Greibach Normal Form for ω-Algebraic Systems and Weighted Simple ω-Pushdown Automata

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Abstract

In weighted automata theory, many classical results on formal languages have been extended into a quantitative setting. Here, we investigate weighted context-free languages of infinite words, a generalization of ω-context-free languages (Cohen, Gold 1977) and an extension of weighted context-free languages of finite words (Chomsky, Schützenberger 1963). As in the theory of formal grammars, these weighted languages, or ω-algebraic series, can be represented as solutions of mixed ω-algebraic systems of equations and by weighted ω-pushdown automata.

In our first main result, we show that mixed ω-algebraic systems can be transformed into *Greibach normal form*. Our second main result proves that *simple ω-reset pushdown automata* recognize all ω-algebraic series that are a solution of an ω-algebraic system in Greibach normal form. Simple reset automata do not use ϵ-transitions and can change the stack only by at most one symbol. These results generalize fundamental properties of context-free languages to weighted languages.

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1 Introduction

Context-free languages provide a fundamental concept for programming languages in computer science. In order to model quantitative properties, already in 1963, Chomsky and Schützenberger [3] introduced weighted context-free languages. The theory of weighted pushdown automata developed quickly; for background, we refer the reader to the survey [19] and the books [21, 20, 16, 10]. In 1977, Cohen and Gold [4] investigated context-free languages of infinite words. Weighted pushdown automata on infinite words were studied more recently by Ésik and Kuich [14].

The goal of this paper is the investigation of weighted context-free languages and weighted pushdown automata on infinite words. As in [20, 16], the weighted context-free languages of finite and infinite words are described by solutions of mixed ω-algebraic systems of equations. In our first main result, we show that these systems can be transformed into a Greibach normal form. In the literature, Greibach normal forms, central for context-free languages of

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finite words, have been established for ω-context-free languages (of infinite words), see [4], and also for algebraic systems of equations for series over finite words [20, 16]; this latter result is employed in our proof. Hence here we extend these classical results to a weighted version for infinite words.

In our second main result, we consider weighted *simple* pushdown automata. These automata do not use ϵ-transitions and utilize only three simple stack commands: popping a symbol, pushing a symbol or leaving the stack unaltered; moreover, it is only possible to read the topmost stack symbol by popping it. Observe that together with the restriction of not allowing ϵ-transitions, these restrictions for the actions on the stack are non-trivial. In our second main result we show that these weighted *simple* pushdown automata still recognize the weighted ω-context-free languages that are a solution of weighted ω-context-free grammars in Greibach normal form. Our proof uses two ingredients. First, weighted ω-pushdown automata are expressively equivalent to mixed ω-algebraic systems of equations, see [8, 9]. Secondly, we apply a recent corresponding expressive equivalence result for weighted simple pushdown automata on finite words from [7] to construct the required weighted simple ω-pushdown automata.

We believe the model of weighted simple ω-pushdown automata to be very natural. Similar expressivity equivalence results in the unweighted case hold for context-free languages of finite words, hidden in a proof by Blass and Gurevich [1], and also for ω-context-free languages, see [6].

After the preliminaries in the next section, Sections 3 and 4 contain our results on the Greibach normal form. Sections 5 and 6 describe weighted simple pushdown automata.

2 Preliminaries

For the convenience of the reader, we recall definitions and results from Ésik, Kuich [16].

A semiring S is called *complete* if it has "infinite sums" (i) that are an extension of the finite sums, (ii) that are associative and commutative and (iii) that satisfy the distribution laws (see Conway [5], Eilenberg [12], Kuich [19]).

A semiring S equipped with an additional unary star operation $^* : S \to S$ is called a *starsemiring*. In complete semirings for each element a, the *star* a^* of a is defined by

$$a^* = \sum_{j \geq 0} a^j \,.$$

Hence, each complete semiring is a starsemiring, called a *complete starsemiring*.

A semiring is called *continuous* if it is ordered, each directed subset has a least upper bound and addition and multiplication preserves the least upper bound of directed sets. Any continuous semiring is complete. See Ésik, Kuich [16] for background.

Suppose that S is a semiring and V is a commutative monoid written additively. We call V a (left) S-semimodule if V is equipped with a (left) action

$$S \times V \ \to \ V, \qquad (s, v) \ \mapsto \ sv$$

subject to the following rules:

$$\begin{aligned} & s(s'v) = (ss')v \,, \quad (s+s')v = sv + s'v \,, \quad s(v+v') = sv + sv' \,, \\ & 1v = v \,, \quad 0v = 0 \,, \quad s0 = 0 \,, \end{aligned}$$

for all $s, s' \in S$ and $v, v' \in V$. If V is an S-semimodule, we call (S, V) a *semiring-semimodule pair*.

Suppose that (S, V) is a semiring-semimodule pair such that S is a starsemiring and S and V are equipped with an omega operation $^{\omega} : S \to V$. Then we call (S, V) a *starsemiring-omegasemimodule pair*. A semiring-semimodule pair (S, V) is called *complete* if S is a complete semiring, V is a complete monoid and the left action of the semimodule is distributive; moreover, it is required that it has "infinite products" mapping infinite sequences over S to V such that the product (i) can be partitioned, (ii) can be extended from the left and (iii) is distributive (see Ésik, Kuich [17]).

Suppose that (S, V) is complete. Then we define

$$s^* = \sum_{i \geq 0} s^i \qquad \text{and} \qquad s^{\omega} = \prod_{i \geq 1} s\,,$$

for all $s \in S$. This turns (S, V) into a starsemiring-omegasemimodule pair. Observe that, if (S, V) is a complete semiring-semimodule pair, then $0^{\omega} = 0$.

A *star-omega semiring* is a semiring S equipped with unary operations * and $^{\omega} : S \to S$. A star-omega semiring S is called *complete* if (S, S) is a complete semiring-semimodule pair, i.e., if S is complete and is equipped with an infinite product operation that satisfies the three conditions stated above. A complete star-omega semiring S is called *continuous* if the semiring S is continuous.

For the definition of quemirings, we refer the reader to [16], page 110. Here we note that a quemiring T is isomorphic to a quemiring $S \times V$ determined by the semiring-semimodule pair (S, V); this is an algebraic structure with an addition given componentwise and a multiplication given by semiring multiplication in the first component and a semidirect product type addition in the second component (since S acts on V); cf. Elgot [13], Ésik, Kuich [16], page 109. Also, one can define a natural star operation on $S \times V$, see [16].

For an alphabet Σ, we call mappings r of Σ^* into S *series*. The collection of all such series r is denoted by $S\langle\langle \Sigma^* \rangle\rangle$. We call the set $\mathrm{supp}(r) = \{w \mid (r, w) \neq 0\}$ the support of a series r. We denote by $S\langle \Sigma \rangle$, $S\langle \{\epsilon\} \rangle$ and $S\langle \Sigma \cup \{\epsilon\} \rangle$ the series with support in Σ, $\{\epsilon\}$ and $\Sigma \cup \{\epsilon\}$, respectively. Mappings of Σ^{ω} into S are called *ω-series* and their collection is denoted by $S\langle\langle \Sigma^{\omega} \rangle\rangle$. See [20, 16] for more information. Examples of series in $S\langle \Sigma^* \rangle$ for a semiring $\langle S, +, \cdot, 0, 1 \rangle$ are 0, w, sw for $s \in S$ and $w \in \Sigma^*$, defined by

$$\begin{aligned} (0, w) &= 0 \text{ for all } w, \\ (w, w) &= 1 \text{ and } (w, w') = 0 \text{ for } w \neq w', \\ (sw, w) &= s \text{ and } (sw, w') = 0 \text{ for } w \neq w'. \end{aligned}$$

Consider a starsemiring-omegasemimodule pair (A, V). Following Bloom, Ésik [2], we define a matrix operation $^{\omega} : A^{n \times n} \to V^{n \times 1}$ on a starsemiring-omegasemimodule pair (A, V) as follows. If $n = 0$, M^{ω} is the unique element of V^0, and if $n = 1$, so that $M = (a)$, for some $a \in A$, $M^{\omega} = (a^{\omega})$. Assume now that $n > 1$ and write M as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where a, b, c and d are submatrices of M, called *blocks* of M. Then

$$M^{\omega} = \begin{pmatrix} (a + bd^*c)^{\omega} + (a + bd^*c)^* bd^{\omega} \\ (d + ca^*b)^{\omega} + (d + ca^*b)^* ca^{\omega} \end{pmatrix}.$$

Following Ésik, Kuich [15], we define matrix operations ${}^{\omega,k}\colon A^{n\times n} \to V^{n\times 1}$ for $0 \le k \le n$ as follows. Assume that $M \in A^{n\times n}$ is decomposed into blocks a, b, c, d as in (1), but with a of dimension $k \times k$ and d of dimension $(n-k)\times(n-k)$. Then

$$M^{\omega,k} = \begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \end{pmatrix}.$$

Observe that $M^{\omega,0} = 0$ and $M^{\omega,n} = M^\omega$. Intuitively, M can be interpreted as an adjacency matrix and $M^{\omega,k}$ are infinite paths where the first k states are repeated states, i.e., states that are Büchi-accepting.

▶ **Example 1.** Formal languages are covered by our model. Let $\langle \mathbb{B}, \vee, \wedge, 0, 1\rangle$ be the Boolean semiring. Then let $0^* = 1^* = 1$ and take infima as infinite products. This makes $\mathbb{B}$ a continuous star-omega and commutative semiring. It then follows that $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle$ is isomorphic to formal languages of finite and infinite words with the usual operations.

The semiring $\langle \mathbb{N}^\infty, +, \cdot, 0, 1\rangle$ with $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ and the natural infinite product operation of numbers is a continuous star-omega and commutative semiring.

The tropical semiring $\langle \mathbb{N}^\infty, \min, +, \infty, 0\rangle$ with the usual infinite sum operation as infinite product is a commutative semiring and a continuous star-omega semiring.

3 Mixed ω-Algebraic Systems

This and the next section describe the Greibach normal form for mixed ω-algebraic systems.

Throughout this paper, S *is a continuous, and therefore complete, star-omega semiring with the underlying semiring* S *being commutative*; and Σ denotes an alphabet.

By Theorem 5.5.5 of Ésik, Kuich [16], $(S\langle\langle\Sigma^*\rangle\rangle, S\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair, hence a Conway semiring-semimodule pair, satisfying $\epsilon^\omega = 0$ (for Conway semiring-semimodule pairs, cf. Ésik, Kuich [16], page 106). Hence, $S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$ is a generalized starquemiring.

In the sequel, x and z denote vectors of dimension n and m, respectively, i.e., $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_m)$. It will be clear from the context whether they are used as row or as column vectors. Similar conventions hold for vectors p, σ and τ. Moreover, X denotes the set of variables $\{x_1, \ldots, x_n\}$ for $S\langle\langle\Sigma^*\rangle\rangle$, while $\{z_1, \ldots, z_m\}$ is the set of variables for $S\langle\langle\Sigma^\omega\rangle\rangle$.

A *mixed ω-algebraic system over the quemiring* $S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$ consists of an algebraic system over $S\langle\langle\Sigma^*\rangle\rangle$

$$x = p(x), \qquad p \in (S\langle(\Sigma \cup X)^*\rangle)^{n\times 1}$$

and a linear system over $S\langle\langle\Sigma^\omega\rangle\rangle$

$$z = \varrho(x)z, \qquad \varrho \in (S\langle(\Sigma \cup X)^*\rangle)^{m\times m}.$$

The pair $(\sigma, \tau) \in (S\langle\langle\Sigma^*\rangle\rangle)^n \times (S\langle\langle\Sigma^\omega\rangle\rangle)^m$ is a *solution* of the mixed ω-algebraic system

$$x = p(x), \quad z = \varrho(x)z\,, \qquad \text{if} \qquad \sigma = p(\sigma), \quad \tau = \varrho(\sigma)\tau\,.$$

Observe that, by Theorem 5.5.7 of Ésik, Kuich [16], $\tau_k = \varrho(\sigma)^{\omega,k}$ for each $1 \le k \le m$ is a solution for the linear system $z = \varrho(\sigma)z$.

A solution $(\sigma_1, \ldots, \sigma_n)$ of the algebraic system $x = p(x)$ is termed *least solution* if

$$\sigma_i \le \tau_i, \qquad \text{for each } 1 \le i \le n,$$

for all solutions $(\tau_1, \ldots, \tau_n)$ of $x = p(x)$.

If σ is the least solution of $x = p(x)$, then $z = \varrho(\sigma)z$ is called an $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$-linear system and $(\sigma, \tau_k) = (\sigma, \varrho(\sigma)^{\omega,k})$, where $k \in \{0, 1, \ldots, m\}$, is called k^{th}-*canonical solution* of $x = p(x)$, $z = \varrho(x)z$. Each k^{th}-canonical solution is also called a *canonical solution*.

Recall that $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$ comprises the components of least solutions of algebraic systems

$$x_i = p_i, \quad (1 \leq i \leq n) \qquad \text{where } p_i \in S\langle(\Sigma \cup X)^*\rangle \text{ for } 1 \leq i \leq n\,.$$

We define $S^{\mathrm{alg}}\langle\langle\Sigma^\omega\rangle\rangle$ to be the collection of all components of vectors $M^{\omega,k}$, where $M \in (S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$, $n \geq 1$, and $k \in \{1, \ldots, n\}$. Moreover, ω-$\mathfrak{Rat}(S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle)$ is defined to be the ω-Kleene closure of (i.e., the generalized starquemiring generated by) $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$.

▶ **Example 2.** We consider the following mixed ω-algebraic system over the quemiring $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle \times \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$ for the tropical semiring $\langle\mathbb{N}^\infty, \min, +, \infty, 0\rangle$

$$\begin{aligned} x_1 &= 1ax_1b + 1ab \qquad & z_1 &= cz_1 \\ & & z_2 &= x_1z_1 + z_1 \end{aligned}$$

where $a, b, c \in \Sigma$ and using the natural number 1.

Then for the algebraic system $x = p(x)$ over $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle$, we get the least solution $\sigma = a^nb^n \mapsto n$. The first canonical solution of the mixed ω-algebraic system $x = p(x)$, $z = \varrho(x)z$ over $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle \times \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$ is then $(\sigma, c^\omega \mapsto 0, a^nb^nc^\omega \mapsto n)$. Hence the series $a^nb^nc^\omega \mapsto n$ is ω-algebraic but it is clearly not recognizable by a weighted automaton without stack.

Now we have the following characterization of algebraic and ω-algebraic series.

▶ **Theorem 3.** *Let S be a continuous complete star-omega semiring with the underlying semiring S being commutative and let Σ be an alphabet. Then the following statements are equivalent for $(s, \upsilon) \in S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$:*

(i) $(s, \upsilon) \in S^{alg}\langle\langle\Sigma^*\rangle\rangle \times S^{alg}\langle\langle\Sigma^\omega\rangle\rangle$,

(ii) $(s, \upsilon) \in \omega\text{-}\mathfrak{Rat}(S^{alg}\langle\langle\Sigma^*\rangle\rangle)$,

(iii) $(s, \upsilon) = \|\mathfrak{A}\|$, *where $\mathfrak{A}$ is a finite $S^{alg}\langle\langle\Sigma^*\rangle\rangle$-automaton over $S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$,*

(iv) $s \in S^{alg}\langle\langle\Sigma^*\rangle\rangle$ *and* $\upsilon = \sum_{1\leq j\leq l} s_jt_j^\omega$ *for some $l \geq 0$, where $s_j, t_j \in S^{alg}\langle\langle\Sigma^*\rangle\rangle$,*

(v) (s, υ) *is component of the automata-theoretic solution of an $S^{alg}\langle\langle\Sigma^*\rangle\rangle$-linear system over $S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$,*

(vi) (s, υ) *is component of the canonical solution of a mixed ω-algebraic system over $S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$.*

Proof. The statements (ii), (iii) and (iv) are equivalent by Theorem 5.4.9 (see also Theorem 5.6.6) of Ésik, Kuich [16]. ◀

4 Greibach Normal Form for Mixed ω-Algebraic Systems

In this section we show that for any element of $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle \times S^{\mathrm{alg}}\langle\langle\Sigma^\omega\rangle\rangle$ there exists a mixed ω-algebraic system in Greibach normal form such that this element is a component of a solution of this mixed ω-algebraic system. Similar to the definition for algebraic systems on finite words (cf. also Greibach [18]), a mixed ω-algebraic system

$$x = p(x), \quad z = \varrho(x)z$$

is in *Greibach normal form* if

$$\begin{aligned} &\mathrm{supp}(p_i(x)) \subseteq \{\epsilon\} \cup \Sigma \cup \Sigma X \cup \Sigma XX, \qquad && \text{for all } 1 \leq i \leq n, \qquad \text{and} \\ &\mathrm{supp}(\varrho_{ij}(x)) \subseteq \Sigma \cup \Sigma X, && \text{for all } 1 \leq i, j \leq m\,. \end{aligned}$$

For the construction of the Greibach normal form we need a corollary to Theorem 3.

▶ **Corollary 4.** *The following statement for* $(s, \upsilon) \in S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^\omega\rangle\rangle$ *is equivalent to the statements* (i) *to* (vi) *of Theorem 3:*
$s \in S^{alg}\langle\langle\Sigma^*\rangle\rangle$ *and* $\upsilon = \sum_{1\le j\le l} s_j t_j^\omega$ *for some* $l \ge 0$*, where* $s_j, t_j \in S^{alg}\langle\langle\Sigma^*\rangle\rangle$ *with* $(t_j, \epsilon) = 0$*; moreover* $(s_j, \epsilon) = 0$ *or* $s_j = (s_j, \epsilon)\epsilon$*.*

Proof. The proof is an easy case distinction. ◀

We now assume that $(s, \upsilon) \in S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle \times S^{\mathrm{alg}}\langle\langle\Sigma^\omega\rangle\rangle$ is given in the form of Corollary 4 with $l = 1$. By Theorem 2.4.10 of Ésik, Kuich [16], there exist algebraic systems in Greibach normal form whose first component of their least solutions equals s_1, t_1.

Firstly, we deal with the case $(s_1, \epsilon) = 0$. Let

$$x_i = p_i(x) + \sum_{1\le j\le n} p_{ij}(x)x_j, \quad \text{for each } 1 \le i \le n, \tag{$*$}$$

where $\mathrm{supp}(p_i(x)) \subseteq \Sigma \cup \Sigma X$, $\mathrm{supp}(p_{ij}(x)) \subseteq \Sigma X$, be the algebraic system in Greibach normal form for s_1 and

$$x'_i = p'_i(x') + \sum_{1\le j\le m} p'_{ij}(x')x'_j, \quad \text{for each } 1 \le i \le m, \tag{$**$}$$

where $\mathrm{supp}(p'_i(x')) \subseteq \Sigma \cup \Sigma X'$, $\mathrm{supp}(p_{ij}(x')) \subseteq \Sigma X'$, be the algebraic system in Greibach normal form for t_1. Let σ and σ' with $\sigma_1 = s_1$ and $\sigma'_1 = t_1$ be the least solutions of $(*)$ and $(**)$, respectively.

Consider now the mixed ω-algebraic system consisting of the algebraic system $(*)$, $(**)$ over $S\langle\langle\Sigma^*\rangle\rangle$ and the linear system over $S\langle\langle\Sigma^\omega\rangle\rangle$

$$\begin{aligned} z'' &= p'_1(x')z'' + \sum_{1\le j\le m} p'_{1j}(x')z'_j\,, \\ z'_i &= p'_i(x')z'' + \sum_{1\le j\le m} p'_{ij}(x')z'_j, \quad \text{for } 1 \le i \le m\,, \\ z_i &= p_i(x)z'' + \sum_{1\le j\le n} p_{ij}(x)z_j, \quad \text{for } 1 \le i \le n\,. \end{aligned} \tag{$***$}$$

Observe that the mixed ω-algebraic system is in Greibach normal form. We then order the variables of the mixed ω-algebraic system $(*)$, $(**)$, $(***)$ as $x_1, \ldots, x_n; x'_1, \ldots, x'_m; z''; z'_1, \ldots, z'_m; z_1, \ldots, z_n$. Observe that $\sigma'_1\sigma'^\omega_1 = \sigma'^\omega_1$.

The next lemma states that the system $(*)$, $(**)$, $(***)$ is the mixed ω-algebraic system in Greibach normal whose canonical solution indeed contains a component $\sigma_1\sigma'^\omega_1 = s_1 t_1^\omega$ as described in the statement of Corollary 4.

▶ **Lemma 5.** *The solution*

$$(\sigma_1, \ldots, \sigma_n; \sigma'_1, \ldots, \sigma'_m; \sigma'_1\sigma'^\omega_1; \sigma'_1\sigma'^\omega_1, \ldots, \sigma'_m\sigma'^\omega_1; \sigma_1\sigma'^\omega_1, \ldots, \sigma_n\sigma'^\omega_1) \tag{2}$$

is the first canonical solution of the mixed ω*-algebraic system* $(*)$*,* $(**)$*,* $(***)$*.*

Secondly, we deal with the case $s_1 = (s_1, \epsilon)\epsilon$. Consider now the mixed ω-algebraic system consisting of $(**)$ and the linear system over $S\langle\langle\Sigma^\omega\rangle\rangle$

$$\begin{aligned} z'' &= p'_1(x')z'' + \sum_{1\le j\le m} p'_{1j}(x')z'_j\,, \\ z'_i &= p'_i(x')z'' + \sum_{1\le j\le m} p'_{ij}(x')z'_j, \ 1 \le i \le m\,, \\ z_1 &= (s_1, \epsilon)p'_1(x')z'' + (s_1, \epsilon) \sum_{1\le j\le m} p'_{1j}(x')z'_j\,. \end{aligned} \tag{$****$}$$

▶ **Lemma 6.** *The solution*

$$(\sigma_1', \ldots, \sigma_m'; \sigma_1'\sigma_1'^{\omega}; \sigma_1'\sigma_1'^{\omega}, \ldots, \sigma_m'\sigma_1'^{\omega}; (s_1, \epsilon)\sigma_1'^{\omega}) \,. \tag{3}$$

is the first canonical solution of the mixed ω-algebraic system (**), (****).

We now consider general sums of series of the above form. The next lemma shows how to construct a mixed ω-algebraic system whose canonical solution is the sum of the canonical solutions of multiple mixed ω-algebraic systems as given in the Lemmas 5 and 6.

▶ **Lemma 7.** *Let $(s, \upsilon) \in S^{alg}\langle\langle\Sigma^*\rangle\rangle \times S^{alg}\langle\langle\Sigma^{\omega}\rangle\rangle$ be given in the form of Corollary 4. Then there exists a mixed ω-algebraic system in Greibach normal form such that υ is a component of its l-th canonical solution.*

Our first main result is the following.

▶ **Theorem 8.** *The following statement for $(s, \upsilon) \in S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^{\omega}\rangle\rangle$ is equivalent to the statements of Theorem 3:*
(s, υ) is component of a canonical solution of a mixed ω-algebraic system over $S\langle\langle\Sigma^*\rangle\rangle \times S\langle\langle\Sigma^{\omega}\rangle\rangle$ in Greibach normal form.

Proof. The above statement trivially implies statement (vi) of Theorem 3. By Corollary 4 and Lemma 7, the statements of Theorem 3 imply the above statement. ◀

5 Simple Reset Pushdown Automata

In this second part of the paper, we want to show that weighted ω-pushdown automata can be transformed into a simple form. The next section will prove this result for ω-algebraic series that are a component of a solution of an ω-algebraic system in Greibach normal form. For the proof, we will need the corresponding result for finite words as an intermediate step. This result, the expressive equivalence of algebraic series (of finite words) and (weighted) simple reset pushdown automata, has been established in [7]. We recall the construction of the weighted simple reset pushdown automata here for the convenience of the reader, as variants of these automata will be used in Section 6 for ω-algebraic series.

Following Kuich, Salomaa [20] and Kuich [19], we introduce pushdown transitions matrices. These matrices can be considered as adjacency matrices of graphs representing automata. A special form, the reset pushdown matrices, is used for pushdown automata starting with an empty stack and allowing the automaton to push onto the empty stack. Here, we are interested in simple reset pushdown matrices, introduced in [7]. This simple form allows the automaton only to push one symbol, to pop one symbol or to ignore the stack. The corresponding automata, the simple reset pushdown automata are a generalization of the unweighted automata used in [6]. They do not use ϵ-transitions and don't allow the inspection of the topmost stack symbol.

Let Γ be an alphabet, called *pushdown alphabet* and let $n \geq 1$. A matrix $\bar{M} \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is called a *pushdown matrix* (with *pushdown alphabet* Γ and *state set* $\{1, \ldots, n\}$) if

(i) for each $p \in \Gamma$ there exist only finitely many blocks $\bar{M}_{p,\pi}$, $\pi \in \Gamma^*$, that are unequal to 0;

(ii) for all $\pi_1, \pi_2 \in \Gamma^*$,

$$\bar{M}_{\pi_1,\pi_2} = \begin{cases} \bar{M}_{p,\pi}, & \text{if there exist } p \in \Gamma, \pi, \pi' \in \Gamma^* \text{ with } \pi_1 = p\pi' \text{ and } \pi_2 = \pi\pi', \\ 0, & \text{otherwise.} \end{cases}$$

Intuitively, the infinite pushdown matrix $\bar{M}$ is (ii) fully represented only by the blocks $\bar{M}_{p,\pi}$ where $p \in \Gamma$, $\pi \in \Gamma^*$ and (i) only finitely many such blocks are nonzero.

A matrix $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is called *row-finite* if $\{\pi' \mid M_{\pi,\pi'} \neq 0\}$ is finite for all $\pi \in \Gamma^*$. Let Γ be a pushdown alphabet and $\{1,\dots,n\}$, $n \geq 1$, be a set of states. A *reset matrix* $M_R \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is a row-finite matrix such that

$$(M_R)_{\pi_1,\pi_2} = 0 \qquad \text{for } \pi_1,\pi_2 \in \Gamma^* \text{ with } \pi_1 \neq \epsilon\,.$$

A *reset pushdown matrix* $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is the sum $M = M_R + \bar{M}$ of a reset matrix M_R and a pushdown matrix $\bar{M}$.

Intuitively, a reset pushdown matrix is similar to a pushdown matrix with the additional possibility to push onto the empty stack, i.e., $M_{\epsilon,\pi}$ is allowed to be nonzero. Note that reset pushdown matrices are still finitely represented because of the row-finiteness.

A reset pushdown matrix M is called *simple* if $M \in \left((S\langle\Sigma\rangle)^{n\times n}\right)^{\Gamma^*\times\Gamma^*}$ for some $n \geq 1$, and for all $p, p_1 \in \Gamma$,

$$M_{p,\epsilon},\ M_{p,p} = M_{\epsilon,\epsilon} \text{ and } M_{p,p_1p} = M_{\epsilon,p_1}\,,$$

are the only blocks $M_{\pi,\pi'}$, where $\pi \in \{\epsilon, p\}$ and $\pi' \in \Gamma^*$, that may be unequal to the zero matrix 0.

Hence, a simple reset pushdown matrix M is defined by its blocks $M_{\epsilon,\epsilon}$ and $M_{p,\epsilon}$, $M_{\epsilon,p}$ ($p \in \Gamma$). Intuitively, the automata will only be allowed to ignore the stack (modeled by $M_{\epsilon,\epsilon}$), pop one symbol ($M_{p,\epsilon}$) or push one symbol ($M_{\epsilon,p}$). Note also that the matrix $M \in ((S\langle\Sigma\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$ forbids ϵ-transitions. Moreover, the equalities $M_{p,p} = M_{\epsilon,\epsilon}$ and $M_{p,p_1p} = M_{\epsilon,p_1}$ imply that the next transition does not depend on the topmost symbol of the stack except when popping it (modeled by $M_{p,\epsilon}$). An example of a simple reset pushdown matrix can be found in Example 14.

A *reset pushdown automaton* (with input alphabet Σ) $\mathfrak{A} = (n, \Gamma, I, M, P)$ is given by

- a *set of states* $\{1,\dots,n\}$, $n \geq 1$,
- a *pushdown alphabet* Γ,
- a reset pushdown matrix $M \in ((S\langle\Sigma \cup \{\epsilon\}\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$ called *transition matrix*,
- a row vector $I \in (S\langle\{\epsilon\}\rangle)^{1\times n}$, called *initial state vector*,
- a column vector $P \in (S\langle\{\epsilon\}\rangle)^{n\times 1}$, called *final state vector*.

The *behavior* $\|\mathfrak{A}\|$ of a reset pushdown automaton $\mathfrak{A}$ is defined by

$$\|\mathfrak{A}\| = I(M^*)_{\epsilon,\epsilon}P\,.$$

A reset pushdown automaton $\mathfrak{A} = (n, \Gamma, I, M, P)$ is called *simple* if M is a simple reset pushdown matrix. Example 14 shows a simple ω-reset pushdown automaton.

Given a series $r \in S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$, we want to construct a simple reset pushdown automaton with behavior r. By Theorems 5.10 and 5.4 of [19], r is a component of the unique solution of a strict algebraic system in Greibach normal form.

We only consider the algebraic series r with $(r, \epsilon) = 0$; cf. [7] for the other case. So we assume without loss of generality that r is the x_1-component of the unique solution of the algebraic system (4) with variables $x_1,\dots,x_n$

$$x_i = p_i,\ \ 1 \leq i \leq n,$$

of the form

$$x_i = \sum_{1\leq j,k\leq n}\sum_{a\in\Sigma}(p_i, ax_jx_k)ax_jx_k + \sum_{1\leq j\leq n}\sum_{a\in\Sigma}(p_i, ax_j)ax_j + \sum_{a\in\Sigma}(p_i, a)a\,. \tag{4}$$

As shown in [7], we can construct the simple reset pushdown automaton $\mathfrak{A}_s = (n+1, \Gamma, I_s, M, P)$, $1 \leq s \leq n$, with $r = \|\mathfrak{A}_1\|$ as follows:
We let $\Gamma = \{x_1, \ldots, x_n\}$; we also denote the state $n+1$ by f; the entries of M of the form $(M_{x_k,x_k})_{i,j}$, $(M_{x_k,\epsilon})_{i,j}$, $(M_{\epsilon,x_k})_{i,j}$, $(M_{\epsilon,\epsilon})_{i,j}$, $(M_{\epsilon,\epsilon})_{i,f}$, where $1 \leq i, j, k \leq n$, that may be unequal to 0 are

$$(M_{\epsilon,x_k})_{i,j} = \sum_{a\in\Sigma} (p_i, a x_j x_k) a\,,$$
$$(M_{x_k,x_k})_{i,j} = (M_{\epsilon,\epsilon})_{i,j} = \sum_{a\in\Sigma} (p_i, a x_j) a\,,$$
$$(M_{x_k,\epsilon})_{i,k} = (M_{x_k,x_k})_{i,f} = (M_{\epsilon,\epsilon})_{i,f} = \sum_{a\in\Sigma} (p_i, a) a\,;$$

we further put $(I_s)_s = \epsilon, (I_s)_i = 0$ for $1 \leq i \leq s-1$ and $s+1 \leq i \leq n+1$; finally let $P_f = \epsilon$ and $P_j = 0$ for $1 \leq j \leq n$;

The following motivation will be essential for our later construction for ω-pushdown automata. Intuitively, the variables in the algebraic system are simulated by states in the simple reset pushdown automaton $\mathfrak{A}_s$. By the Greibach normal form, only two variables on the right-hand side are allowed. The first is modeled directly by changing the state, the second is pushed to the pushdown tape and the state is changed to it later when the variable is popped again. The special final state f will only be used as the last state.

Note that $(M_{x_k,x_k})_{i,f}$ allows the automaton to change to the final state with a non-empty pushdown tape. This is an artificial addition to fit the definition of simple reset pushdown matrices. If the simple reset automaton is not popping a symbol from the pushdown tape, it cannot distinguish between different pushdown states. Even though the automaton can enter the final state too early, it can not continue from there as it is a sink.

Observe that $\|\mathfrak{A}_s\| = ((M^*)_{\epsilon,\epsilon})_{s,f}$ for all $1 \leq s \leq n$.

This simple reset pushdown matrix M is called the simple pushdown matrix *induced* by the Greibach normal form (4). The simple reset pushdown automata $\mathfrak{A}_s$, $1 \leq s \leq n$, are called the simple reset pushdown automata *induced* by the Greibach normal form (4).

The following (main) theorem of [7] states that the behavior of the simple reset pushdown automata induced by the Greibach normal form (4) is the unique solution of the original algebraic system (4).

▶ **Theorem 9** (Theorem 11 of [7]). *The unique solution of the algebraic system* (4) *is*

$$(\|\mathfrak{A}_1\|, \ldots, \|\mathfrak{A}_n\|) = (((M^*)_{\epsilon,\epsilon})_{1,f}, \ldots, ((M^*)_{\epsilon,\epsilon})_{n,f})\,.$$

▶ **Corollary 10** (Corollary 12 of [7]). *Let $r \in S^{alg}\langle\langle\Sigma^*\rangle\rangle$. Then there exists a simple reset pushdown automaton with behavior r.*

6 Simple ω-Reset Pushdown Automata

This section will prove that simple ω-reset pushdown automata can be obtained from ω-algebraic systems in Greibach normal form. We first prove some results for infinite applications of simple reset pushdown matrices. Then we introduce simple ω-reset pushdown automata and the main theorem will show that they can recognize all ω-algebraic series that are solutions of ω-algebraic systems in Greibach normal form.

In the sequel, (S, V) *is a complete semiring-semimodule pair.*

We will use sets P_l comprising infinite sequences over $\{1, \ldots, n\}$ as defined in [8]:

$$P_l = \{(j_1, j_2, \ldots) \in \{1, \ldots, n\}^\omega \mid j_t \leq l \text{ for infinitely many } t \geq 1\}\,.$$

Observe the following summation identity: Assume that $A_1, A_2, \ldots$ are matrices in $S^{n\times n}$. Then for $0 \leq l \leq n$, $1 \leq j \leq n$, and $m \geq 1$, we have

$$\sum_{(j_1,j_2,\ldots)\in P_l} (A_1)_{j,j_1}(A_2)_{j_1,j_2}\cdots = \sum_{1\leq j_1,\ldots,j_m\leq n} (A_1)_{j,j_1}\cdots(A_m)_{j_{m-1},j_m} \sum_{(j_{m+1},j_{m+2},\ldots)\in P_l} (A_{m+1})_{j_m,j_{m+1}}\cdots\,.$$

By Theorem 5.5.1 of Ésik, Kuich [16] we obtain, for a finite matrix A and for $0 \leq l \leq n$, the equality $AA^{\omega,l} = A^{\omega,l}$. By Theorem 6 of Droste, Ésik, Kuich [8], we have a similar result for pushdown matrices. We will now show the same equality for a reset pushdown matrix M.

▶ **Theorem 11.** *Let (S, V) be a complete semiring-semimodule pair and $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a reset pushdown matrix. Then*
- **(i)** $M^{\omega,l} = MM^{\omega,l}$*, for each* $0 \leq l \leq n$*,*
- **(ii)** $(M^\omega)_p = (\bar{M}^\omega)_p + (\bar{M}^*)_{p,\epsilon}(M^\omega)_\epsilon$*, for any* $p \in \Gamma$*,*
- **(iii)** $(M^{\omega,l})_p = (\bar{M}^{\omega,l})_p + (\bar{M}^*)_{p,\epsilon}(M^{\omega,l})_\epsilon$*, for each* $0 \leq l \leq n$ *and* $p \in \Gamma$*.*

Proof.
- **(i)** The proof is similar to the proof of Theorem 6 of [8] but we also need to handle empty pushdown tapes.
- **(ii)** We obtain, for $p \in \Gamma$,

$$\begin{aligned}(M^\omega)_p &= \sum_{\pi_1,\pi_2,\cdots\in\Gamma^+} M_{p,\pi_1}M_{\pi_1,\pi_2}\cdots + \sum_{t\geq1}\sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^+} M_{p,\pi_1}\cdots M_{\pi_{t-1},\epsilon}(M^\omega)_\epsilon \\ &= (\bar{M}^\omega)_p + \sum_{t\geq1}(\bar{M}^t)_{p,\epsilon}(M^\omega)_\epsilon \\ &= (\bar{M}^\omega)_p + (\bar{M}^*)_{p,\epsilon}(M^\omega)_\epsilon\,.\end{aligned}$$

- **(iii)** The proof is similar to the proof of (ii) but more technical as it needs to consider the repeated states. ◀

▶ **Lemma 12.** *Let (S, V) be a complete semiring-semimodule pair. Let M be a simple reset pushdown matrix. Then,*
- **(i)** $(M^\omega)_p = (M^\omega)_\epsilon + (M^*)_{\epsilon,\epsilon}M_{p,\epsilon}(M^\omega)_\epsilon$ *for* $p \in \Gamma$*,*
- **(ii)** $(M^{\omega,l})_p = (M^{\omega,l})_\epsilon + (M^*)_{\epsilon,\epsilon}(M_{p,\epsilon})(M^{\omega,l})_\epsilon$ *for each* $0 \leq l \leq n$ *and* $p \in \Gamma$*.*

Proof. Only (i): We obtain, for $p \in \Gamma$,

$$\begin{aligned}(M^\omega)_p &= \sum_{\pi_1,\pi_2,\cdots\in\Gamma^*} M_{p,\pi_1}M_{\pi_1,\pi_2}\cdots \\ &= \sum_{\pi_1,\pi_2,\cdots\in\Gamma^*} M_{p,\pi_1 p}M_{\pi_1 p,\pi_2 p}\cdots + \sum_{t\geq0}\sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} M_{p,\pi_1 p}\cdots M_{\pi_{t-1}p,p}M_{p,\epsilon}(M^\omega)_\epsilon \\ &= (M^\omega)_\epsilon + \Big(\sum_{t\geq0}\sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} M_{\epsilon,\pi_1}\cdots M_{\pi_{t-1},\epsilon}\Big)M_{p,\epsilon}(M^\omega)_\epsilon \\ &= (M^\omega)_\epsilon + \sum_{t\geq0}(M^t)_{\epsilon,\epsilon}M_{p,\epsilon}(M^\omega)_\epsilon \\ &= (M^\omega)_\epsilon + (M^*)_{\epsilon,\epsilon}M_{p,\epsilon}(M^\omega)_\epsilon\,.\end{aligned}$$ ◀

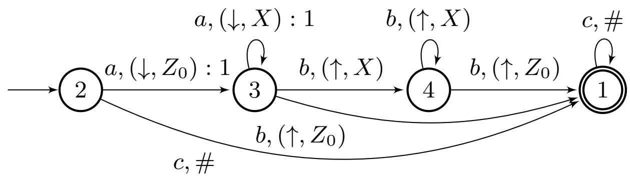

Figure 1 Example 14: Weighted simple ω-pushdown automaton, where $(\downarrow, X)$ means push symbol X, $(\uparrow, X)$ means pop X, and $\#$ leaves the stack unaltered. All shown transitions have a weight equal to the natural number 0 except the two transitions reading letter a and pushing a symbol onto the stack that have weight 1. All other possible transitions have weight ∞.

▶ **Lemma 13.** *Let M be induced by the Greibach normal form (4). Then, for all $1 \le j, k \le n$ and $0 \le l \le n$,*

$$((M^{\omega,l})_{x_k})_j = ((M^{\omega,l})_\epsilon)_j + ((M^*)_{\epsilon,\epsilon})_{j,f}((M^{\omega,l})_\epsilon)_k \,.$$

Proof. By Lemma 12(ii), we have

$$\begin{aligned}((M^{\omega,l})_{x_k})_j &= \left[(M^{\omega,l})_\epsilon + (M^*)_{\epsilon,\epsilon}(M_{x_k,\epsilon})(M^{\omega,l})_\epsilon\right]_j \\ &= ((M^{\omega,l})_\epsilon)_j + \left[(M^*)_{\epsilon,\epsilon}(M_{x_k,\epsilon})(M^{\omega,l})_\epsilon\right]_j\end{aligned}$$

Then for $1 \le j, k \le n$, we have

$$\begin{aligned}\left((M^*)_{\epsilon,\epsilon}(M_{x_k,\epsilon})(M^{\omega,l})_\epsilon\right)_j &= \sum_{1\le t_1,t_2\le f} ((M^*)_{\epsilon,\epsilon})_{j,t_1}(M_{x_k,\epsilon})_{t_1,t_2}((M^{\omega,l})_\epsilon)_{t_2} \\ &= \sum_{1\le t_1\le f} ((M^*)_{\epsilon,\epsilon})_{j,t_1}(M_{\epsilon,\epsilon})_{t_1,f}((M^{\omega,l})_\epsilon)_k \\ &= ((M^*)_{\epsilon,\epsilon}M_{\epsilon,\epsilon})_{j,f}((M^{\omega,l})_\epsilon)_k \\ &= ((M^*)_{\epsilon,\epsilon})_{j,f}((M^{\omega,l})_\epsilon)_k \,.\end{aligned}$$

The second equality holds because we defined $(M_{x_k,\epsilon})_{t_1,t_2} = 0$ for $t_2 \ne k$ and $(M_{x_k,\epsilon})_{t_1,k} = (M_{\epsilon,\epsilon})_{t_1,f}$ for induced simple pushdown matrices. The result follows. ◀

Next, an ω-reset pushdown automaton

$$\mathfrak{A} = (n, \Gamma, I, M, P, l)$$

is given by a reset pushdown automaton (n, Γ, I, M, P) and an integer l with $0 \le l \le n$, which indicates that $1, \ldots, l$ are the repeated states of $\mathfrak{A}$. The behavior $\|\mathfrak{A}\|$ of this ω-reset pushdown automaton $\mathfrak{A}$ is defined by

$$\|\mathfrak{A}\| = I(M^*)_{\epsilon,\epsilon}P + I(M^{\omega,l})_\epsilon \,.$$

The ω-reset pushdown automaton $\mathfrak{A} = (n, \Gamma, I, M, P, l)$ is called *simple* if M is a simple reset pushdown matrix.

▶ **Example 14.** Figure 1 shows a simple ω-reset pushdown automaton $\mathcal{A} = (4, \Gamma, I, M, P, 1)$ over the quemiring $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle{\times}\mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$ for the tropical semiring $\langle\mathbb{N}^\infty, \min, +, \mathbb{0} = \infty, \mathbb{1} = 0\rangle$ with $\Sigma = \{a, b, c\}$, $\Gamma = \{Z_0, X\}$, $I_2 = 0$, $I_i = \infty$ for $i \ne 2$ and $P_i = \infty$ for all $1 \le i \le 4$. The adjacency matrix M of the automaton is a simple reset pushdown matrix. As an indication, M is defined with $(M_{\epsilon,\epsilon})_{1,1} = (M_{\epsilon,\epsilon})_{2,1} = 0c$, $(M_{\epsilon,Z_0})_{2,3} = 1a$, etc., resulting in e.g.,

$$M_{\epsilon,\epsilon} = \begin{pmatrix} 0c & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 0c & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \end{pmatrix} \text{ and finally } M = \begin{pmatrix} M_{\epsilon,\epsilon} & M_{\epsilon,Z_0} & M_{\epsilon,X} & \cdots \\ M_{Z_0,\epsilon} & M_{\epsilon,\epsilon} & \mathbb{0} & \cdots \\ M_{X,\epsilon} & \mathbb{0} & M_{\epsilon,\epsilon} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the excluded part of M can be derived from the rules of pushdown and simple reset pushdown matrices. The automaton $\mathcal{A}$ has the behavior $a^n b^n c^\omega \mapsto n$, similar to the mixed ω-algebraic system in Example 2.

Now, for a series $r \in S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle \times S^{\mathrm{alg}}\langle\langle\Sigma^\omega\rangle\rangle$, we want to construct a simple ω-reset pushdown automaton with behavior r. For our construction, r must be a component of a solution of an ω-algebraic system in Greibach normal form. An ω-algebraic system consists of only one system over the quemiring variables $\{y_1, \ldots, y_n\}$. See [16], pp. 136 for details.

Similar to the definition for mixed ω-algebraic systems, an ω-algebraic system

$$y = p(y)$$

is in *Greibach normal form* if

$$\mathrm{supp}(p_i(y)) \subseteq \{\epsilon\} \cup \Sigma \cup \Sigma Y \cup \Sigma YY, \qquad \text{for all } 1 \leq i \leq n\,.$$

Let r be a component of a solution of the ω-algebraic system (5) in Greibach normal form over the complete semiring-semimodule pair (S, V), i.e., over the quemiring $S \times V$,

$$y_i = \sum_{1 \leq j,k \leq n} \sum_{a \in \Sigma} (p_i, a y_j y_k) a y_j y_k + \sum_{1 \leq j \leq n} \sum_{a \in \Sigma} (p_i, a y_j) a y_j + \sum_{a \in \Sigma} (p_i, a) a\,. \tag{5}$$

The variables of this system are y_i, $(1 \leq i \leq n)$; they are variables for (S, V). The system (5) induces the following mixed ω-algebraic system:

$$x_i = \sum_{1 \leq j,k \leq n} \sum_{a \in \Sigma} (p_i, a y_j y_k) a x_j x_k + \sum_{1 \leq j \leq n} \sum_{a \in \Sigma} (p_i, a y_j) a x_j + \sum_{a \in \Sigma} (p_i, a) a\,, \tag{4}$$

and

$$z_i = \sum_{1 \leq j,k \leq n} \sum_{a \in \Sigma} (p_i, a y_j y_k) a (z_j + x_j z_k) + \sum_{1 \leq j \leq n} \sum_{a \in \Sigma} (p_i, a y_j) a z_j\,. \tag{6}$$

Let now, for $1 \leq s \leq n$ and $0 \leq l \leq n$,

$$\mathfrak{A}_s^l = (n+1, \Gamma, I_s, M, P, l)$$

be the simple ω-reset pushdown automata such that $(n+1, \Gamma, I_s, M, P)$, for $1 \leq s \leq n$, are induced by the Greibach normal form (4).

The following theorem states that the induced simple ω-reset pushdown automata behave similar to the solution of system (5). Note that in the semimodule part $(M^{\omega,l})_\epsilon$ of the behavior, state f will never be reached.

▶ **Theorem 15.** *Let (S, V) be a complete semiring-semimodule pair. Let the simple ω-reset pushdown automata $\|\mathfrak{A}_s^l\|$ for $1 \leq s \leq n$ and $0 \leq l \leq n$ be induced by the Greibach normal form (4). Then, for $0 \leq l \leq n$,*

$$(\|\mathfrak{A}_1^l\|, \ldots, \|\mathfrak{A}_n^l\|) = \big(((M^*)_{\epsilon,\epsilon})_{1,f} + ((M^{\omega,l})_\epsilon)_1, \ldots, ((M^*)_{\epsilon,\epsilon})_{n,f} + ((M^{\omega,l})_\epsilon)_n\big)$$

is a solution of (5).

Proof. By Theorem 9, $(((M^*)_{\epsilon,\epsilon})_{1,f}, \ldots, ((M^*)_{\epsilon,\epsilon})_{n,f})$ is a solution of (4). We show that $(((M^{\omega,l})_\epsilon)_1, \ldots, ((M^{\omega,l})_\epsilon)_n)$ is a solution of (6) and substitute it into the right sides of (6):

$$\begin{aligned}
&\sum_{1\leq j,k\leq n}(M_{\epsilon,y_k})_{i,j}\Big(((M^{\omega,l})_\epsilon)_j + ((M^*)_{\epsilon,\epsilon})_{j,f}((M^{\omega,l})_\epsilon)_k\Big) + \sum_{1\leq j\leq n}(M_{\epsilon,\epsilon})_{i,j}((M^{\omega,l})_\epsilon)_j \\
&= \sum_{1\leq j,k\leq n} (M_{\epsilon,y_k})_{i,j}((M^{\omega,l})_{y_k})_j + \sum_{1\leq j\leq n} (M_{\epsilon,\epsilon})_{i,j}((M^{\omega,l})_\epsilon)_j \\
&= \sum_{1\leq k\leq n} (M_{\epsilon,y_k}(M^{\omega,l})_{y_k})_i + (M_{\epsilon,\epsilon}(M^{\omega,l})_\epsilon)_i \\
&= ((MM^{\omega,l})_\epsilon)_i = ((M^{\omega,l})_\epsilon)_i, \qquad \text{for each } 1\leq i\leq n\,.
\end{aligned}$$

The first equality is by Lemma 13, the last equality by Theorem 11(i). The result follows. ◀

The following is now immediate by Theorem 15 and our previous discussion.

▶ **Corollary 16.** *Let $r \in S^{alg}\langle\langle\Sigma^*\rangle\rangle \times S^{alg}\langle\langle\Sigma^\omega\rangle\rangle$ such that r is a component of a solution of an ω-algebraic system in Greibach normal form. Then there exists a simple ω-reset pushdown automaton with behavior r.*

7 Discussion

We have extended the characterization of ω-algebraic series so that we can use the ω-Kleene closure to transfer the property of Greibach normal form from algebraic systems to mixed ω-algebraic ones. This generalizes a fundamental property from context-free languages.

We believe that the same technique can be used to transfer other properties of algebraic systems to infinite words. Cohen, Gold [4] use this technique also for the elimination of chain rules, for the Chomsky normal form and for effective decision methods of emptiness, finiteness and infiniteness.

The second part transforms ω-algebraic series into simple ω-reset pushdown automata. Simple ω-reset pushdown automata do not use ϵ-transitions; in the literature, this is also called a *realtime* pushdown automaton. Realtime pushdown automata read a symbol of the input word in every transition - exactly like context-free grammars in Greibach normal form generate a letter in every derivation step. Additionally, each derivation step of context-free grammars in Greibach normal form increases the number of non-terminals in the sentential form by at most one. We showed that for realtime pushdown automata it suffices to handle at most one stack symbol per transition. Here the Greibach normal form provides exactly the properties needed to construct simple ω-reset pushdown automata.

As the first part applies only to mixed ω-algebraic systems, we could not use this result in the second part where the Greibach normal form is needed for ω-algebraic systems.

The model of simple ω-reset pushdown automata seems to be very natural. They occur when applying general homomorphisms to nested-word automata [1]. Their unweighted counterparts have been used for a Büchi-type logical characterization of timed pushdown languages [11] and ω-context-free languages [6]. A corresponding result for weighted ω-context-free languages is currently in development and uses the simple ω-reset pushdown automata introduced here.

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