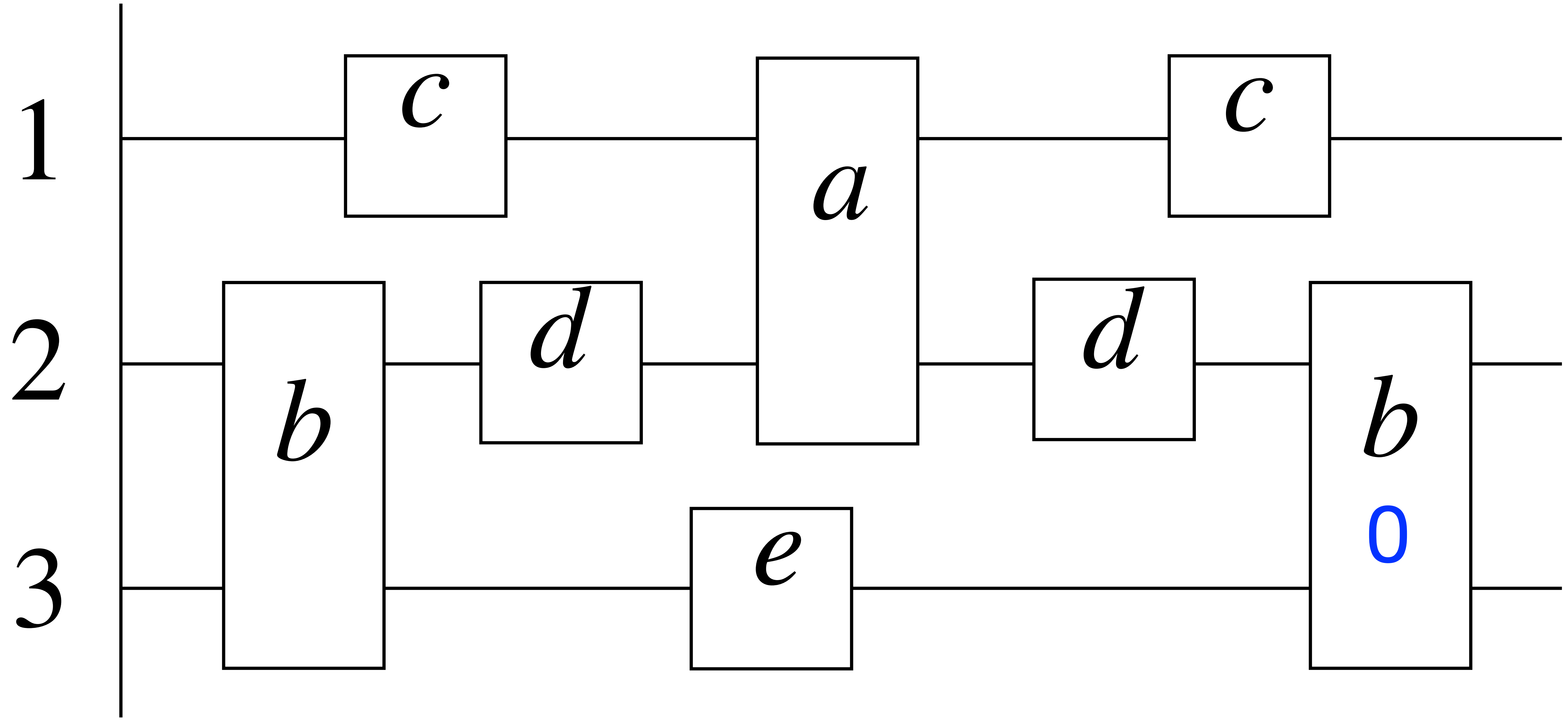
$\Gamma = \{0,1\}$

$Y_3 \leq Y_2$: In the strict past,

the last event on process 3 is below

the last event on process 2

Labelling function



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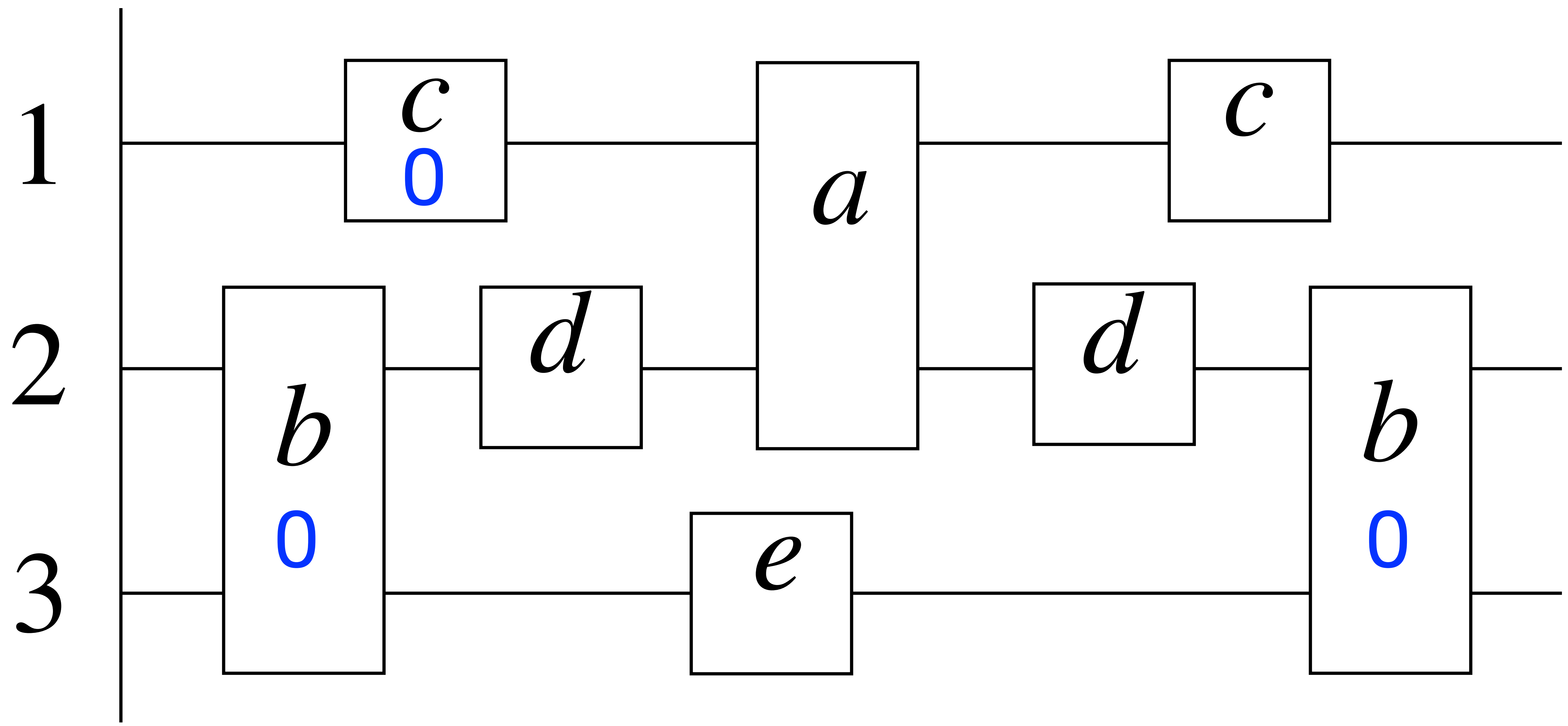

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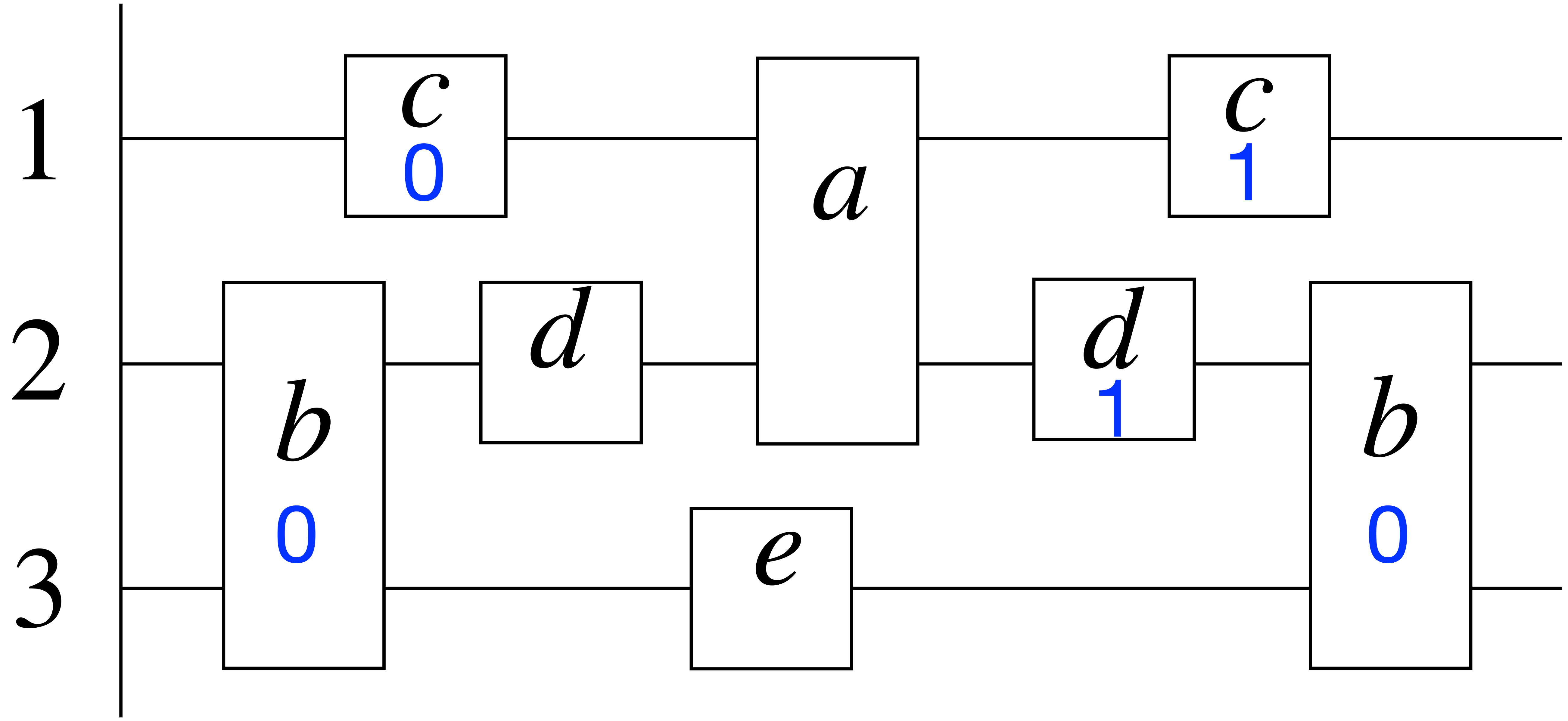

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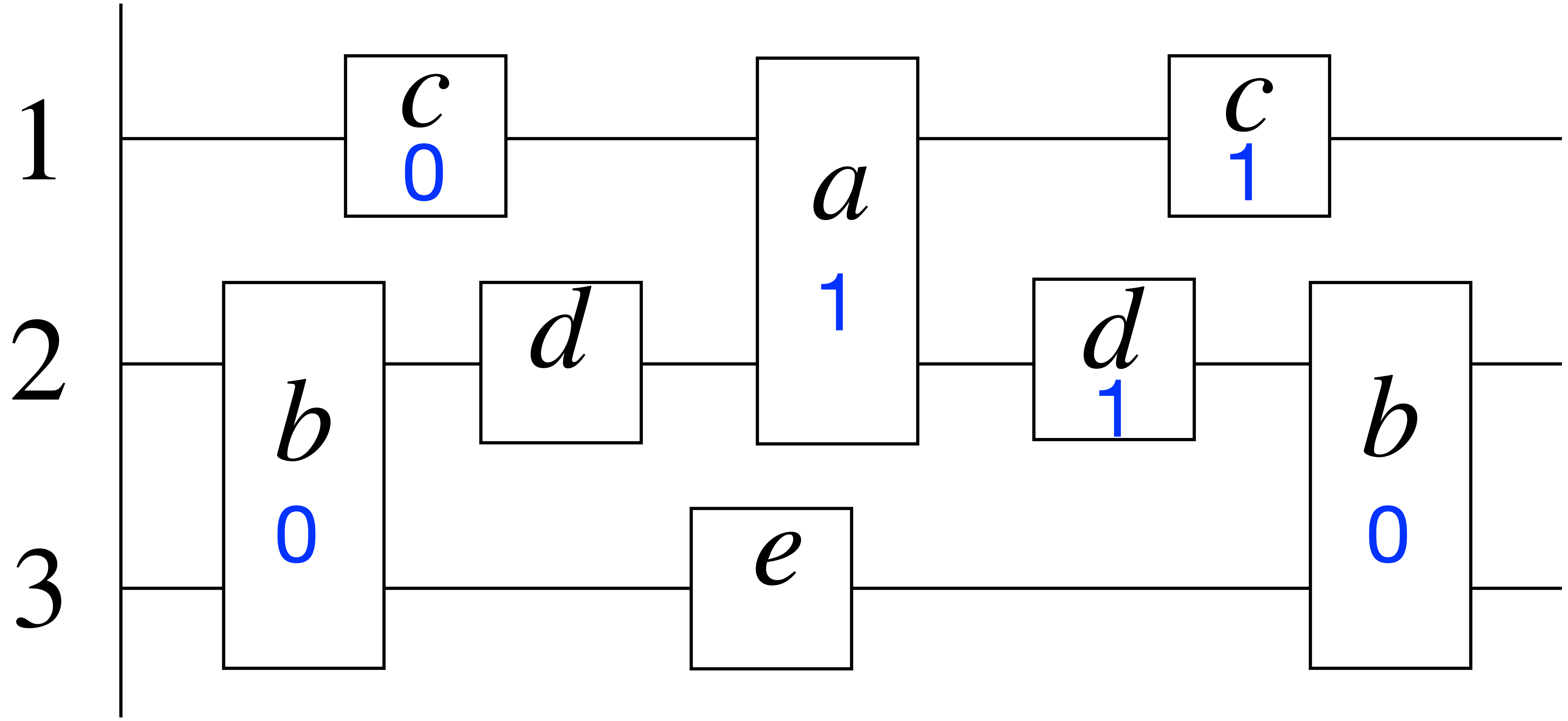

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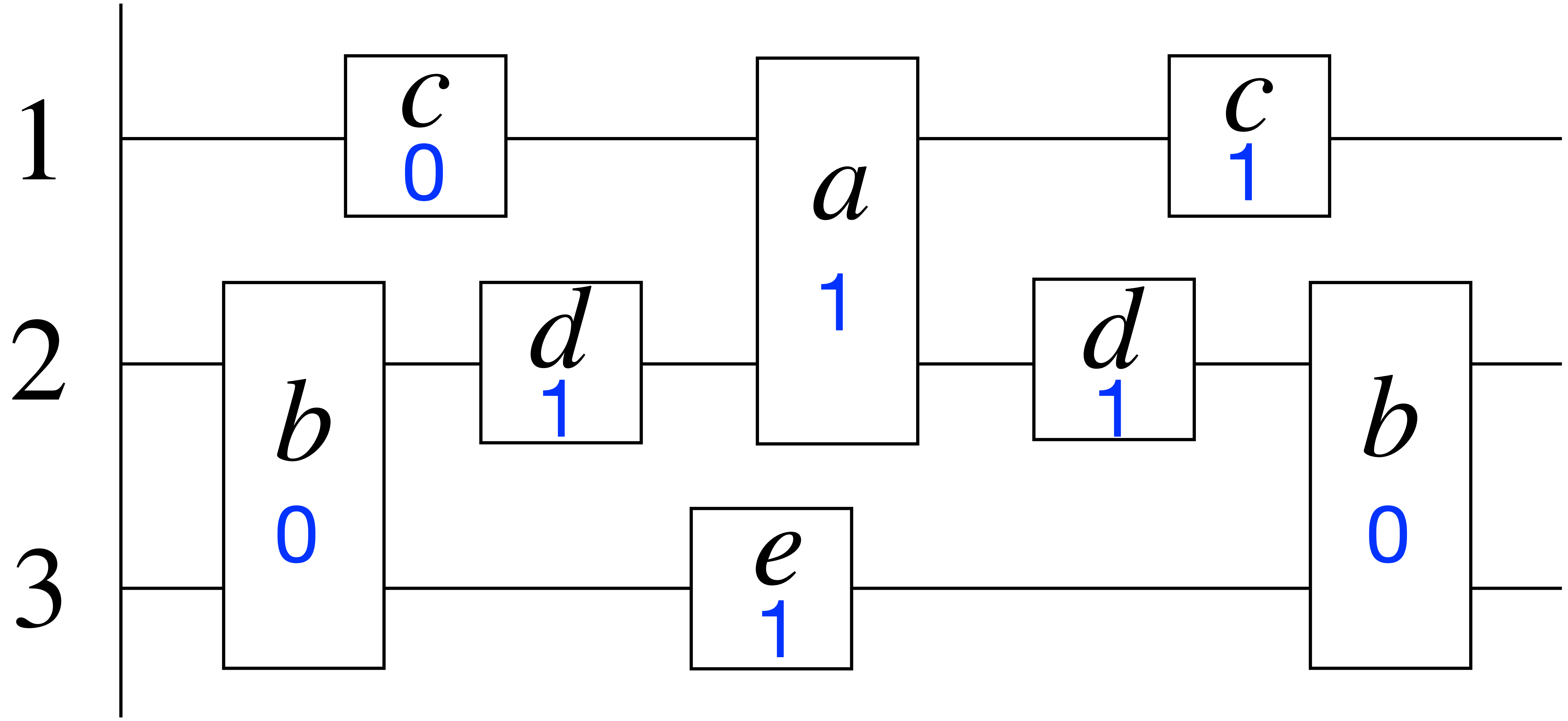

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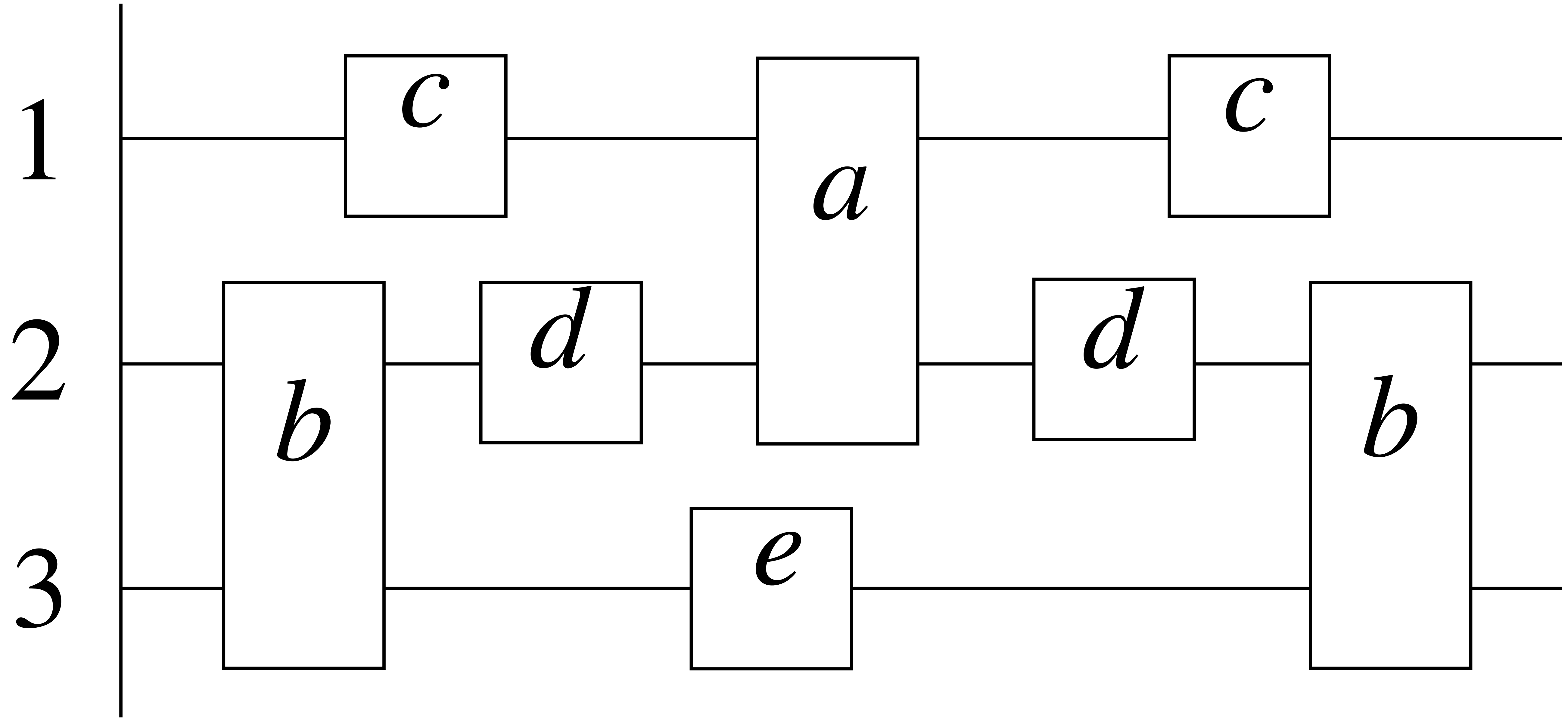
Asynchronous Labelling function



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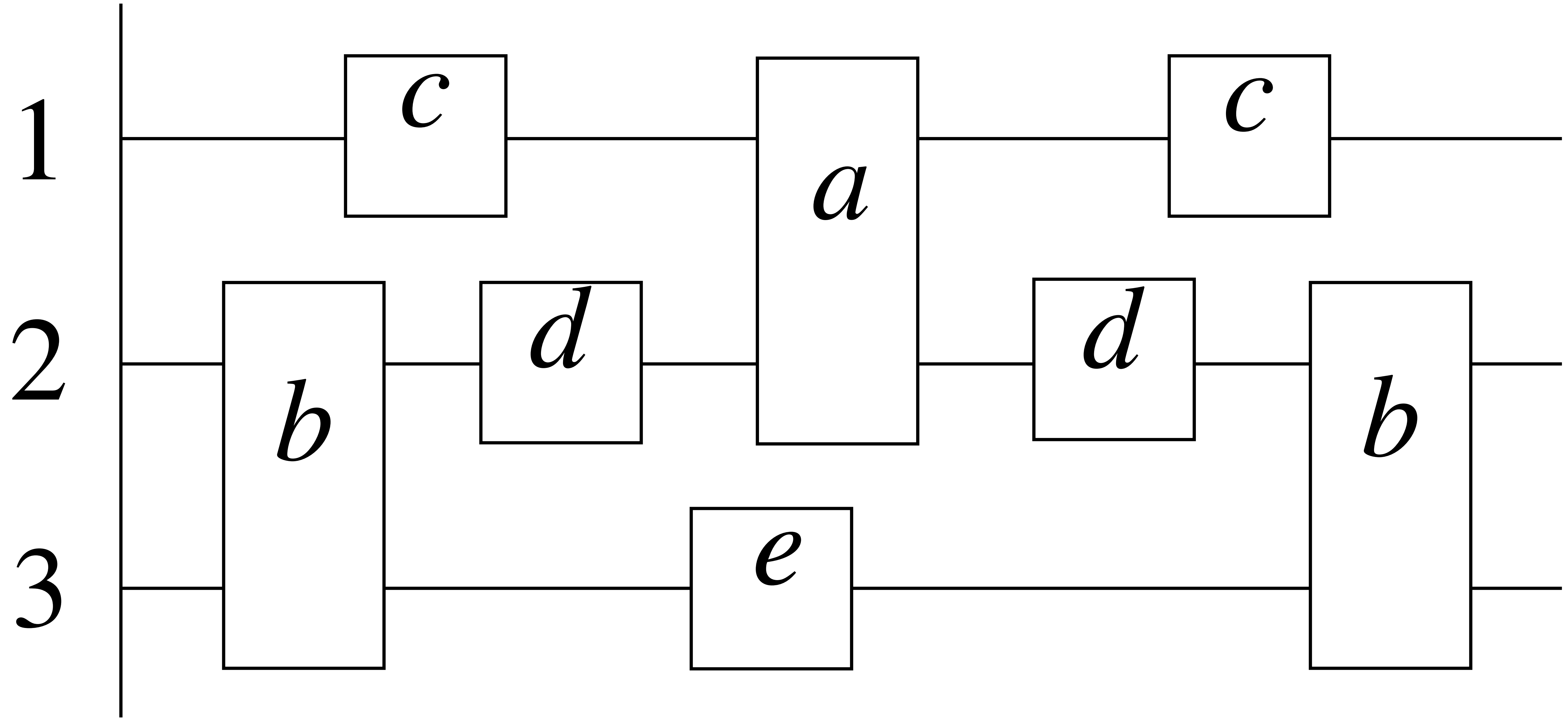

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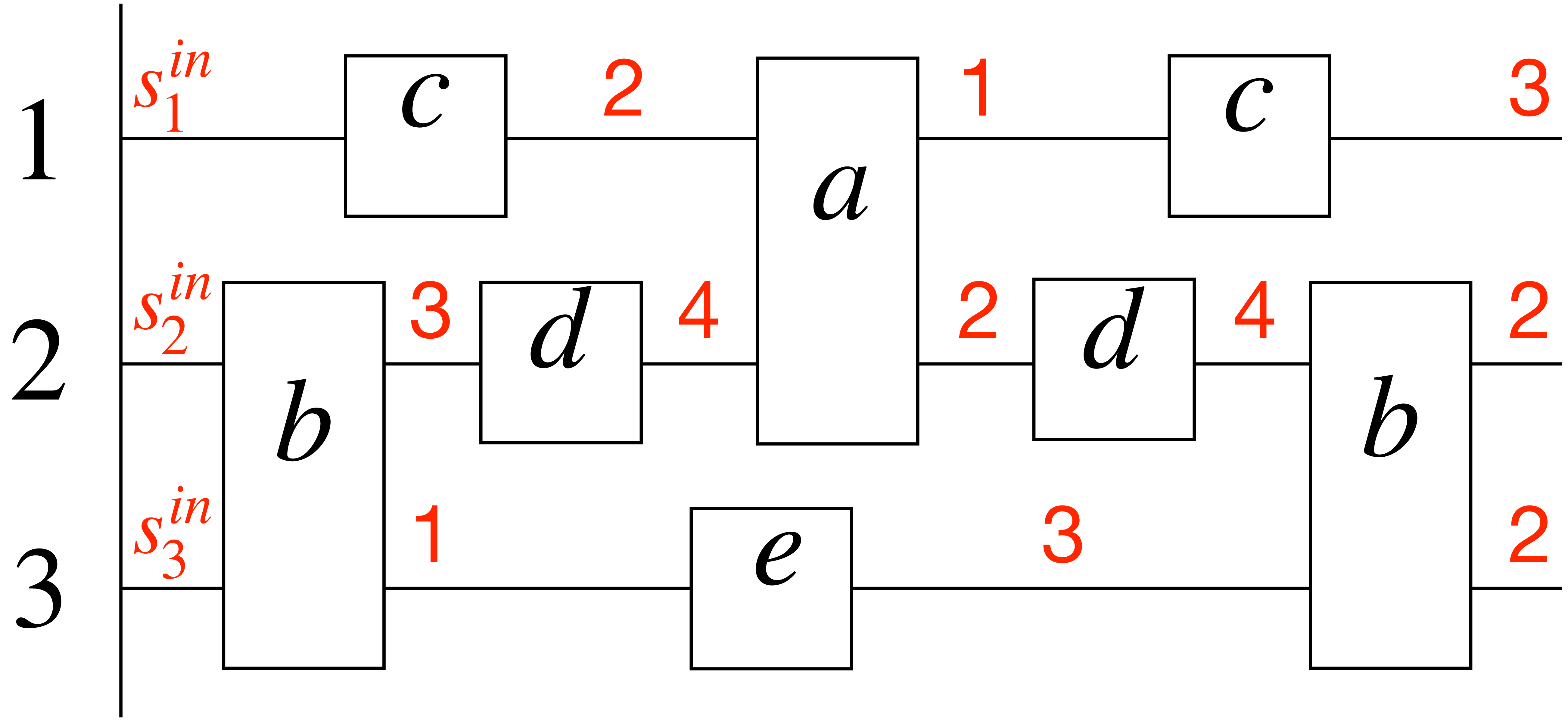

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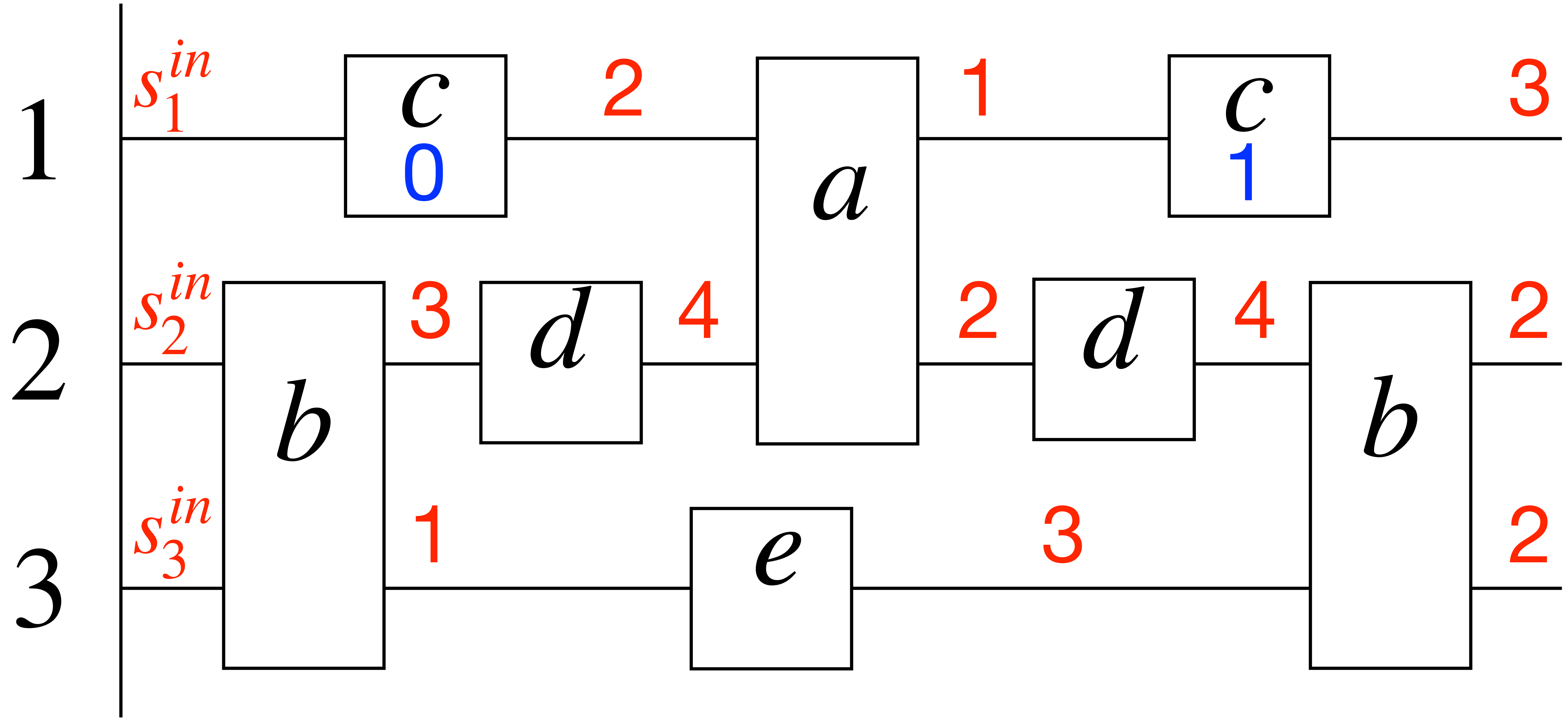

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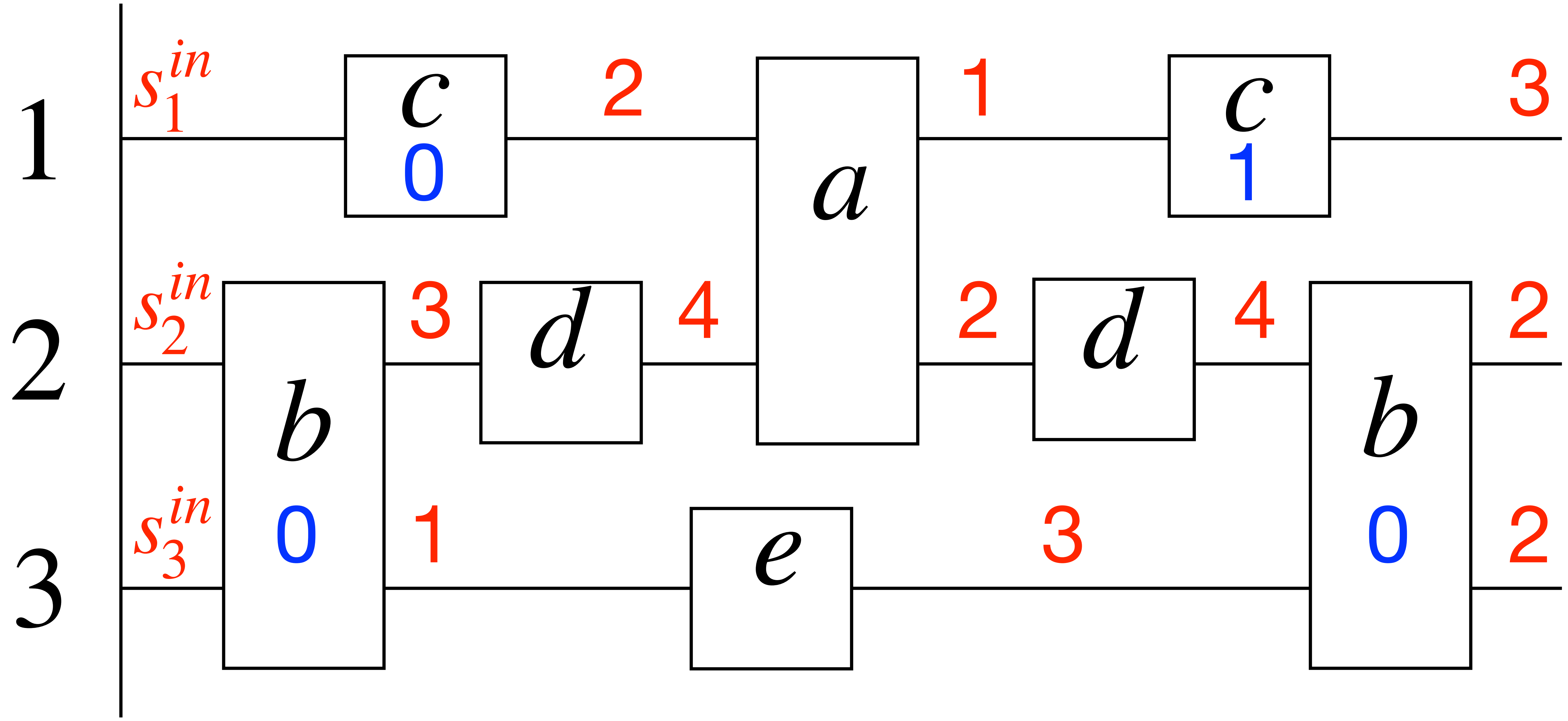

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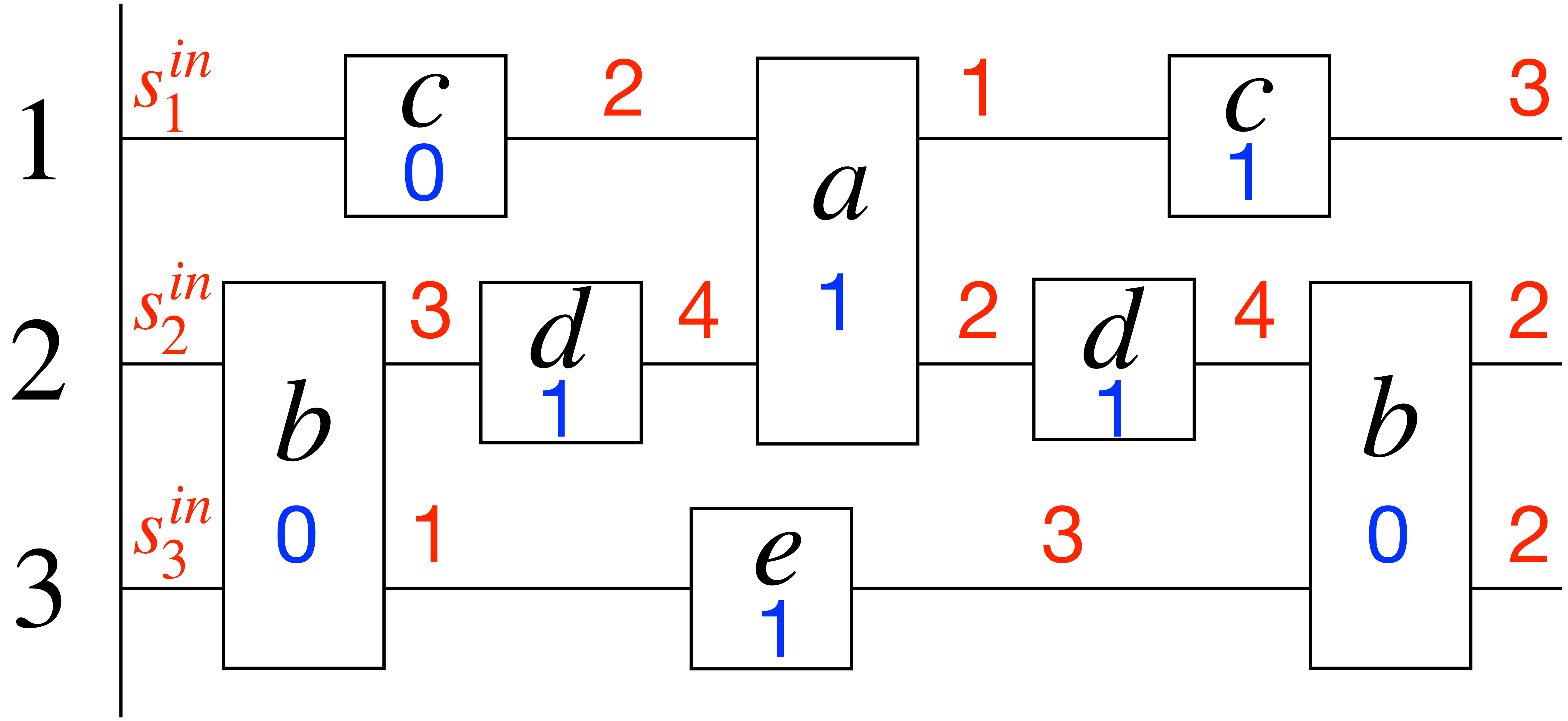

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Composition - Cascade product

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- Composition of labelling functions

$$Tr(\Sigma) \xrightarrow{\theta_1} Tr(\Sigma \times \Gamma) \xrightarrow{\theta_2} Tr(\Sigma \times \Pi)$$

Composition - Cascade product

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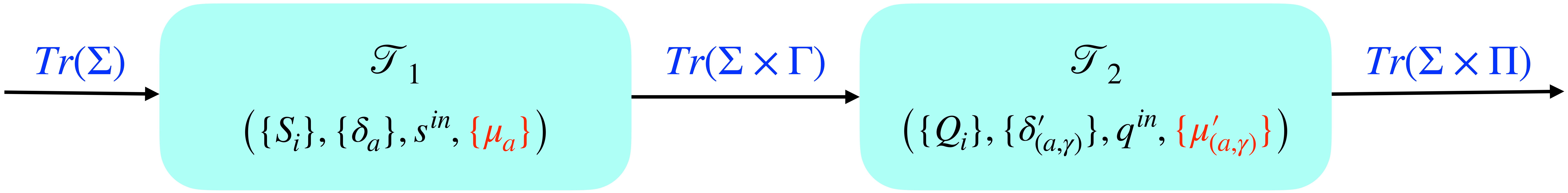
- Cascade product of asynchronous (letter-to-letter) transducers



Composition - Cascade product

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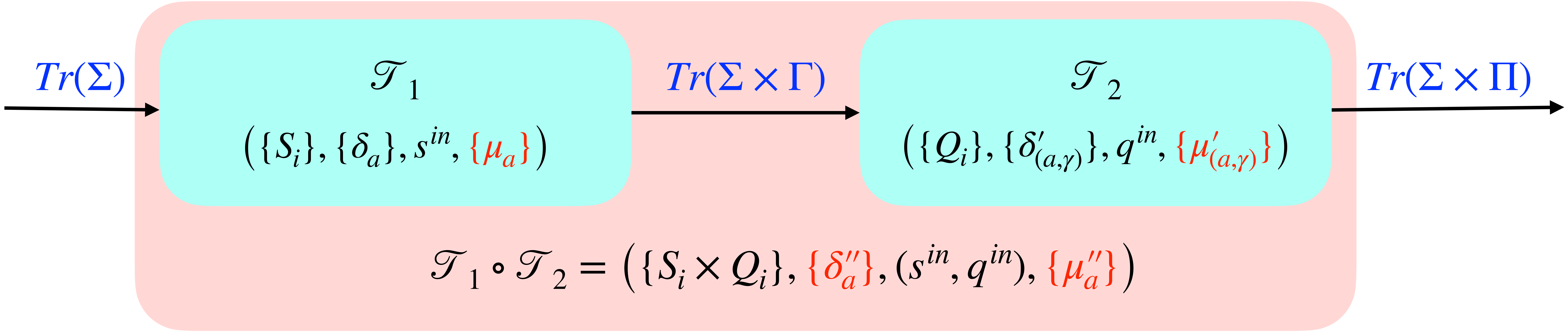
$$Tr(\Sigma) \xrightarrow{\theta_1} Tr(\Sigma \times \Gamma) \xrightarrow{\theta_2} Tr(\Sigma \times \Pi)$$

$$\delta''_a(s,q) = \left(\delta_a(s), \delta'_{(a,\mu_a(s))}(q)\right)$$

$$\mu''_a(s,q) = \mu'_{(a,\mu_a(s))}(q)$$

- Cascade product of asynchronous (letter-to-letter) transducers



Cascade Decomposition

Theorem (main)

Any asynchronous labelling function can be realised by a cascade product of local asynchronous transducers:

$$\mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_n$$

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Corollary: Zielonka's theorem

Asynchronous Automata = Regular Trace Languages

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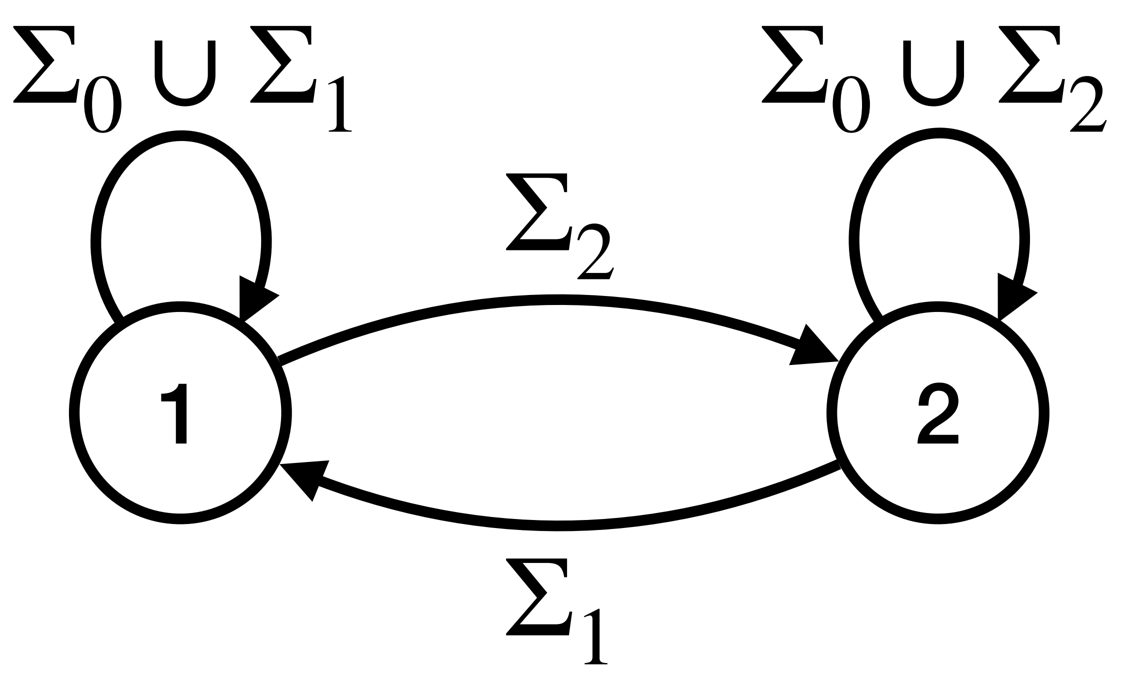
$$\mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_n$$

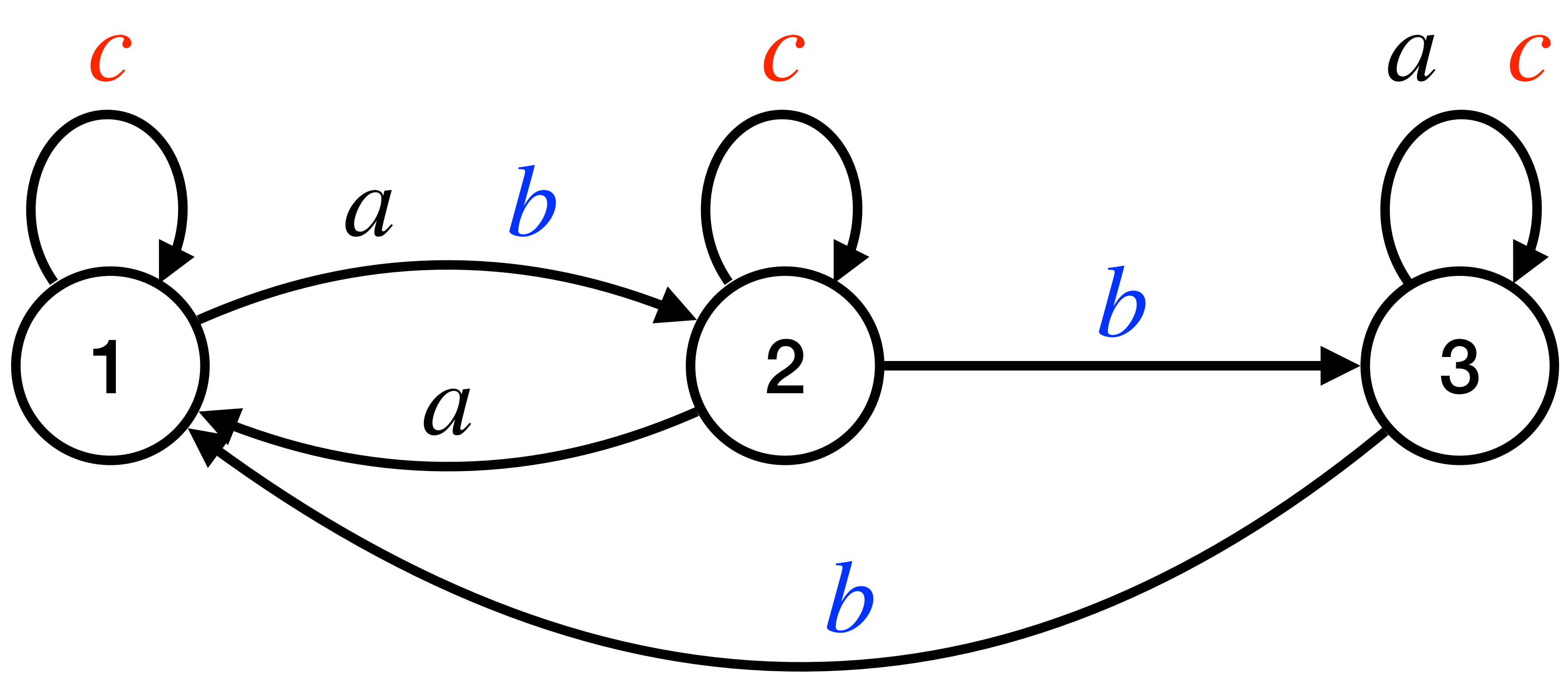
Bonus: Using Krohn-Rhodes theorem

Each local asynchronous transducer $\mathcal{T}$ can be chosen to be (on its non-trivial component)

Reset - $\mathcal{U}_2$



Permutation



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Proof sketch:

- Design a local and past propositional dynamic logic (locPastPDL)
  - State/Event formulas $\varphi ::= a \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle \pi \rangle \varphi$
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- For each event formula φ, construct by structural induction a cascade product of local asynchronous transducers computing its labelling function θ_φ (easier)

Outline

- ✅ Labelling functions, sequential transducers and cascade product
- ✅ Krohn-Rhodes theorem for aperiodic/regular word languages
- ✅ Model of concurrency: Mazurkiewicz traces and asynchronous Zielonka automata
- ✅ Asynchronous labelling functions, transducers and cascade product
- Propositional dynamic logic for traces
- Conclusion

Propositional Dynamic Logic

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- First introduced to reason about programs (Fischer, Ladner 1979)
  - State formulas $\varphi ::= p \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\varphi$
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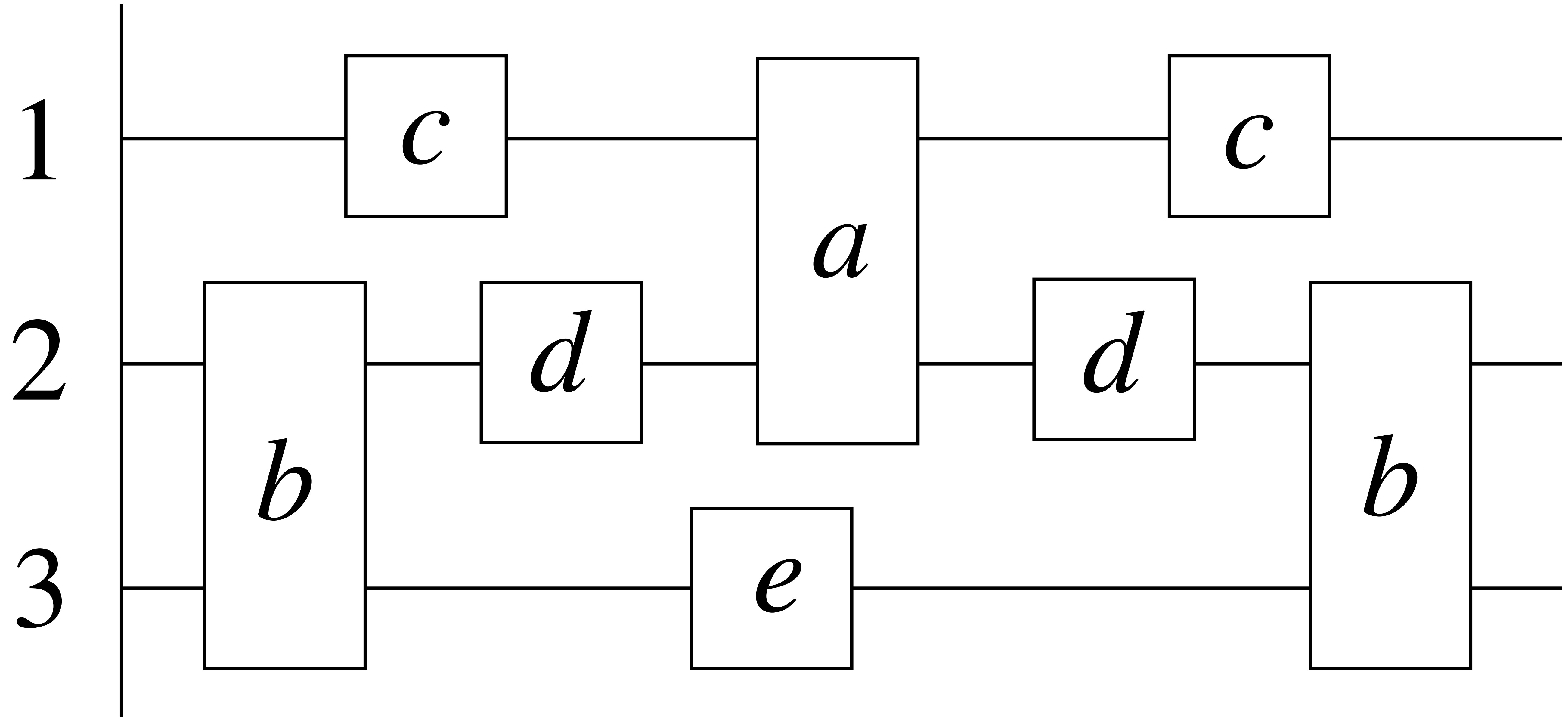
- Interpretation over words: Linear Dynamic Logic (Giacomo, Vardi 2013)

Regular word languages = MSO definable = LDL definable

Past PDL for Traces



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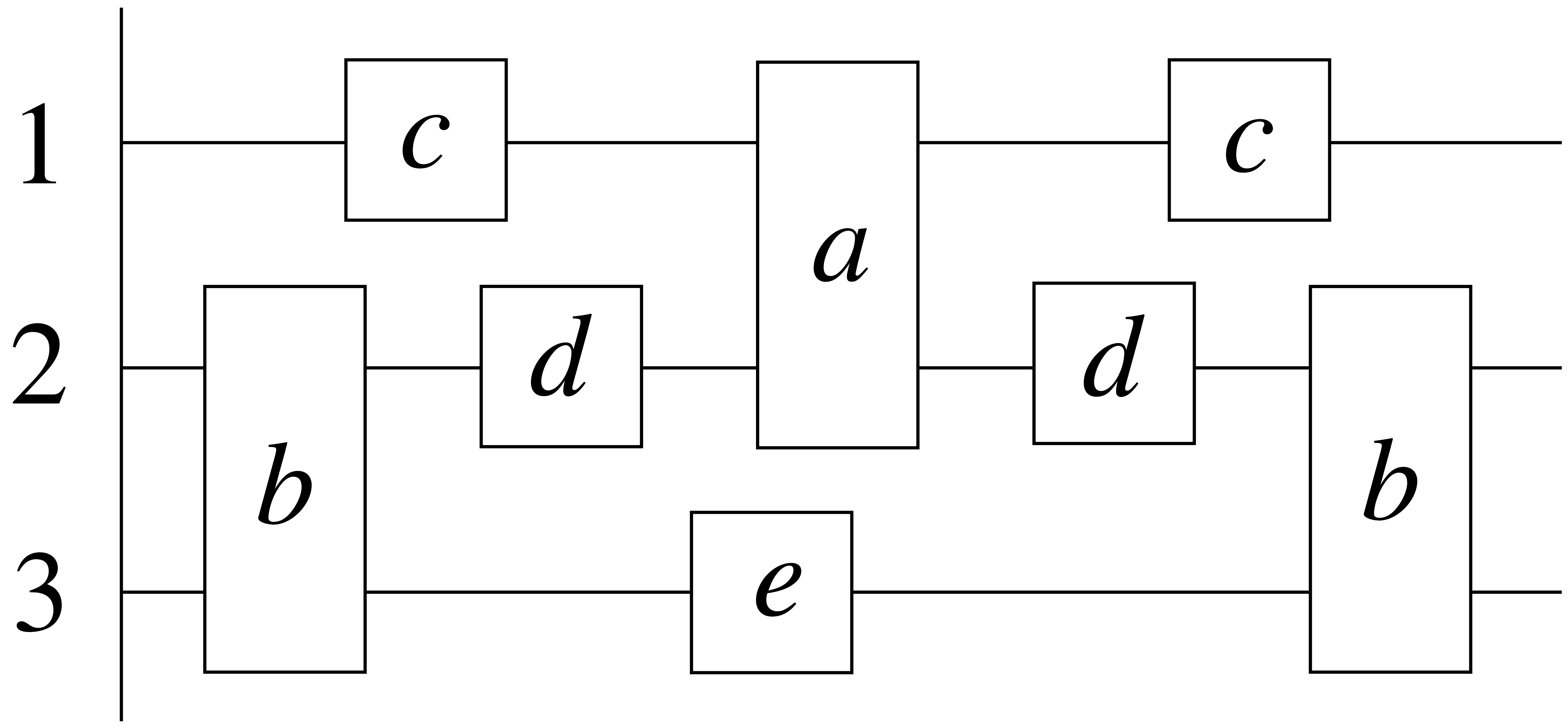

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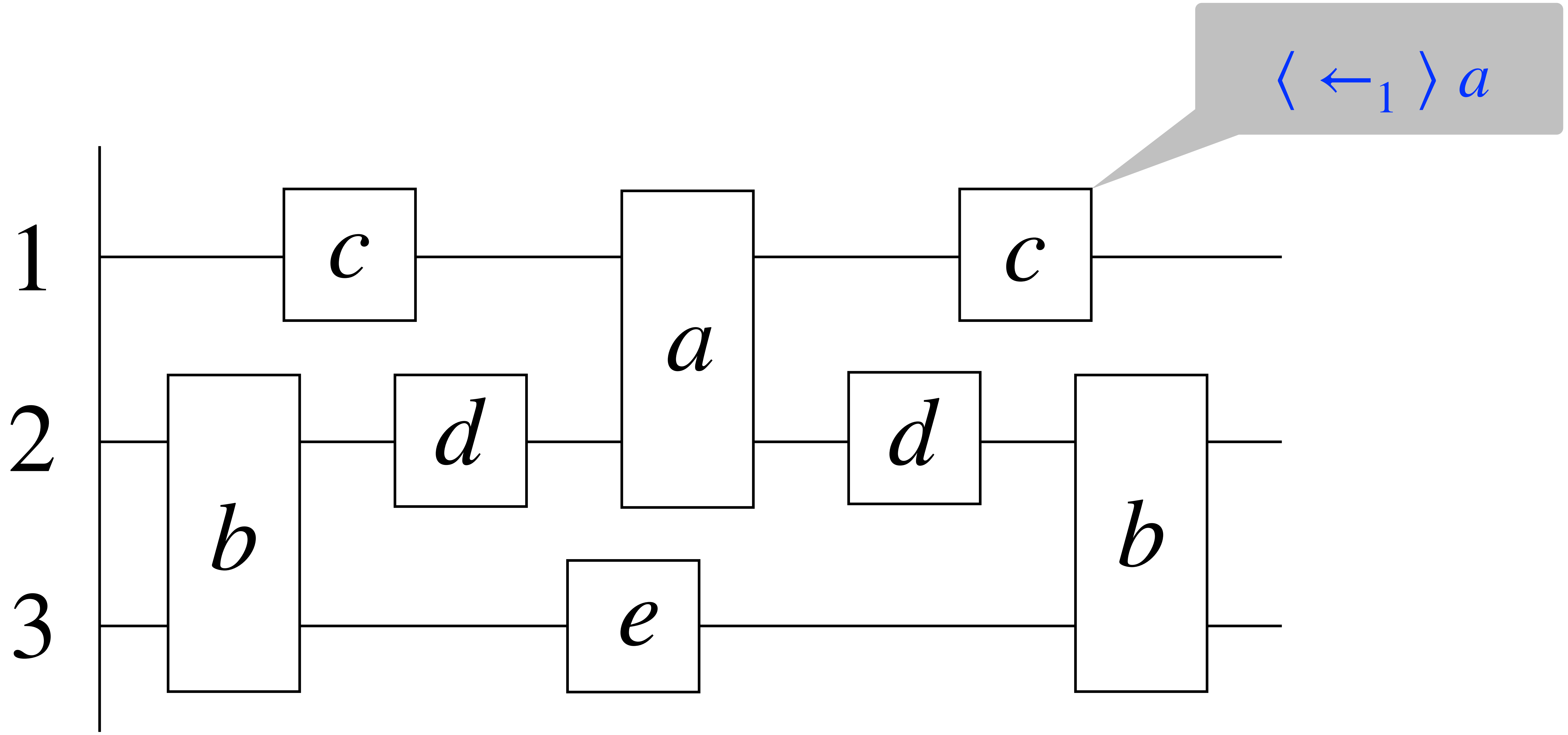

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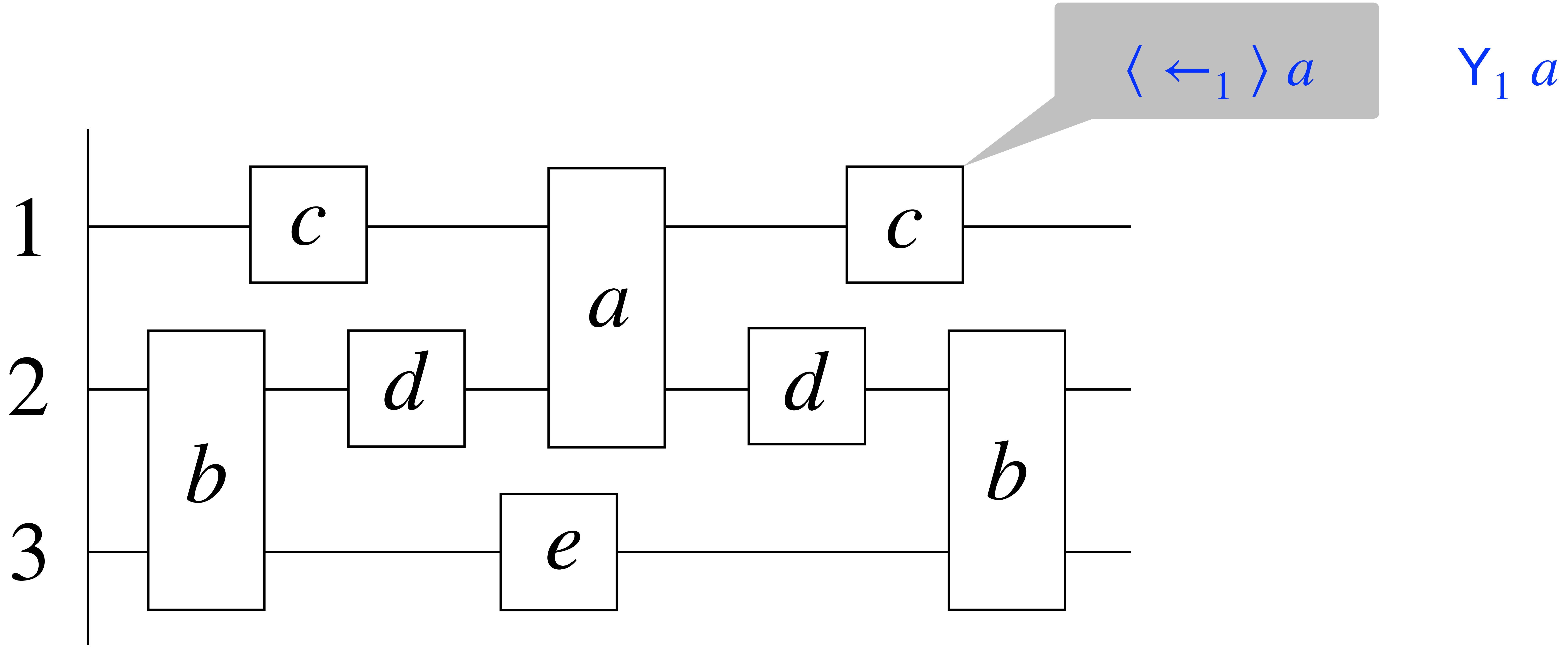

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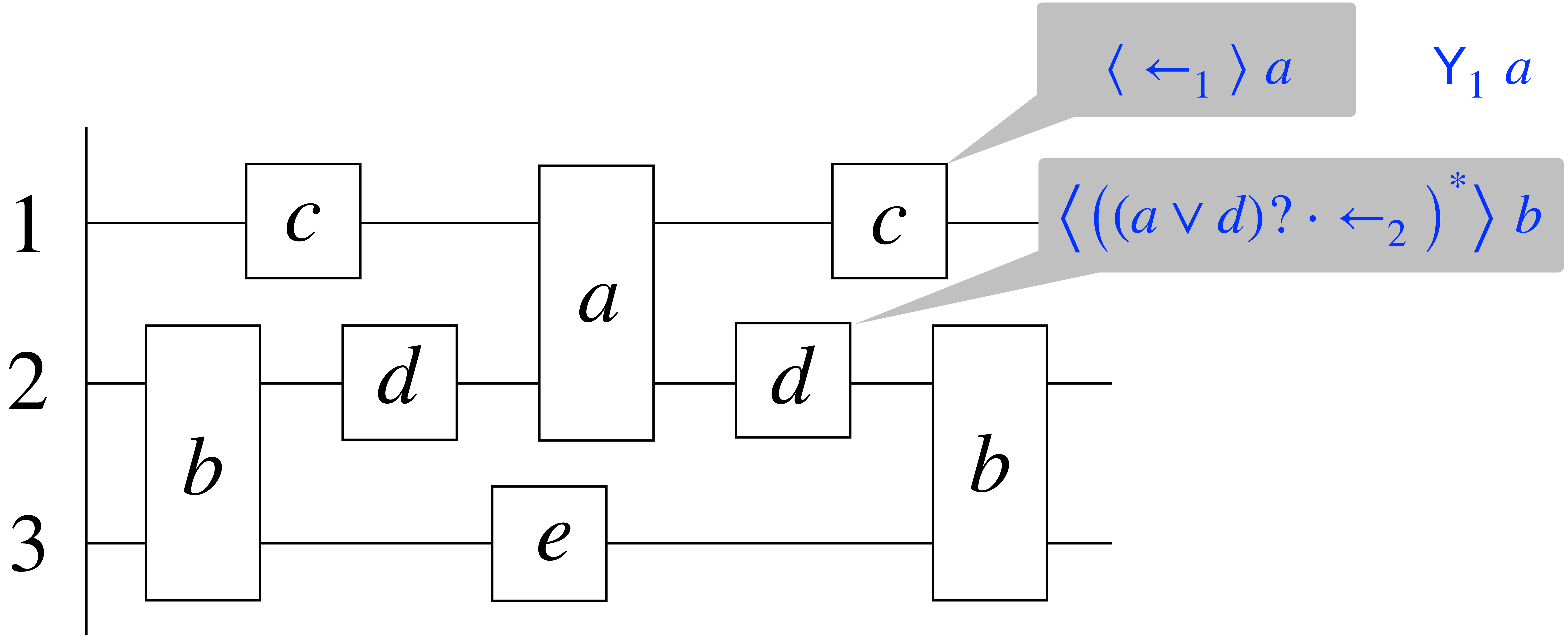

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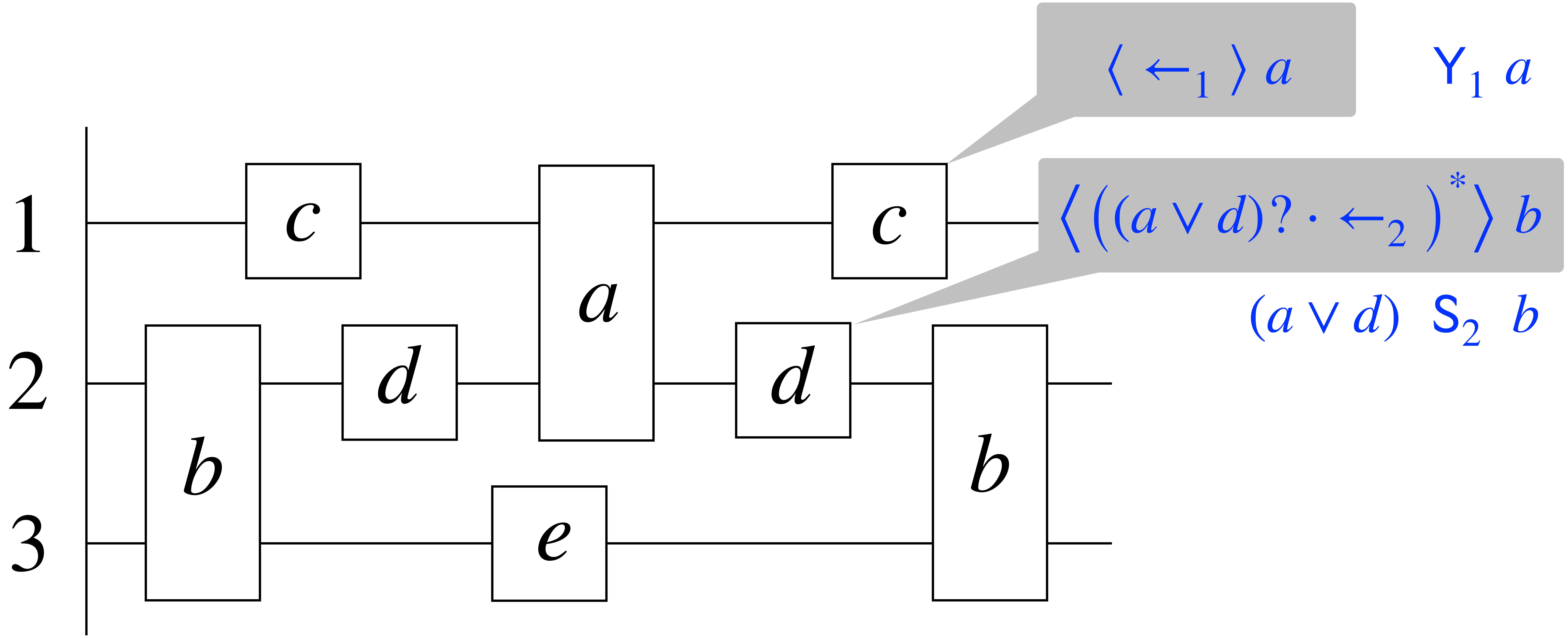

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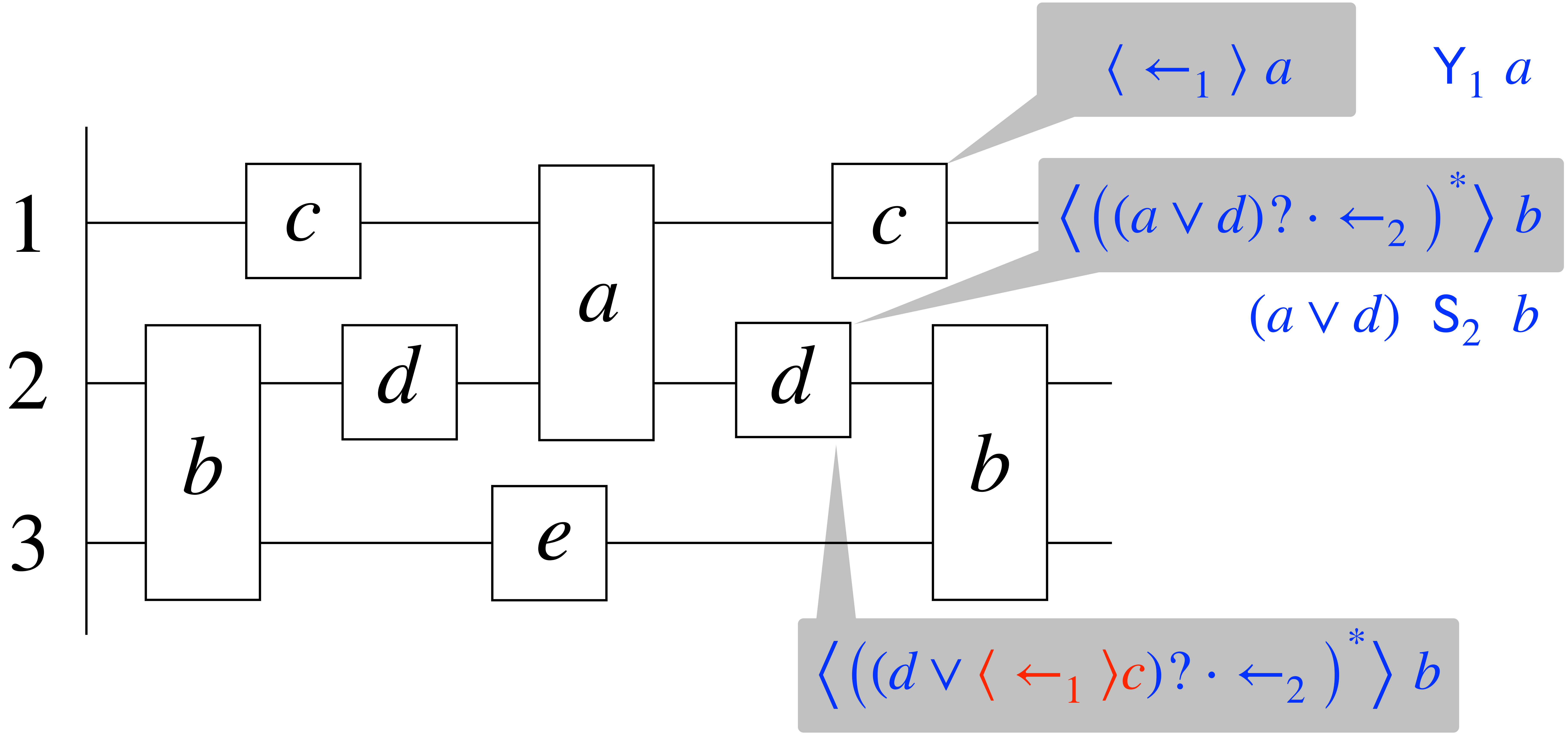

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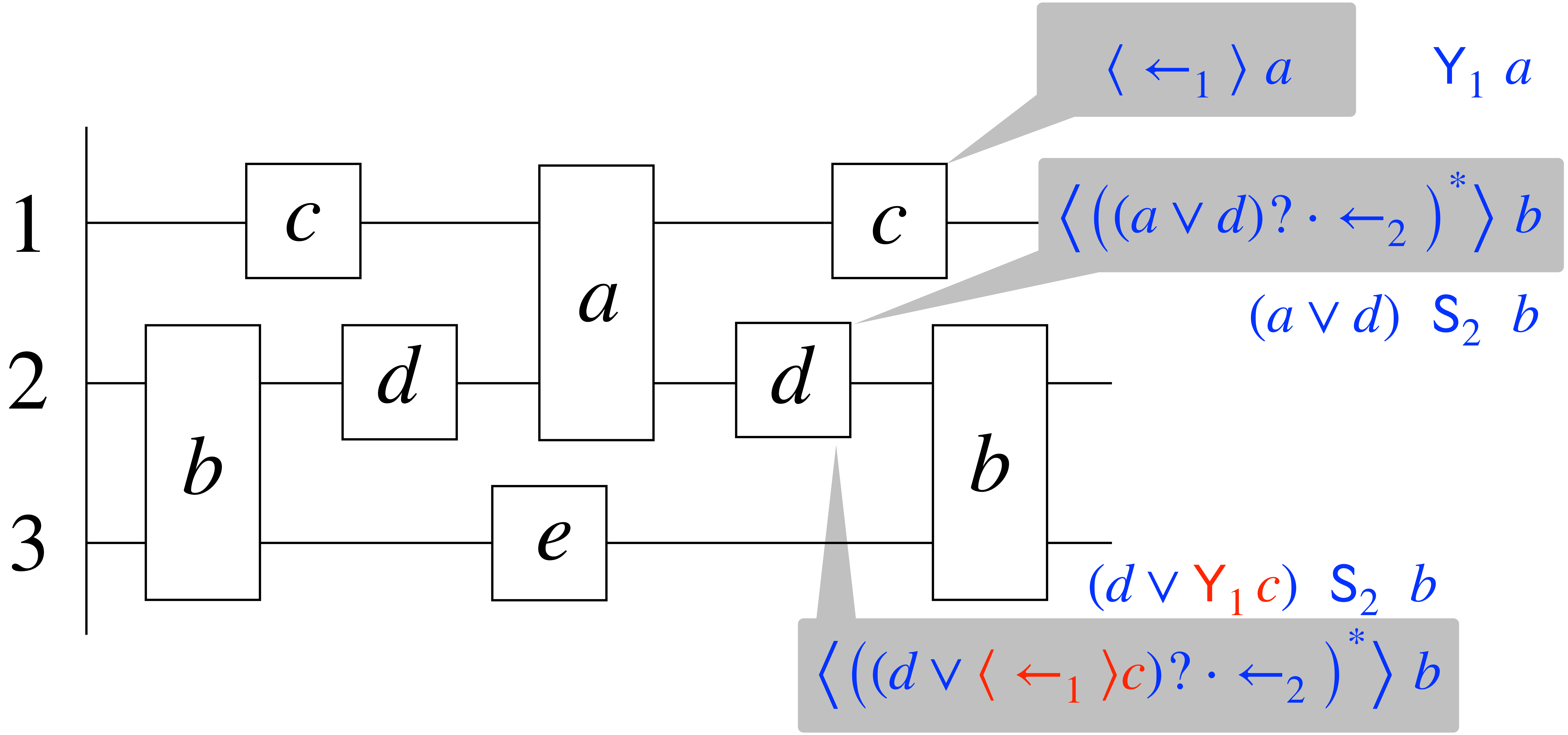

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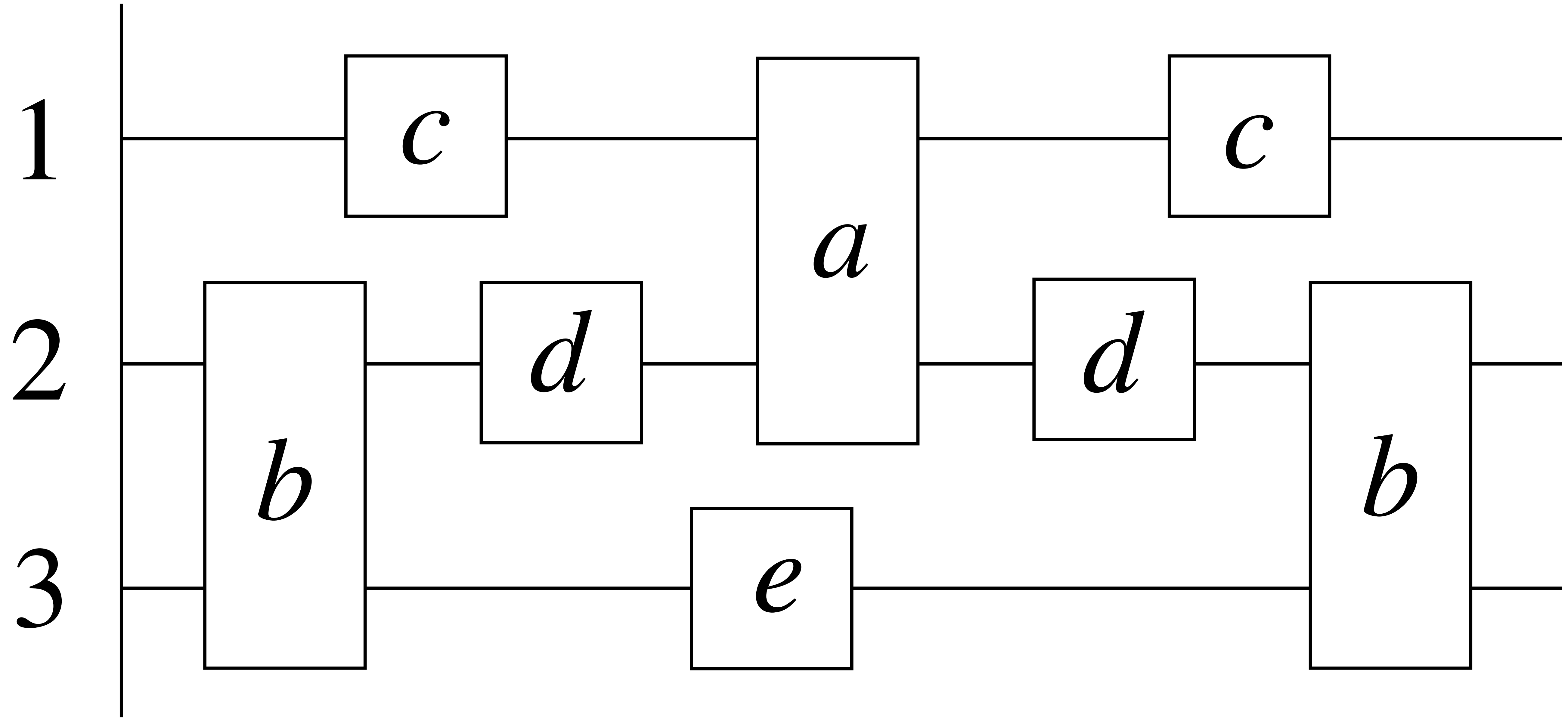

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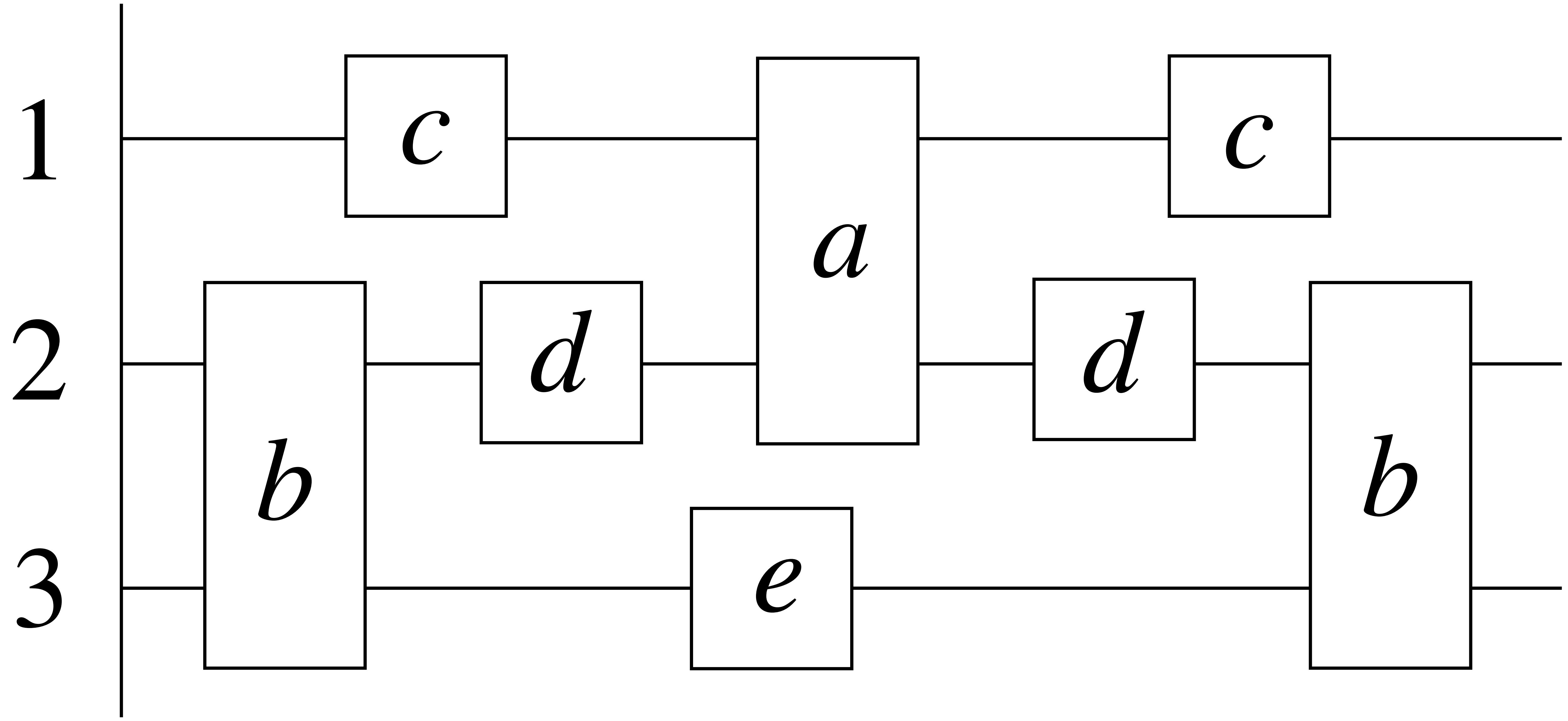

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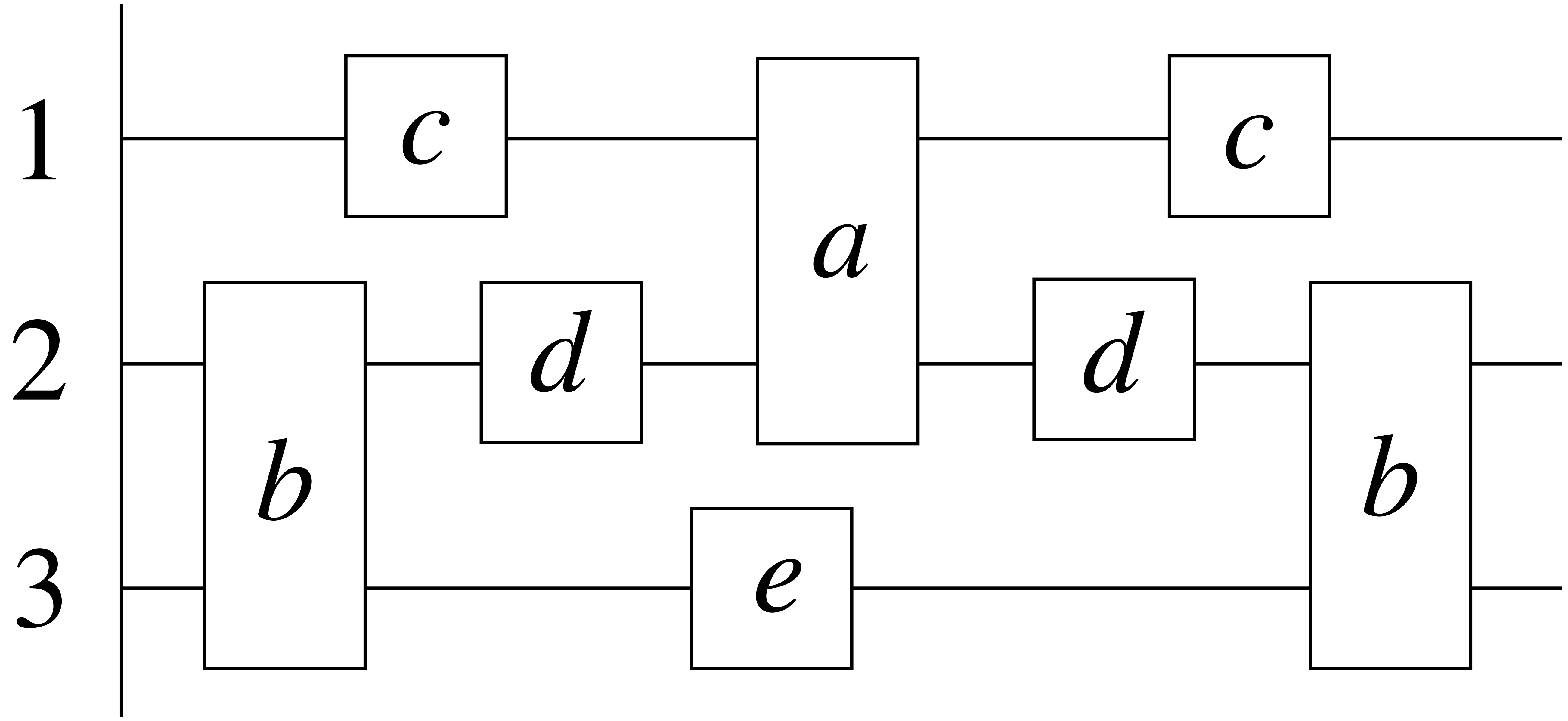
$T \models \mathsf{EM}_1\, c \wedge \mathsf{EM}_3 \Big(b \wedge \langle \leftarrow_2 \cdot ((a \vee d)? \cdot \leftarrow_2)^* \rangle b\Big)$



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Induction on the number of processes

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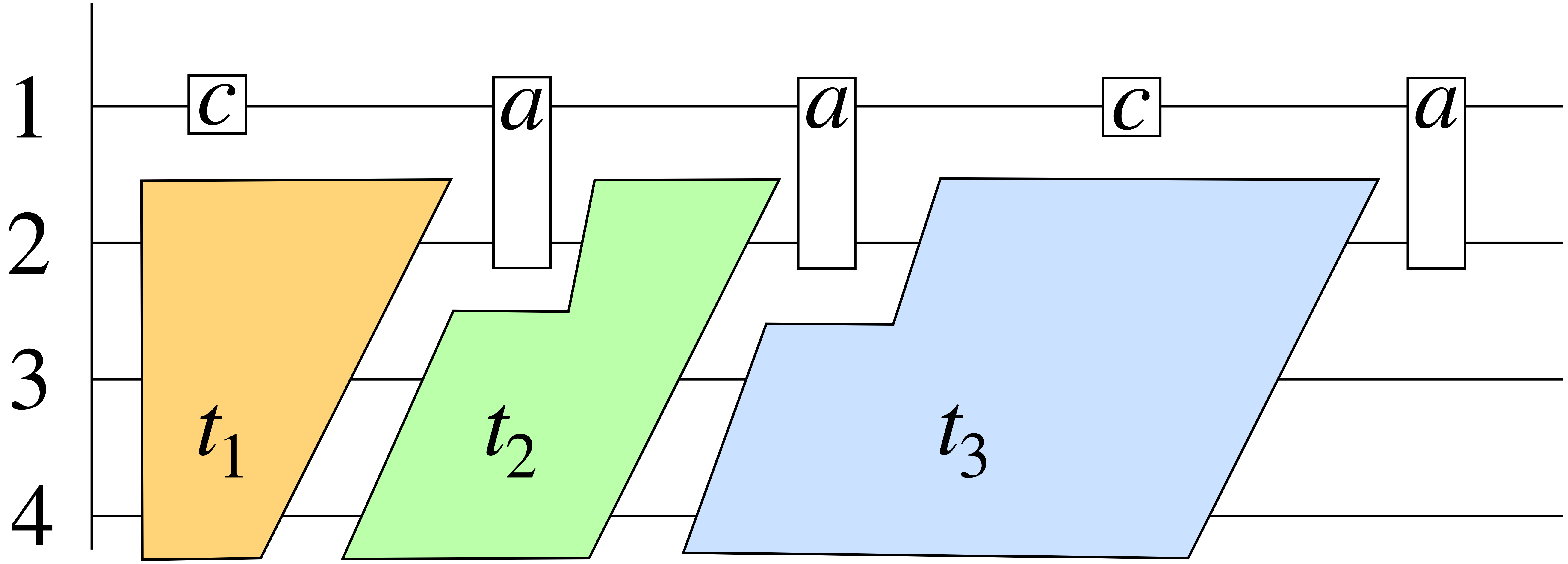

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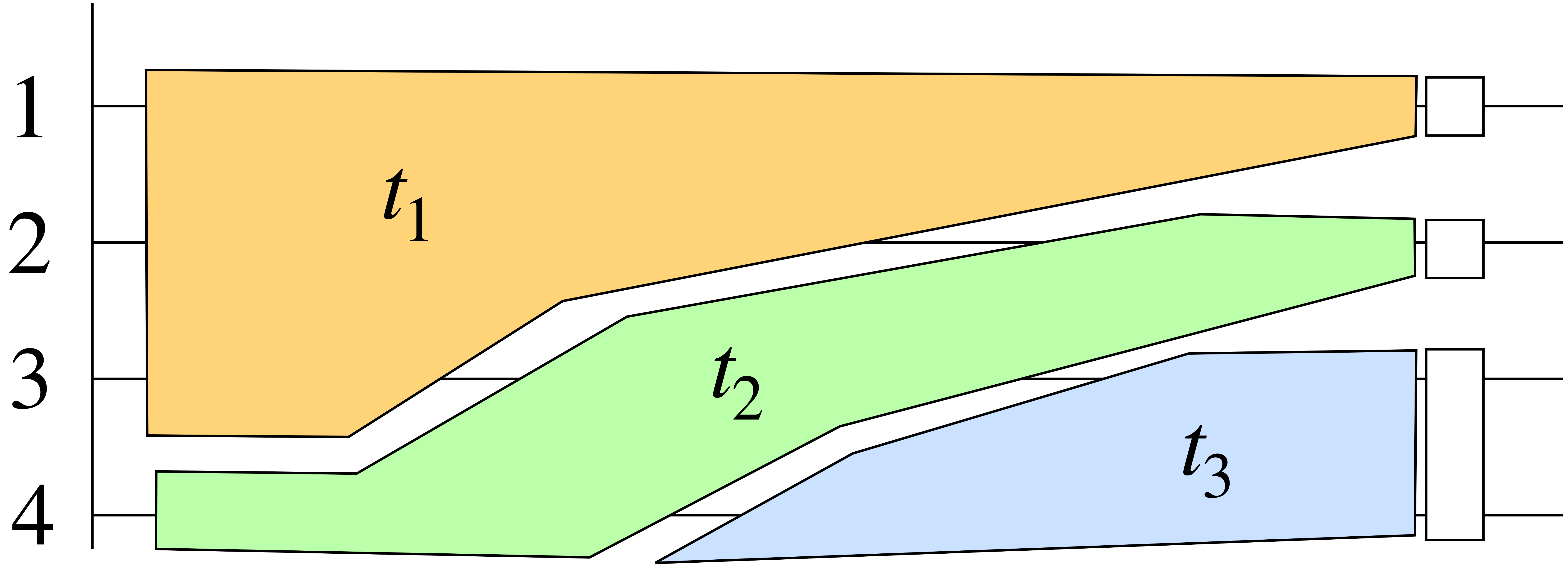

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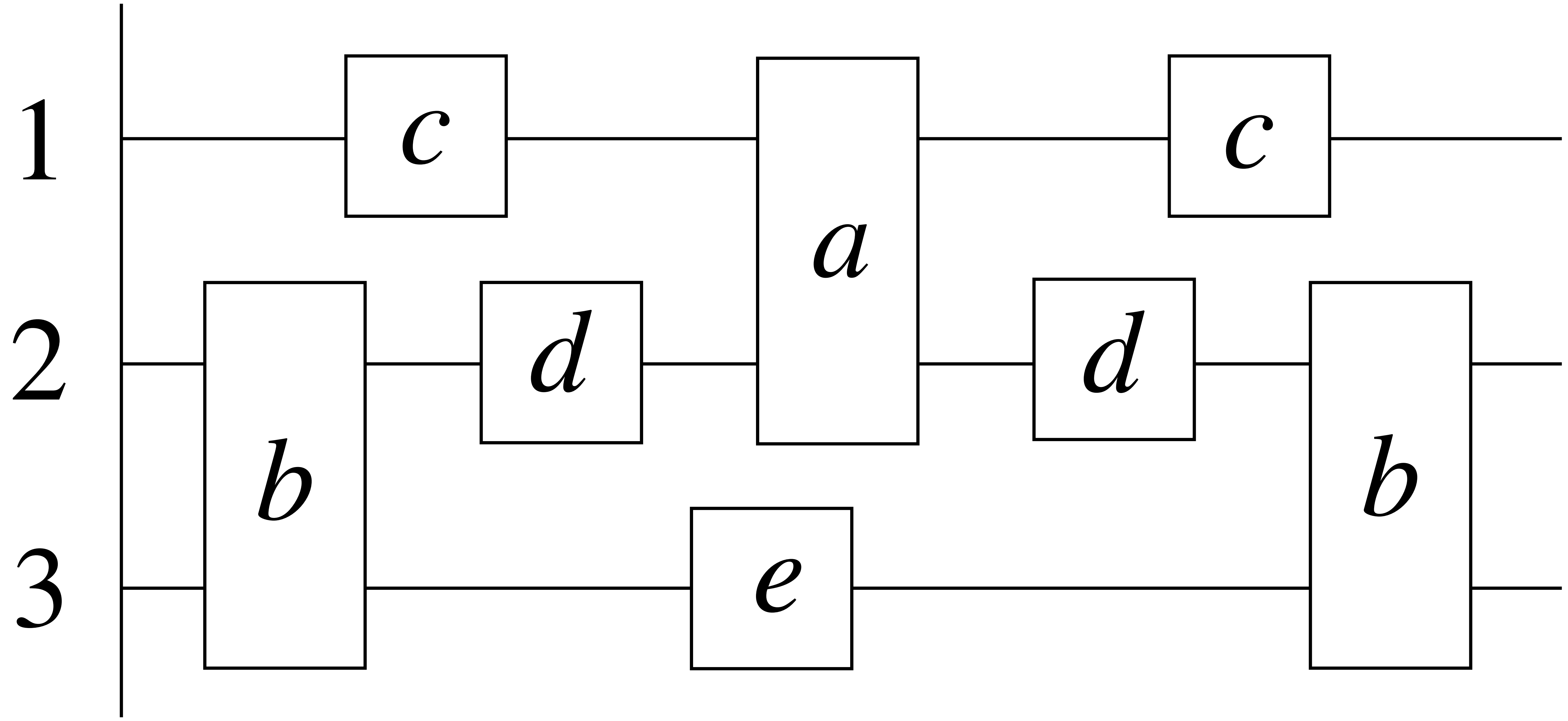
Decompositions depend crucially on

$Y_i \leq Y_j$, $Y_{i,k} \leq Y_j$ and $L_i \leq L_j$, $L_{i,k} \leq L_j$

Extended locPastPDL for Traces



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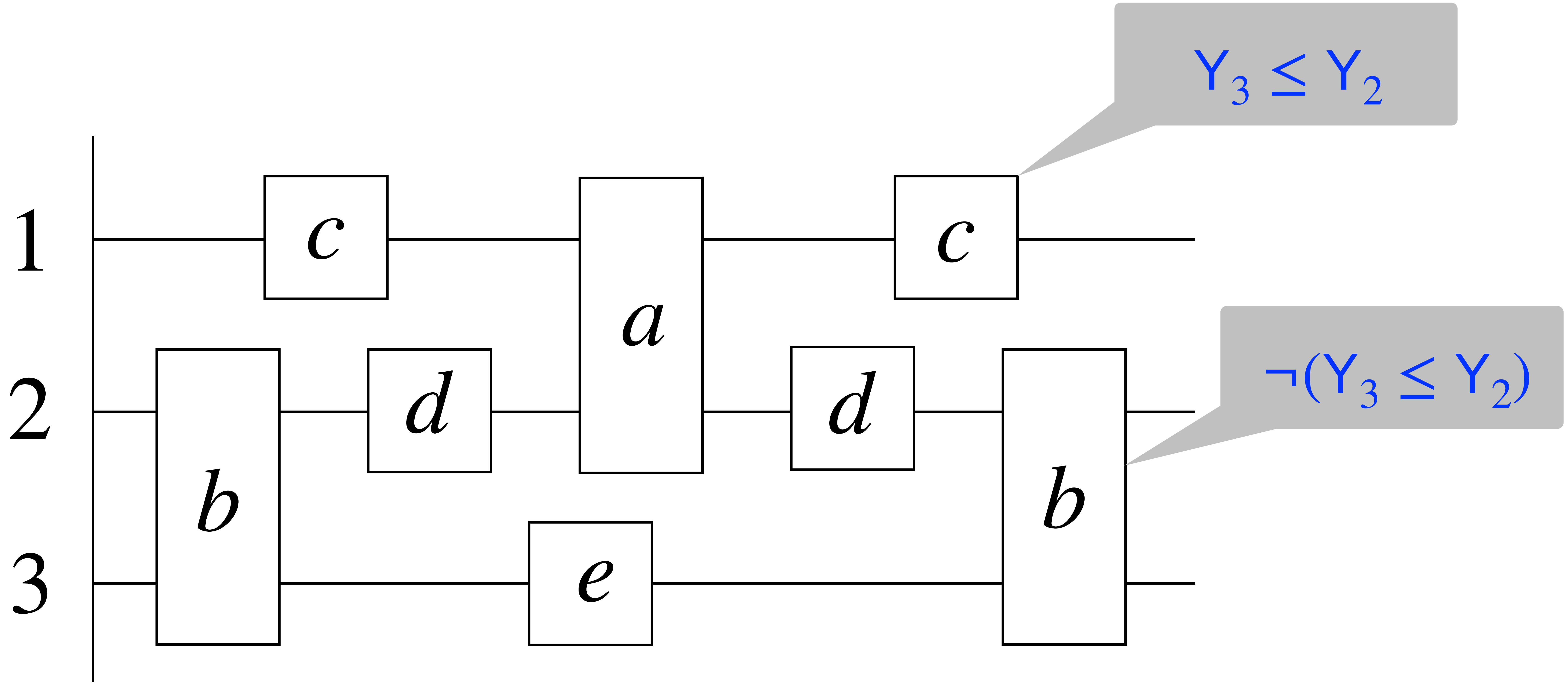

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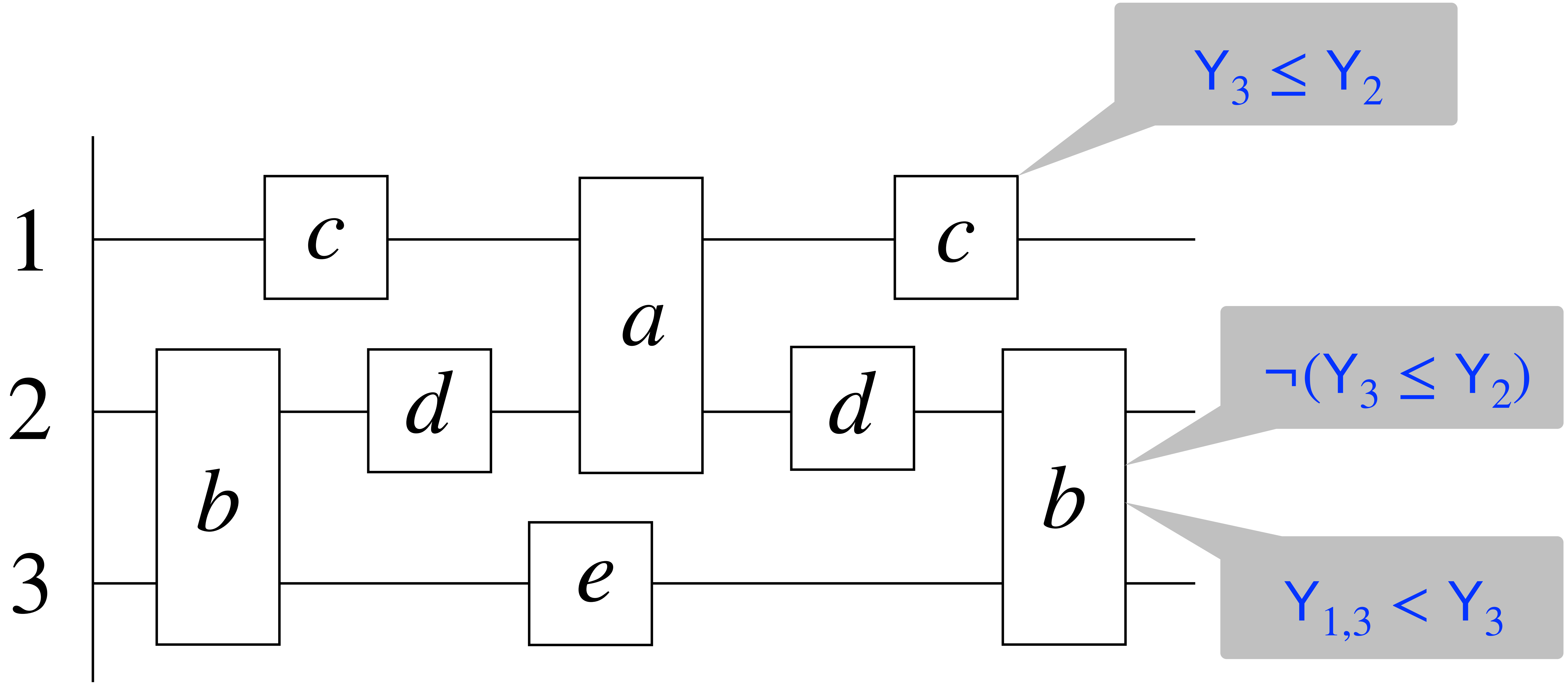

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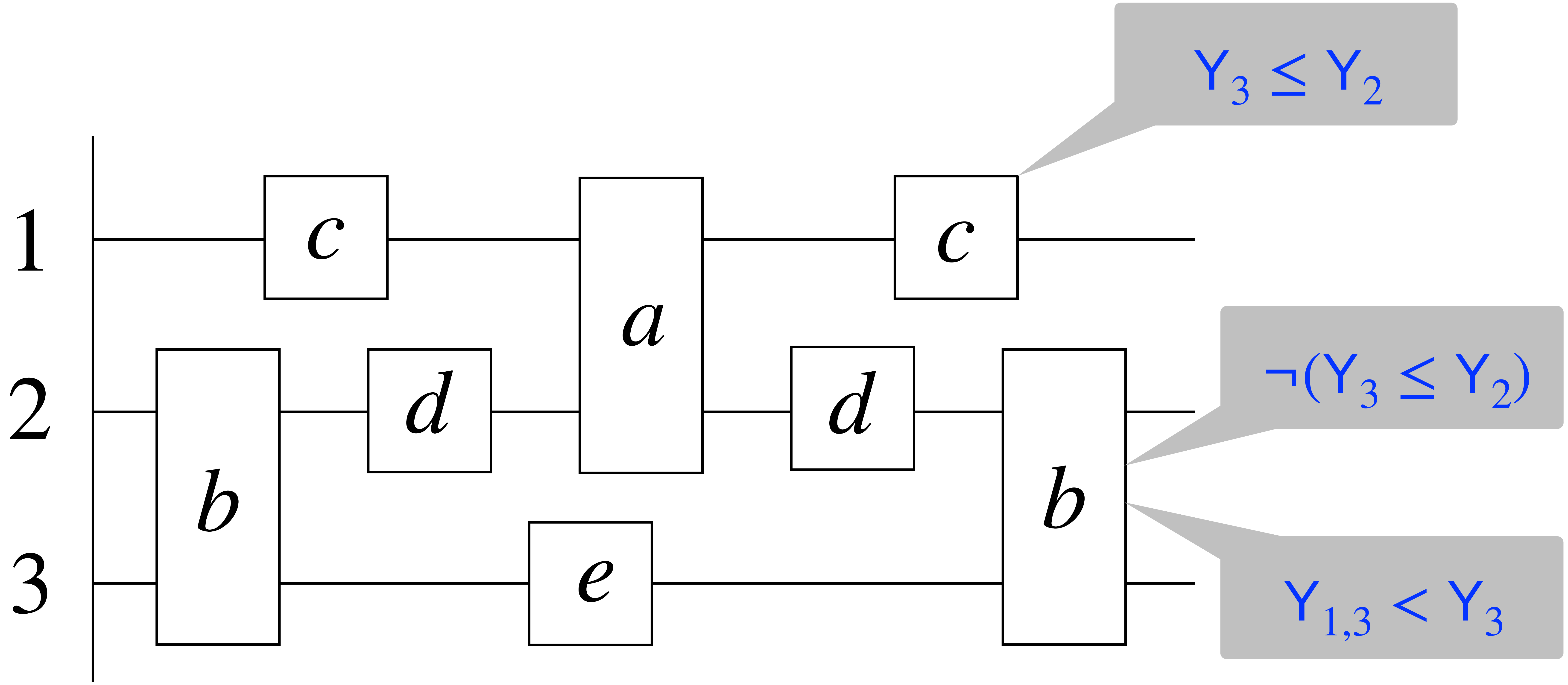

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Extended locPastPDL for Traces

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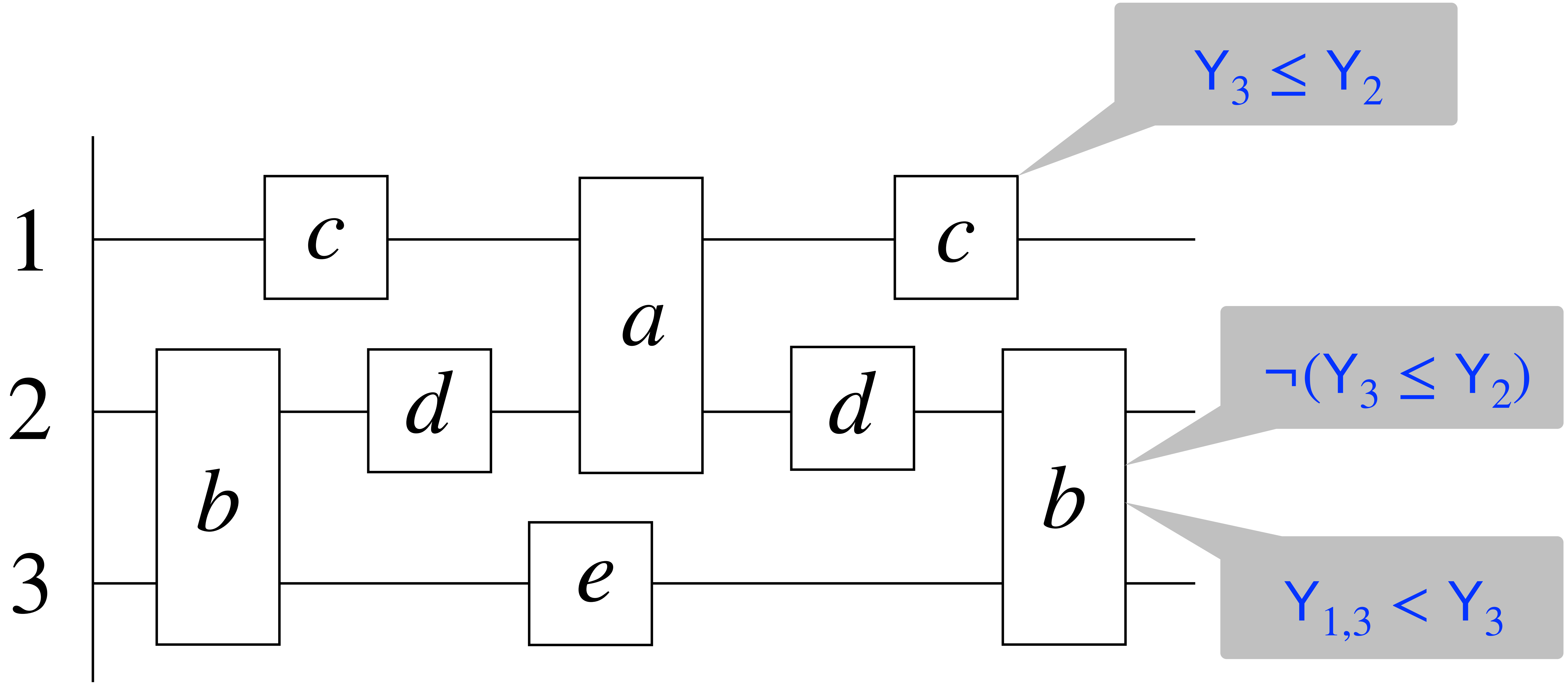

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Extended locPastPDL for Traces

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Theorem (Adsul, Gastin, Sarkar, Weil — CONCUR'22)

- Extended locPastPDL is expressively complete for regular trace languages

Extended locPastPDL for Traces

Theorem (Adsul, Gastin, Sarkar, Weil — CONCUR'22)

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Theorem (Mukund-Sohoni 1997)

There is an asynchronous letter-to-letter transducer $\mathscr{G}$ which computes the truth values of the constants from

$$\mathscr{Y} = \{\mathsf{Y}_i \leq \mathsf{Y}_j \, , \mathsf{Y}_{i,k} \leq \mathsf{Y}_j \ \mid \ i, j, k \in \mathscr{P}\}.$$

Extended locPastPDL for Traces

Theorem (Adsul, Gastin, Sarkar, Weil — CONCUR'22)

- Extended locPastPDL is expressively complete for regular trace languages
- Any regular trace language is accepted by a cascade product of the gossip transducer followed by a sequence of local asynchronous transducers:

$$\mathcal{G} \circ \mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_n$$

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Extended locPastPDL = locPastPDL

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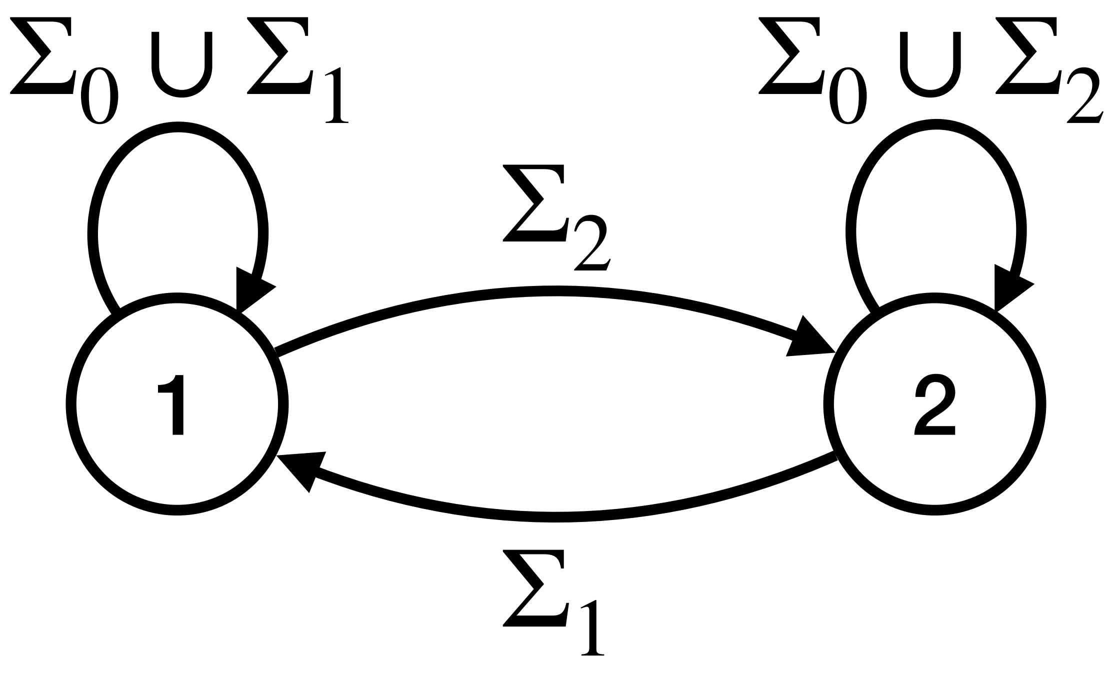
Aperiodic = FO-definable

Theorem [Adsul, Gastin, Sarkar, Weil — Concur'20, LMCS'22]

Any aperiodic (FO) trace language is accepted by a cascade product of the gossip transducer followed by a sequence of local reset transducers:

$$\mathcal{G} \circ \mathcal{U}_2 \circ \mathcal{U}_2 \circ \cdots \circ \mathcal{U}_2$$

Reset - $\mathcal{U}_2$



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Direct proof (Not using Krohn-Rhodes theorem)

based on a past temporal logic $\mathsf{LTL}\left(\mathsf{Y}_i \leq \mathsf{Y}_j, \mathsf{S}_i\right)$ proved expressively complete for FO

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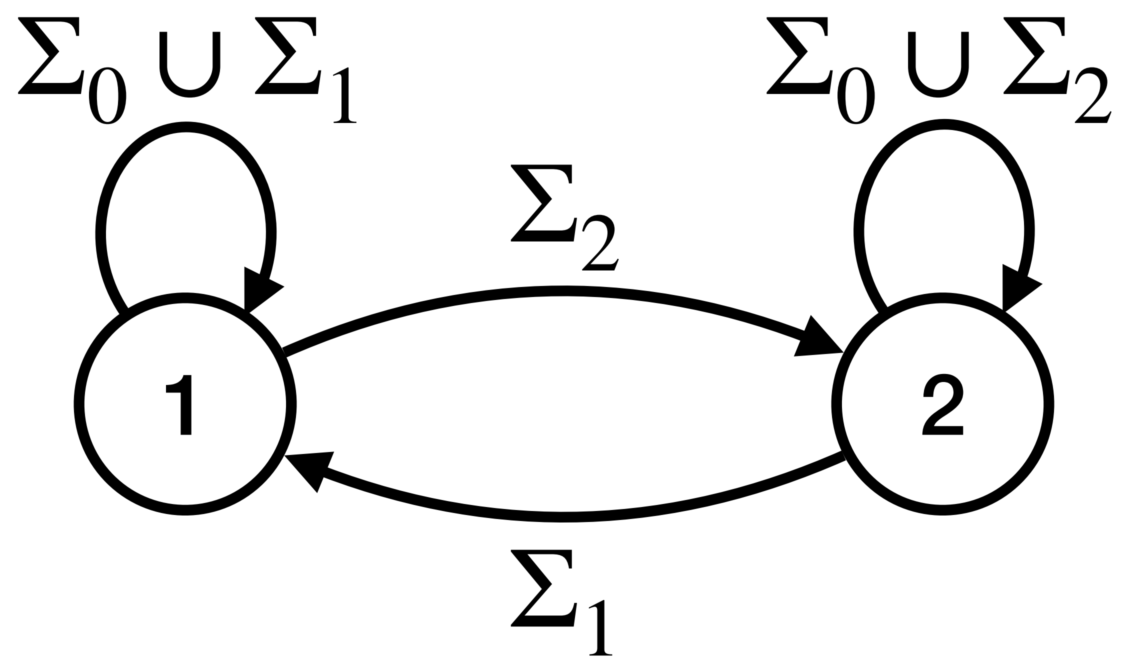

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- ✅ Labelling functions, sequential transducers and cascade product
- ✅ Krohn-Rhodes theorem for aperiodic/regular word languages
- ✅ Model of concurrency: Mazurkiewicz traces and asynchronous Zielonka automata
- ✅ Asynchronous labelling functions, transducers and cascade product
- ✅ Propositional dynamic logic for traces
- Conclusion

Conclusion

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Main results

- (Local) (past) Propositional dynamic logic expressively complete for regular trace languages
  - Specification language: natural, easy, expressive, good complexity

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Open problem

- Generalisation to other structures, eg, Message sequence charts & Message passing automata?

Thank you for your attention!

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