Existential Second-Order Logic and Modal Logic with Quantified Accessibility Relations

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Abstract

This article investigates the role of *arity* of second-order quantifiers in existential second-order logic, also known as Σ^1_1. We identify fragments L of Σ^1_1 where second-order quantification of relations of arity $k > 1$ is (nontrivially) vacuous in the sense that each formula of L can be translated to a formula of (a fragment of) *monadic* Σ^1_1. Let *polyadic Boolean modal logic with identity* ($\mathrm{PBML}^=$) be the logic obtained by extending standard polyadic multimodal logic with built-in identity modalities and with constructors that allow for the Boolean combination of accessibility relations. Let $\Sigma^1_1(\mathrm{PBML}^=)$ be the extension of $\mathrm{PBML}^=$ with existential prenex quantification of accessibility relations and proposition symbols. The principal result of the article is that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into *monadic* Σ^1_1. As a corollary, we obtain a variety of decidability results for multimodal logic. The translation can also be seen as a step towards establishing whether every property of finite directed graphs expressible in $\Sigma^1_1(\mathrm{FO}^2)$ is also expressible in monadic Σ^1_1. This question was left open by Grädel and Rosen (1999).

1 Introduction

Properties of existential second-order logic have been widely studied in finite model theory. Existential second-order logic captures the complexity class NP, and there exists a large body of results concerning the expressive power of different fragments of the logic (see e.g. [1, 5, 13, 15, 19]). However, there are several issues related to the expressivity of Σ^1_1 that are not understood well. Most notably, Fagin's *spectrum arity hierarchy conjecture* (see [7, 8]) remains a longstanding difficult open problem in finite model theory. Fagin's question is whether there exist sets of positive integers (spectra) definable by first-order sentences[1] with predicates of maximum arity $k + 1$, but not definable by sentences with predicates of arity k.

In this article we investigate *arity reduction* of formulae of existential second-order logic: we identify fragments L of Σ^1_1 where second-order quantification of relations of arity $k > 1$ is (nontrivially) vacuous in the sense that each formula of L can be translated into a formula of (a fragment of) monadic Σ^1_1, also known as $\exists$MSO. Our work is directly related to a novel perspective on *modal correspondence theory*, and our investigations lead to a variety of *decidability results* concerning multimodal logics over classes of frames with built-in relations. Our work also aims to provide a stepping stone towards a solution of an open problem of Grädel and Rosen posed in [14].

The objective of modal correspondence theory (see [3]) is to classify formulae of modal logic according to whether they define elementary classes of Kripke frames.[2] On the level of frames, modal logic can be considered a fragment of monadic Π^1_1, also known as $\forall$MSO, and therefore correspondence theory studies a special fragment of $\forall$MSO.

When a modal formula is inspected from the point of view of Kripke frames, the *proposition symbols* occurring in the formula are quantified universally; it is natural to ask what happens if one also quantifies *binary relation symbols* occurring in (the standard translation of) a modal formula. This question is investigated in [20], where the focus is on the expressivity of multimodal logic with universal prenex quantification of (some of) the binary and unary relation symbols occurring in a formula. A question that immediately suggests itself is whether there exists any class of multimodal frames definable in this logic, let us call it $\Pi^1_1(\mathrm{ML})$, but not definable in monadic second-order logic MSO. The question can be regarded as a question of

[1] The spectrum of a sentence φ is the set of positive integers n such that φ has a model of the size n.

[2] It is well known that if a class of Kripke frames is definable by a modal formula, then the class is definable by a set of FO formulae iff it is definable by a single FO formula. See [11] for example. Therefore it makes no difference *here* whether the term "elementary" is taken to mean definability by a single first-order formula or definability by a set of first-order formulae.

modal correspondence theory. Here, however, the correspondence language is MSO rather than FO. For further investigations that involve quantification of binary relations in modal logic, see for example [4, 21].

In the current article we investigate two multimodal logics with existential second-order prenex quantification of accessibility relations and proposition symbols, $\Sigma^1_1(\mathrm{PBML}^=)$ and $\Sigma^1_1(\mathrm{ML})$. The logic $\Sigma^1_1(\mathrm{ML})$ is the extension of ordinary multimodal logic with existential second-order prenex quantification of binary accessibility relations and proposition symbols. $\mathrm{PBML}^=$ is the logic obtained by extending standard polyadic[3] multimodal logic by built-in identity modalities and by constructors that allow for the Boolean combination of accessibility relations (see Subsection 2.1). Obviously $\Sigma^1_1(\mathrm{PBML}^=)$ is the extension of $\mathrm{PBML}^=$ with existential second-order prenex quantification of accessibility relations and proposition symbols.

We warm up by showing that $\Sigma^1_1(\mathrm{ML})$ translates into *monadic* $\Sigma^1_1(\mathrm{MLE})$, which is the extension of multimodal logic with the global modality and existential second-order prenex quantification of *only* proposition symbols. The method of proof is based on the notion of a *largest filtration* (see [3] for the definition). We then push the method and show that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into monadic Σ^1_1. Note that both of these results immediately imply that $\Pi^1_1(\mathrm{ML})$ translates into $\forall$MSO, and therefore show that MSO would be a somewhat dull correspondence language for correspondence theory of $\Pi^1_1(\mathrm{ML})$.

The logic $\mathrm{PBML}^=$ contains a wide variety of logics used in different applications of modal logic. It could be argued that $\{\neg, \cup, \cap, \circ, {}^*, {}^\smile, E, D\}$ is more or less the core collection of operations on binary relations used in extensions of modal logic defined for the purposes of applications. Here $\neg$, $\cup$, $\cap$, $\circ$, *, ${}^\smile$ denote the complement, union, intersection, composition, transitive reflexive closure and converse operations, respectively. The symbols E and D denote the global modality and difference modality. Logics using some of these core operations include for example propositional dynamic logic PDL [9, 16] and its extensions, Boolean modal logic [10, 23], description logics [2, 18, 25], modal logic with the global modality [12] and modal logic with the difference modality [28]. The operations $\neg, \cup, \cap, E, D$ are part of $\mathrm{PBML}^=$. One of our principal motivations for studying $\mathrm{PBML}^=$ is that the logic subsumes a large number of typical extensions of modal logic. Our translation from

[3]Modal logics with accessibility relations of arities greater than two are called polyadic. See Section 2.1 for the related definitions.

$\Sigma^1_1(\mathrm{PBML}^=)$ into $\exists$MSO gives as a direct corollary a wide range of decidability results for extensions of multimodal logic over various classes of Kripke frames with built-in relations; see Theorem 4.10 below.

In addition to applied modal logics, the investigations in this article are directly related to an interesting open problem concerning two-variable logics. Grädel and Rosen ask in [14] the question whether there exists any class of finite directed graphs that is definable in $\Sigma^1_1(\mathrm{FO}^2)$ but not in $\exists$MSO. Let $\mathrm{BML}^=$ denote ordinary Boolean modal logic with a built-in identity relation, i.e., $\mathrm{BML}^=$ is the restriction of $\mathrm{PBML}^=$ to binary relations. Lutz, Sattler and Wolter show in the article [24] that $\mathrm{BML}^=$ extended with the converse operator is expressively complete for FO^2. Therefore, in order to prove that $\Sigma^1_1(\mathrm{FO}^2) \leq \exists$MSO, one would have to modify our translation from $\Sigma^1_1(\mathrm{BML}^=)$ into $\exists$MSO such that it takes into account the possibility of using the converse operation. We have succeeded neither in this nor in identifying a $\Sigma^1_1(\mathrm{FO}^2)$ definable class of directed graphs that is not definable in $\exists$MSO. However, we find modal logic a promising framework for working on the problem.

This article is the journal version of the conference article [17].

2 Preliminary definitions

In this section we discuss technical notions that occupy a central role in the rest of the article.

2.1 Syntax and semantics of $\Sigma^1_1(\mathrm{PBML}^=)$

The semantics of $\mathrm{PBML}^=$—defined in detail below—is obtained by combining the semantics of Boolean modal logic with the standard generalization of Kripke semantics to polyadic modal contexts.

Let V be a vocabulary containing relation symbols only. A *V-model*, or a *model of the vocabulary V*, is an ordinary first-order model (see [6]) that gives an interpretation to exactly all the symbols in V. We use this notion of a model in both predicate logic and modal logic. If M is a V-model and w a point in the domain of M, then the pair (M, w) is a *pointed model* of the vocabulary V.

Below we shall exclusively consider vocabularies V containing relation symbols only. We let V_1 denote the subset of V containing exactly all the unary relation symbols in V, and we let V_h be the subset of V containing exactly all the relation symbols in V of higher arities, i.e., arities greater or equal to two. We define the set $\mathrm{MP}(V)$ of *modal parameters* over V to be the smallest set S satisfying the following conditions.

1. For each $k \in \mathbb{N}_{\geq 2}$, let id_k be a symbol. We assume that none of the symbols id_k is in V. We have $id_k \in S$ for all $k \in \mathbb{N}_{\geq 2}$. The symbol id_k is called the *k-ary identity symbol.*

2. If $R \in V_h$, then $R \in S$.

3. If $\mathcal{M} \in S$, then $\neg\mathcal{M} \in S$.

4. If $\mathcal{M} \in S$ and $\mathcal{N} \in S$, then $(\mathcal{M} \cap \mathcal{N}) \in S$.

Each modal parameter $\mathcal{M}$ is associated with an arity $Ar(\mathcal{M})$ defined as follows.

1. If $\mathcal{M} = id_k$, then $Ar(\mathcal{M}) = k$.

2. If $\mathcal{M} = R \in V_h$, then the $Ar(\mathcal{M})$ is equal to the arity of R.

3. If $\mathcal{M} = \neg\mathcal{N}$, then $Ar(\mathcal{M}) = Ar(\mathcal{N})$.

4. If $\mathcal{M} = (\mathcal{N}_1 \cap \mathcal{N}_2)$ and $Ar(\mathcal{N}_1) = Ar(\mathcal{N}_2)$, then $Ar(\mathcal{M}) = Ar(\mathcal{N}_1)$. If $Ar(\mathcal{N}_1) \neq Ar(\mathcal{N}_2)$, then $Ar(\mathcal{M}) = 2$.

The set of formulae of $\mathrm{PBML}^=$ of the vocabulary V (V-formulae) is defined to be the smallest set F satisfying the following conditions.

1. If $P \in V_1$, then $P \in F$.

2. If $\varphi \in F$, then $\neg\varphi \in F$.

3. If $\varphi_1, \varphi_2 \in F$, then $(\varphi_1 \wedge \varphi_2) \in F$.

4. If $\varphi_1, ..., \varphi_k \in F$ and if $\mathcal{M} \in \mathrm{MP}(V)$ is a $(k+1)$-ary modal parameter, then $\langle\mathcal{M}\rangle(\varphi_1, ..., \varphi_k) \in F$.

Operators $\langle\mathcal{M}\rangle$ are called *diamonds*. The *modal depth* $Md(\varphi)$ of a formula φ is the maximum nesting depth of diamonds in φ, defined as follows.

1. $Md(P) = 0$ for $P \in V_1$.

2. $Md(\neg\varphi) = Md(\varphi)$.

3. $Md\big((\varphi_1 \wedge \varphi_2)\big) \;=\; max(\,\{Md(\varphi_1), Md(\varphi_2)\}\,)$.

4. $Md\big(\,\langle\mathcal{M}\rangle(\varphi_1, ..., \varphi_k\,)\big) \;=\; 1 \,+\, max(\,\{Md(\varphi_1), ..., Md(\varphi_k)\}\,)$.

Let M be a V-model with the domain A. The *extension* $\mathcal{M}^M$ of a modal parameter $\mathcal{M}$ over M is a relation of the arity $Ar(\mathcal{M})$ over A. The extension of $R \in V_h$ over M is simply the interpretation R^M of the symbol R. For each $k \in \mathbb{N}_{\geq 2}$, the extension id_k^M of the symbol id_k is the set

$$\{\,(w_1, ..., w_k) \in A^k \mid w_i = w_j \text{ for all } i, j \in \{1, ..., k\}\,\}.$$

Other modal parameters are interpreted recursively such that the following conditions hold.

1. If $\mathcal{M} = \neg\mathcal{N}$, then $\mathcal{M}^M \;=\; A^{Ar(\mathcal{M})} \setminus \mathcal{N}^M$.

2. If $\mathcal{M} = (\mathcal{N}_1 \cap \mathcal{N}_2)$, then $\mathcal{M}^M \;=\; \mathcal{N}_1^M \cap \mathcal{N}_2^M$.

Note that if $Ar(\mathcal{N}_1) \neq Ar(\mathcal{N}_2)$, then $(\mathcal{N}_1 \cap \mathcal{N}_2)^M = \emptyset$.

The satisfaction relation $\Vdash$ for PBML$^=$ formulae of the vocabulary V is defined with respect to pointed V-models as follows.

1. If $P \in V_1$, then
$$(M, w) \Vdash P \;\Leftrightarrow\; w \in P^M.$$

2. For other formulae, the satisfaction relation is interpreted according to the following recursive clauses.

$$\begin{array}{lcl} (M, w) \Vdash \neg\varphi & \Leftrightarrow & (M, w) \nVdash \varphi. \\ (M, w), \Vdash (\varphi_1 \wedge \varphi_2) & \Leftrightarrow & (M, w) \Vdash \varphi_1 \text{ and } (M, w) \Vdash \varphi_2. \\ (M, w) \Vdash \langle\mathcal{M}\rangle\,(\varphi_1, ..., \varphi_k) & \Leftrightarrow & \text{there exist } u_1, ..., u_k \in Dom(M) \\ & & \text{such that } (w, u_1, ..., u_k) \in \mathcal{M}^M \text{ and} \\ & & (M, u_i) \Vdash \varphi_i \text{ for all } i \in \{1, ..., k\}. \end{array}$$

For each V-model M and each formula φ of the vocabulary V, we let $\|\,\varphi\,\|^M$ denote the set

$$\{\; w \in Dom(M) \mid (M, w) \Vdash \varphi \;\}.$$

The set $\|\,\varphi\,\|^M$ is called the *extension* of φ over M. When φ and ψ are formulae of the vocabulary V, we write $\varphi \Vdash \psi$ if

$$(M, w) \Vdash \varphi \;\Rightarrow\; (M, w) \Vdash \psi$$

for all pointed V-models (M, w).

Let V be a vocabulary containing relation symbols only; V may be empty, and V may contain relation symbols of any finite positive arity. A formula φ of $\Sigma_1^1(\mathrm{PBML}^=)$ of the vocabulary V (V-formula) is a formula of the type

$$\exists S_1 ... \exists S_n \ \psi,$$

where the variables S_i are relation symbols (of any positive arity) and ψ is a $\mathrm{PBML}^=$ formula of the vocabulary $V \cup \{S_1, ..., S_n\}$. The set $\{S_1, ..., S_n\}$ is allowed to be empty, so $\mathrm{PBML}^=$ is a fragment of $\Sigma_1^1(\mathrm{PBML}^=)$. The sets V and $\{S_1, ..., S_n\}$ are always assumed to be disjoint. Let (M, w) be a pointed V-model. We define $(M, w) \Vdash \varphi$ if there exists an expansion

$$M' = (M, S_1^{M'}, ..., S_n^{M'})$$

of the model M such that $(M', w) \Vdash \psi$. The set of *non-logical symbols* of a $\Sigma_1^1(\mathrm{PBML}^=)$ formula $\exists S_1 ... \exists S_n \, \chi$ of the vocabulary V is the set of relation symbols (of any arity) that belong to V and also actually occur in χ. The symbols id_k are not considered to be non-logical symbols.

Let $\mathrm{BML}^=$ be the fragment of $\mathrm{PBML}^=$ where each modal parameter occurring in a formula is required to be *binary*. The logic ML is the fragment of $\mathrm{BML}^=$ where the modal parameters $\mathcal{M}$ defining diamonds $\langle \mathcal{M} \rangle$ are required to be *atomic* binary relation symbols that belong to the vocabulary considered. Note that the modal parameter id_2 is not considered to be part of the vocabulary. The logic MLE is the extension of ML with the global diamond $\langle E \rangle$, i.e., the diamond $\langle \neg (id_2 \cap \neg id_2) \rangle$. Logics $\Sigma_1^1(\mathrm{ML})$ and $\Sigma_1^1(\mathrm{MLE})$ are the fragments of $\Sigma_1^1(\mathrm{PBML}^=)$ defined by extending ML and MLE with existential prenex quantification of binary and unary relation symbols. *Monadic* $\Sigma_1^1(\mathrm{MLE})$ is the fragment of $\Sigma_1^1(\mathrm{MLE})$ where we only allow second-order quantifiers quantifying *unary* relation symbols.

The logics $\Pi_1^1(\mathrm{PBML}^=)$, $\Pi_1^1(\mathrm{ML})$ and $\Pi_1^1(\mathrm{MLE})$ are the counterparts of the logics $\Sigma_1^1(\mathrm{PBML}^=)$, $\Sigma_1^1(\mathrm{ML})$ and $\Sigma_1^1(\mathrm{MLE})$, but with universal second-order quantifiers instead of existential ones.

Let V be a vocabulary containing relation symbols only. The set of formulae of the vocabulary V (V-formulae) of *existential second-order logic*, or Σ_1^1, is the set of formulae of the type $\exists S_1 ... S_n \, \chi$, where χ is a first-order (with equality) formula of the vocabulary V; the sets $\{S_1, ..., S_n\}$ and V are always assumed to be disjoint, and the set of non-logical symbols of $\exists S_1 ... S_n \, \chi$ is the set of relation symbols that belong to V and also actually occur in χ. Equality is not considered to be a non-logical symbol. For the semantics of Σ^1_1, see for example [22]. Monadic Σ^1_1 is the fragment of Σ^1_1 where the second-order relation variables S_i are unary.

Let φ be a formula of $\Sigma^1_1(\mathrm{PBML}^=)$ or $\Pi^1_1(\mathrm{PBML}^=)$ of the vocabulary V. Let $\psi(x)$ be a V-formula of Σ^1_1 with exactly one free variable, the first-order variable x. The formulae φ and $\psi(x)$ of are called *V-equivalent* if for all pointed V-models (M, w), we have

$$(M, w) \Vdash \varphi \quad \Leftrightarrow \quad M, \frac{w}{x} \models \psi(x),$$

where $M, \frac{w}{x} \models \psi(x)$ means that the model M satisfies the formula $\psi(x)$ of predicate logic when x is interpreted to be w. The formulae $\psi(x)$ and φ are *uniformly equivalent* if they have the same set U of non-logical symbols and if the formulae are U-equivalent.[4] Two $\Sigma^1_1(\mathrm{PBML}^=)$ formulae φ_1 and φ_2 of the vocabulary V are V-equivalent if they are satisfied by exactly the same pointed V-models. The formulae φ_1 and φ_2 are uniformly equivalent if they have exactly the same set U of non-logical symbols and if the formulae are U-equivalent. Two V-sentences of predicate logic are uniformly equivalent if they have exactly the same set U of non-logical symbols and if they are satisfied by exactly the same U-models.

The reason we have chosen to define $\mathrm{PBML}^=$ exactly the way defined above, is relatively simple. Firstly, $\mathrm{BML}^=$ extended with the converse modality is expressively complete for FO^2. We do not know whether $\Sigma^1_1(\mathrm{FO}^2)$ is contained in $\exists\mathrm{MSO}$, but we will show below that $\Sigma^1_1(\mathrm{BML}^=) \leq \exists\mathrm{MSO}$ by establishing that even the extension $\Sigma^1_1(\mathrm{PBML}^=)$ of $\Sigma^1_1(\mathrm{BML}^=)$ with polyadic modalities is indeed contained in $\exists\mathrm{MSO}$. Finally, the reason we have included the modalities id_k for $k \geq 3$ in the language of $\mathrm{PBML}^=$ is mostly due to technical presentation related issues. The reader may, indeed, think that the modalities id_k for $k \geq 3$ are not very canonical. The modalities do, however, have some interesting features. Notice for example that we can easily eliminate the use of conjunction from $\mathrm{PBML}^=$. We shall not make any use of this feature below, however.

We shall next establish that there is an algorithm that decides, when given a $\mathrm{PBML}^=$ formula φ of any relational vocabulary V, whether there exists a pointed V-model (M, w) such that $(M, w) \models \varphi$. In other words, $\mathrm{PBML}^=$

[4] For example the formulae $x = x \wedge \exists y R(y, y)$ and $\exists S\, \exists P\, \langle S \rangle \langle id_2 \cap R \rangle P$ are uniformly equivalent. The set of non-logical symbols of both formulae is $\{R\}$.

is decidable. This result will be needed below in order to ensure that the translation of $\Sigma^1_1(\mathrm{PBML}^=)$ into $\exists$MSO is effective. There are several simple ways of establishing the decidability of $\mathrm{PBML}^=$. One of them involves *fluted logic* [26]. Fluted logic is a fragment of first-order logic, where the first-order variables in atomic formulae are are always written in the same order, without ever permuting them. See [26] for the exact definition of the logic. It is easy to see that $\mathrm{PBML}^=$ translates into fluted logic (with identity) simply by using a straightforward generalization of the *standard translation* (see [3]) of modal logic. The article [26] constructs an algorithm that decides, given any formula φ of fluted logic (with identity) of any relational vocabulary, whether φ is satisfiable. Therefore $\mathrm{PBML}^=$ is decidable.

2.2 Types

In the current subsection we define the notion of a *type* for the logic $\mathrm{PBML}^=$. Let V be a finite vocabulary such that $V_1 \neq \emptyset$. Let m' be the maximum arity of the modal parameters in V_h. In the case $V_h = \emptyset$, let $m' = 0$. Let m be an integer such that $2 \leq m$ and $m' \leq m$. Define the set

$$S_V \;=\; V_h \;\cup\; \{\, \neg R \;\mid\; R \in V_h \,\} \;\cup\; \{\, id_k,\ \neg id_k \mid 2 \leq k \leq m \,\}$$

of at most m-ary atomic and negated atomic modal parameters over V. Let k be an integer such that $2 \leq k \leq m$. Let $S_V(k)$ be the set that contains as elements exactly the k-ary modal parameters in S_V. Notice that $S_V(k) \neq \emptyset$. Let $T_V(k)$ denote the set whose elements are exactly the subsets $T \subseteq S_V(k)$ such that the following conditions are satisfied.

1. Exactly one of the modal parameters id_k and $\neg id_k$ is in the set T.
2. If $R \in V_h$ is k-ary, then exactly one of the modal parameters R and $\neg R$ is in the set T.

Let f be a function with the domain $T_V(k)$ that maps each $T \in T_V(k)$ to an intersection $\mathcal{N} \in \mathrm{MP}(V)$ of the elements of T. (There may be several ways to choose the order of the members of T and bracketing when writing the modal parameter $\mathcal{N}$. The order and bracketing that f chooses does not matter.) The set

$$\{\ f(T) \mid T \in T_V(k)\ \}$$

of modal parameters is the set of k-ary *access types* over V. We let $\mathrm{ATP}_V(k)$ denote the set of k-ary access types over V.

Let $\mathcal{M}$ be a k-ary access type over V, and let $R \in V_h \cup \{id_k\}$ be a k-ary atomic modal parameter. We write $R \in \mathcal{M}$ if $\neg R$ does *not* occur in $\mathcal{M}$. Let $U \subseteq V$ and let $\mathcal{N}$ be a k-ary access type over U. We say that $\mathcal{N}$ is *consistent* with $\mathcal{M}$ (or alternatively, $\mathcal{M}$ is consistent with $\mathcal{N}$), if for all k-ary symbols $R \in U_h \cup \{id_k\}$, we have $R \in \mathcal{M}$ iff $R \in \mathcal{N}$.

Let (M, w) be a pointed model of the vocabulary V. We define

$$\tau^0_{(M,w),m} := \bigwedge_{\substack{P \,\in\, V_1, \\ (M,w) \,\Vdash\, P}} P \quad \wedge \bigwedge_{\substack{Q \,\in\, V_1, \\ (M,w) \,\nVdash\, Q}} \neg Q.$$

The formula $\tau^0_{(M,w),m}$ is the *type* of (M, w) of the modal depth 0 and up to the arity m. We choose the bracketing and ordering of conjuncts of the formulae $\tau^0_{(M,w),m}$ such that if for some pointed V-models (N, v) and (N', v') the types $\tau^0_{(N,v),m}$ and $\tau^0_{(N',v'),m}$ are uniformly equivalent, then actually $\tau^0_{(N,v),m} = \tau^0_{(N',v'),m}$. In other words, if two types of pointed V-models of the modal depth 0 and up to the arity m are uniformly equivalent, then they are in fact the one and the same formula. We let $\mathrm{TP}^0_{V,m}$ denote the set containing exactly the formulae τ such that for some pointed model (M, w) of the vocabulary V, the formula τ is the type of (M, w) of the modal depth 0 and up to the arity m. Clearly the set $\mathrm{TP}^0_{V,m}$ is finite.

Let $n \in \mathbb{N}$ and assume we have defined formulae $\tau^n_{(M,w),m}$ for all pointed models (M, w), and assume also that $\mathrm{TP}^n_{V,m}$ is a finite set containing exactly all these formulae. We define

$$\begin{aligned}
\tau^{n+1}_{(M,w),m} \;:=\;& \tau^n_{(M,w),m} \\
& \wedge \bigwedge \{\, \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k) \;\mid\; 1 \leq k \leq m-1,\ \mathcal{M} \in \mathrm{ATP}_V(k+1),\ \sigma_1, ..., \sigma_k \in \mathrm{TP}^n_{V,m},\ (M, w) \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k) \,\} \\
& \wedge \bigwedge \{\, \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k) \;\mid\; 1 \leq k \leq m-1,\ \mathcal{M} \in \mathrm{ATP}_V(k+1),\ \sigma_1, ..., \sigma_k \in \mathrm{TP}^n_{V,m},\ (M, w) \nVdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k) \,\}.
\end{aligned}$$

The formula $\tau^{n+1}_{(M,w),m}$ is the *type* of (M, w) of the modal depth $n+1$ and up to the arity m. Again we assume some standard ordering of the conjuncts and some standard bracketing, so that if two types $\tau^{n+1}_{(M,w),\,m}$ and $\tau^{n+1}_{(N,v),\,m}$ of pointed V-models (M,w) and (N,v) are uniformly equivalent, then the types are the same formula. We let $\mathrm{TP}^{n+1}_{V,m}$ be the set containing exactly the formulae τ such that for some pointed model (M,w) of the vocabulary V, the formula τ is the type of (M,w) of the modal depth $n+1$ and up to the arity m. We observe that the set $\mathrm{TP}^{n+1}_{V,m}$ is finite. Since $\mathrm{PBML}^{=}$ is decidable, there is an algorithm that constructs for each triple (V,m,n) the set $\mathrm{TP}^{n}_{V,m}$. This fact is used in the proof below establishing that $\Sigma^1_1(\mathrm{PBML}^{=})$ translated effectively into $\exists\mathrm{MSO}$.

We list a number of properties of types that are straightforward to prove. Let (M,w) be a pointed model of the vocabulary U, where U may be infinite. Assume that $U_1 \neq \emptyset$. Let $V \subseteq U$ be a finite vocabulary and let m be as defined above, i.e., m is at least two and greater or equal to the maximum arity of the symbols in V_h. Assume that $V_1 \neq \emptyset$. Let $n \in \mathbb{N}$. Firstly, (M,w) satisfies exactly one type in $\mathrm{TP}^{n}_{V,m}$. Also, for all $\tau \in \mathrm{TP}^{n}_{V,m}$ and all $l \leq n$, there exists exactly one type $\sigma \in \mathrm{TP}^{l}_{V,m}$ such that $\tau \Vdash \sigma$. Notice also that for each type $\tau \in \mathrm{TP}^{n}_{V,m}$, there exists some pointed V-model that satisfies τ. Let $\alpha \in \mathrm{TP}^{n}_{V,m}$ and let ψ be an arbitrary formula of the vocabulary V and of some modal depth $n' \leq n$. Assume that the maximum arity of the modal parameters that occur in ψ is at most m. Now either $\alpha \Vdash \psi$ or $\alpha \Vdash \neg\psi$, and thus, for all points $u, v \in \parallel \alpha \parallel^{M}$, we have $(M,u) \Vdash \psi$ iff $(M,v) \Vdash \psi$. Finally, ψ is V-equivalent to $\bigvee\{\alpha \in \mathrm{TP}^{n}_{V,m} \mid \alpha \Vdash \psi\}$. Notice that $\bigvee\emptyset = \bot$, where $\bot$ is defined to be the formula $(P \wedge \neg P)$ for some $P \in V_1$.

3 $\Sigma^1_1(\mathrm{ML})$ translates into monadic $\Sigma^1_1(\mathrm{MLE})$

In this subsection we show how to translate $\Sigma^1_1(\mathrm{ML})$ formulae to uniformly equivalent formulae of monadic $\Sigma^1_1(\mathrm{MLE})$. The translation is based on the notion of a *largest filtration* (see [3] for the definition). The principal idea is to fix a fresh unary predicate P_α for each subformula α of the quantifier free part of the $\Sigma^1_1(\mathrm{ML})$ formula to be translated. The translation is given in full detail below, but intuitively, the fresh predicates P_α encode extensions of the formulae α, and information concerning extensions of the formulae α can be recursively recovered from the information encoded by the unary predicates P_α.

We begin by fixing a $\Sigma^1_1(\mathrm{ML})$ formula φ. We will first show how to translate φ to a uniformly equivalent formula $\varphi^*(x)$ of $\exists\mathrm{MSO}$. We will then establish that that the first-order part of $\varphi^*(x)$ translates to a uniformly equivalent formula of MLE.

Let $\varphi := \overline{Q}\,\psi$, where $\overline{Q}$ is a string of existential second-order quantifiers and ψ a formula of ML. Let V_1^ψ and V_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in ψ. Define

$$V^\psi = V_1^\psi \cup V_2^\psi.$$

Let Q_1^ψ and Q_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in $\overline{Q}$. Define

$$Q^\psi = Q_1^\psi \cup Q_2^\psi.$$

Let SUB_ψ denote the set of subformulae of the formula ψ.

We fix a unary relation symbol P_α for each formula $\alpha \in \mathrm{SUB}_\psi$. The symbols P_α are assumed not to occur in φ. We then define a collection of auxiliary formulae needed in order to define the translated formula $\varphi^*(x)$. Let

$$P',\ \neg\alpha,\ (\beta \wedge \gamma),\ \langle R\rangle\rho,\ \langle S\rangle\sigma \quad \in \quad \mathrm{SUB}_\psi,$$

where $P' \in V_1^\psi$, $R \in V_2^\psi \setminus Q_2^\psi$ and $S \in Q_2^\psi$. We define

$$\begin{aligned}
\psi_{P'} \quad &:= \quad \forall x\Big(P_{P'}(x) \leftrightarrow P'(x)\Big),\\
\psi_{\neg\alpha} \quad &:= \quad \forall x\Big(P_{\neg\alpha}(x) \leftrightarrow \neg P_\alpha(x)\Big),\\
\psi_{(\beta\wedge\gamma)} \quad &:= \quad \forall x\Big(P_{(\beta\wedge\gamma)}(x) \leftrightarrow (P_\beta(x) \wedge P_\gamma(x))\Big),\\
\psi_{\langle R\rangle\rho} \quad &:= \quad \forall x\Big(P_{\langle R\rangle\rho}(x) \leftrightarrow \exists y(R(x,y) \wedge P_\rho(y))\Big),\\
\psi_{\langle S\rangle\sigma} \quad &:= \quad \forall x\Big(P_{\langle S\rangle\sigma}(x) \leftrightarrow \exists y(Access_S(x,y) \wedge P_\sigma(y))\Big),
\end{aligned}$$

where

$$Access_S(x,y) \quad := \quad \bigwedge_{\langle S\rangle\chi \,\in\, \mathrm{SUB}_\psi} \Big(\ P_\chi(y) \rightarrow P_{\langle S\rangle\chi}(x)\ \Big).$$

Finally, we define

$$\delta_\psi \quad := \quad \bigwedge_{\alpha \,\in\, \mathrm{SUB}_\psi} \psi_\alpha$$

and

$$\varphi^*(x) \ := \ \overline{Q}^*(\delta_\psi \ \wedge \ P_\psi(x)),$$

where $\overline{Q}^*$ is a string of existential quantifiers that quantify the predicate symbols $P \in Q_1^\psi$ and also the symbols P_α such that $\alpha \in \mathrm{SUB}_\psi$.

We then prove that $(M, w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$. Assume that $(M, w) \Vdash \varphi$. Therefore there exists an expansion M_2 of M by interpretations of the binary and unary symbols in Q^ψ such that we have $(M_2, w) \Vdash \psi$. We define an expansion M_1 of M by interpretations of the unary symbols occurring in $\overline{Q}^*$. For the symbols $P \in Q_1^\psi$, we let $P^{M_1} = P^{M_2}$. For the symbols P_α, where $\alpha \in \mathrm{SUB}_\psi$, we define $P_\alpha^{M_1} = \parallel \alpha \parallel^{M_2}$.

Lemma 3.1. *Let $\langle S\rangle\sigma \in \mathrm{SUB}_\psi$, where $S \in Q_2^\psi$, and let $v \in Dom(M)$. Then $(M_2, v) \Vdash \langle S\rangle\sigma$ iff $M_1, \frac{v}{x} \models \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big)$.*

Proof. Assume $(M_2, v) \Vdash \langle S\rangle\sigma$. Thus $(v, u) \in S^{M_2}$ for some point

$$u \ \in \parallel \sigma \parallel^{M_2} = P_\sigma^{M_1}.$$

To establish that

$$M_1, \frac{v}{x} \ \models \ \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big),$$

it therefore suffices to prove that for all $\langle S\rangle\chi \in \mathrm{SUB}_\psi$, if $u \in P_\chi^{M_1}$, then $v \in P_{\langle S\rangle\chi}^{M_1}$. Therefore assume that $u \in P_\chi^{M_1}$ for some formula $\langle S\rangle\chi \in \mathrm{SUB}_\psi$. As $\parallel \chi \parallel^{M_2} = P_\chi^{M_1}$, we have $u \in \parallel \chi \parallel^{M_2}$. Since $(v, u) \in S^{M_2}$, we have $(M_2, v) \Vdash \langle S\rangle\chi$. As $\parallel \langle S\rangle\chi \parallel^{M_2} = P_{\langle S\rangle\chi}^{M_1}$, we must have $v \in P_{\langle S\rangle\chi}^{M_1}$, as desired.

Assume then that

$$M_1, \frac{v}{x} \ \models \ \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big).$$

Hence $M_1, \frac{v}{x}\frac{u}{y} \models Access_S(x, y)$ for some $u \in P_\sigma^{M_1} = \parallel \sigma \parallel^{M_2}$. Now, by the definition of the formula $Access_S(x, y)$, we observe that $v \in P_{\langle S\rangle\sigma}^{M_1}$. As $\parallel \langle S\rangle\sigma \parallel^{M_2} = P_{\langle S\rangle\sigma}^{M_1}$, we have $v \in \parallel \langle S\rangle\sigma \parallel^{M_2}$. Therefore $(M_2, v) \Vdash \langle S\rangle\sigma$, as desired. $\square$

Lemma 3.2. *Under the assumption $(M, w) \Vdash \varphi$, we have $M, \frac{w}{x} \models \varphi^*(x)$.*

Proof. We establish the claim of the lemma by proving that

$$M_1, \frac{w}{x} \models \delta_\psi \wedge P_\psi(x).$$

Since $(M_2, w) \Vdash \psi$ and $\| \psi \|^{M_2} = P_\psi^{M_1}$, we have $M_1, \frac{w}{x} \models P_\psi(x)$. The non-trivial part in the argument establishing that $M_1 \models \delta_\psi$ involves showing that $M_1 \models \psi_{\langle S\rangle\sigma}$ for each $\langle S\rangle\sigma \in \mathrm{SUB}_\psi$, where $S \in Q_2^\psi$. This follows directly by Lemma 3.1, as $P^{M_1}_{\langle S\rangle\sigma} = \| \langle S\rangle\sigma \|^{M_2}$. □

We then establish that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. Therefore we assume that $M, \frac{w}{x} \models \varphi^*(x)$. Therefore there exists an expansion M_1' of M by interpretations of the unary symbols occurring in $\overline{Q}^*$ such that $M_1', \frac{w}{x} \models \delta_\psi \wedge P_\psi(x)$. We define an expansion M_2' of M by interpretations of the binary and unary symbols that occur in $\overline{Q}$. For the symbols $P \in Q_1^\psi$, we define $P^{M_2'} = P^{M_1'}$. For the symbols $S \in Q_2^\psi$, we let $(v, u) \in S^{M_2'}$ if and only if $M_1', \frac{v}{x}\frac{u}{y} \models \mathit{Access}_S(x, y)$.

Lemma 3.3. *Let $\alpha \in \mathrm{SUB}_\psi$ and $v \in Dom(M)$. We have $(M_2', v) \Vdash \alpha$ iff $M_1', \frac{v}{x} \models P_\alpha(x)$.*

Proof. We establish the claim of the lemma by induction on the structure of α. Since $M_1' \models \delta_\psi$, the claim holds trivially for all atomic formulae $P \in V_1^\psi$. Also, the cases where α is of form $\neg\beta$, $(\beta \wedge \gamma)$ or $\langle R\rangle\beta$, where $R \in V_2^\psi \setminus Q_2^\psi$, are straightforward since $M_1' \models \delta_\psi$.

Assume that $(M_2', v) \Vdash \langle S\rangle\sigma$, where $S \in Q_2^\psi$ and $\langle S\rangle\sigma \in \mathrm{SUB}_\psi$. Therefore $(v, u) \in S^{M_2'}$ for some $u \in \| \sigma \|^{M_2'}$. Hence $M_1', \frac{v}{x}\frac{u}{y} \models \mathit{Access}_S(x, y)$ by the definition of $S^{M_2'}$. We also have $P_\sigma^{M_1'} = \| \sigma \|^{M_2'}$ by the induction hypothesis. Therefore $u \in P_\sigma^{M_1'}$, whence we have

$$M_1', \frac{v}{x} \;\models\; \exists y\big(\mathit{Access}_S(x, y) \wedge P_\sigma(y)\big).$$

Therefore, as $M_1' \models \psi_{\langle S\rangle\sigma}$, we have $M_1', \frac{v}{x} \models P_{\langle S\rangle\sigma}(x)$.

For the converse, we assume that $M_1', \frac{v}{x} \models P_{\langle S\rangle\sigma}(x)$. As $M_1' \models \psi_{\langle S\rangle\sigma}$, we have

$$M_1', \frac{v}{x} \;\models\; \exists y\big(\mathit{Access}_S(x, y) \wedge P_\sigma(y)\big).$$

Hence there exists some element $u \in P_\sigma^{M_1'}$ such that $M_1', \frac{v}{x}\frac{u}{y} \models \mathit{Access}_S(x, y)$. Therefore $(v, u) \in S^{M_2'}$ by the definition of $S^{M_2'}$. Since $u \in P_\sigma^{M_1'}$ and as

$\| \sigma \|^{M_2'} = P_\sigma^{M_1'}$ by the induction hypothesis, we may therefore conclude that $(M_2', v) \Vdash \langle S \rangle \sigma$. □

By Lemma 3.3 we immediately observe that since $M_1', \frac{w}{x} \models P_\psi(x)$, we must have $(M_2', w) \Vdash \psi$. Therefore $(M, w) \Vdash \varphi$. This, together with Lemma 3.2, justifies the following conclusion.

Theorem 3.4. *Each formula of* $\Sigma_1^1(\mathrm{ML})$ *translates to a uniformly equivalent formula of* $\exists\mathrm{MSO}$. *The translation is effective.*

We then establish that $\varphi^*(x)$ is in fact expressible in monadic $\Sigma_1^1(\mathrm{MLE})$. This is easy. Fix a symbol $S \in Q_2^\psi$ and let A be the subset of SUB_ψ that contains exactly all the formulae of the form $\langle S \rangle \chi$. The formula

$$\exists y \big(Access_S(x, y) \wedge P_\sigma(y) \big)$$

is uniformly equivalent to the following formula of MLE.

$$\bigvee_{B \subseteq A} \Big(\bigwedge_{\langle S \rangle \chi \, \in \, B} P_{\langle S \rangle \chi} \ \wedge \ \langle E \rangle \Big(P_\sigma \ \wedge \bigwedge_{\langle S \rangle \chi \, \in \, B} P_\chi \ \wedge \bigwedge_{\langle S \rangle \chi \, \in \, A \setminus B} \neg P_\chi \Big) \Big)$$

Thus we see that for each sentence ψ_α, where $\alpha \in \mathrm{SUB}_\psi$, there exists a formula of MLE that is uniformly equivalent to the formula $x = x \wedge \psi_\alpha$. We may therefore draw the following conclusion.

Theorem 3.5. *Each formula of* $\Sigma_1^1(\mathrm{ML})$ *translates to a uniformly equivalent formula of monadic* $\Sigma_1^1(\mathrm{MLE})$. *The translation is effective.*

The following corollaries are immediate.

Corollary 3.6. *Each formula of* $\Pi_1^1(\mathrm{ML})$ *translates to a uniformly equivalent formula of monadic* $\Pi_1^1(\mathrm{MLE})$. *The translation is effective.*

Corollary 3.7. *Let $\mathcal{C}$ be a class of unimodal Kripke frames (W, R_0) with a binary relation R_0. Let I be a set of indices such that $0 \in I$, and define*

$$\mathcal{D} \ = \ \{ \ (W, \{R_i\}_{i \in I}) \mid R_i \subseteq W \times W, \ (W, R_0) \in \mathcal{C} \ \}.$$

If the satisfiability problem of MLE *w.r.t. the class $\mathcal{C}$ is decidable, then the satisfiability problem of* ML *w.r.t. $\mathcal{D}$ is decidable.*

4 $\Sigma_1^1(\mathrm{PBML}^=)$ translates into $\exists$MSO

In this section we prove that each formula of $\Sigma_1^1(\mathrm{PBML}^=)$ can be translated to a uniformly equivalent formula of $\exists$MSO. The translation bears some similarity to the translation of $\Sigma_1^1(\mathrm{ML})$ into monadic $\Sigma_1^1(\mathrm{MLE})$, but is much more complicated. Instead of using the notion of a largest filtration and subformulae, the translation from $\Sigma_1^1(\mathrm{PBML}^=)$ into $\exists$MSO is based on types. One of the main ideas in the translation is to use fresh unary predicates P_τ in order to encode extensions of types τ of $\mathrm{PBML}^=$ formulae. In addition to types, the translation also uses fresh unary predicates that encode information concerning extensions of access types.

4.1 An effective translation

In the current subsection we define an effective translation of formulae of $\Sigma_1^1(\mathrm{PBML}^=)$ to uniformly equivalent formulae of $\exists$MSO. Effectivity of the translation follows from the decidability of $\mathrm{PBML}^=$.

Let us fix a $\Sigma_1^1(\mathrm{PBML}^=)$ formula φ and show how it is translated. Let $\varphi := \overline{Q}\,\psi$, where $\overline{Q}$ is vector of existential second-order quantifiers and ψ a formula of $\mathrm{PBML}^=$. For presentation related results, assume w.l.o.g. that $Md(\psi) \geq 2$ and that each symbol in $\overline{Q}$ occurs in ψ. We let m denote the maximum arity of the modal parameters that occur in ψ. Since $Md(\psi) \geq 2$, the formula ψ must contain diamonds, and therefore m exists and $m \geq 2$.

Let V_1^ψ denote the set of unary relation symbols that occur in ψ, and let V_h^ψ be the set of relation symbols of higher arities occurring in ψ. Let

$$V^\psi = V_1^\psi \cup V_h^\psi.$$

Some of the relation symbols in V^ψ may occur in the quantifier prefix $\overline{Q}$ and some may not. Let Q_1^ψ denote the set of unary relation symbols that occur in $\overline{Q}$. The set of relation symbols of higher arities occurring in $\overline{Q}$ is denoted by Q_h^ψ. Let

$$Q^\psi = Q_1^\psi \cup Q_h^\psi.$$

For each $k \in \mathbb{N}_{\geq 2}$, we let $\mathrm{ATP}_\psi(k)$ denote the set containing exactly the k-ary access types over V^ψ. For each $n \in \mathbb{N}$, we let TP_ψ^n denote the set $\mathrm{TP}_{V^\psi,m}^n$ of types. We define

$$\mathrm{TP}_\psi \;=\; \bigcup_{i\,\leq\,Md(\psi)} \mathrm{TP}_\psi^i.$$

We then fix a set of fresh (i.e., not occurring in φ) unary predicate symbols. We fix a unique unary predicate symbol P_τ for each $\tau \in \mathrm{TP}_\psi$. We also fix a unary predicate symbol $P_{(\mathcal{M},\overline{\beta})}$ for each pair $(\mathcal{M},\, \overline{\beta})$ such that for some $k \in \{1, ..., m-1\}$, we have $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and $\overline{\beta} \in (\mathrm{TP}_\psi^{Md(\psi)-1})^k$.

The translation $\varphi^*(x)$ of φ is the formula

$$\Big(\exists P\Big)_{P\,\in\, Q_1^\psi}\Big(\exists P_\tau\Big)_{\tau\,\in\,\mathrm{TP}_\psi}\Big(\exists P_{(\mathcal{M},\,\overline{\beta})}\Big)_{\substack{k\,\in\,\{1,\ldots,m-1\}\\ \mathcal{M}\,\in\,\mathrm{ATP}_\psi(k+1),\\ \overline{\beta}\,\in\,(\mathrm{TP}_\psi^{Md(\psi)-1})^k}} \psi^*(x)\ ,$$

where $\psi^*(x)$ is a first-order formula—to be defined below—in one free variable, x. We let $\overline{Q}^*$ denote the above vector of monadic existential second-order quantifiers.

One fundamental idea in the translation we will define is that the symbols P_τ are used in order to encode the extensions of the types $\tau \in \mathrm{TP}_\psi$. This is manifest in the way the model M_1 is defined below and also in the content of Lemma 4.6. While the symbols P_τ store information about extensions of types, the symbols $P_{(\mathcal{M},\,\overline{\beta})}$ are used in order to encode information about the extensions of the access types $\mathcal{M} \in \mathrm{ATP}_\psi(Ar(\mathcal{M}))$. We use the symbols $P_{(\mathcal{M},\,\overline{\beta})}$ when we define the formulae $Access_\mathcal{M}(x, y_1, ..., y_k)$ below. The formulae $Access_\mathcal{M}(x, y_1, ..., y_k)$ encode information about the extensions of the access types $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ in a way made explicit in Lemmata 4.1 and 4.5.

Before fixing the translation $\varphi^*(x)$ of φ, we define a number of auxiliary formulae. The first formula we define ensures that for all $n \in \{0, ..., Md(\psi)\}$, the extensions of the predicate symbols P_τ, where $\tau \in \mathrm{TP}_\psi^n$, always cover all of the domain of any model and never overlap each other. We define

$$\psi_{uniq} := \forall x\Big(\bigwedge_{0\,\leq\, i\,\leq\, Md(\psi)}\Big(\bigvee_{\tau\,\in\,\mathrm{TP}_\psi^i}\Big(P_\tau(x)\ \wedge \bigwedge_{\substack{\sigma\,\in\,\mathrm{TP}_\psi^i,\\ \sigma\,\neq\,\tau}}\neg P_\sigma(x)\Big)\Big)\Big).$$

The next formula asserts that each symbol P_β, where $\beta \in \mathrm{TP}_\psi^{Md(\psi)-1}$, must be interpreted such that for all symbols P_τ, where $Md(\tau) < Md(\beta)$, the extension of P_β is either fully included in the extension of P_τ or does not overlap with it. We let

$$\psi_{pack} := \forall x \forall y \bigwedge_{\beta \,\in\, \mathrm{TP}_\psi^{Md(\psi)-1}} \Big(\big(P_\beta(x) \wedge P_\beta(y)\big) \rightarrow \bigwedge_{\tau \,\in\, \mathrm{TP}_\psi^{<Md(\psi)-1}} \big(P_\tau(x) \leftrightarrow P_\tau(y)\big)\Big).$$

Let k be an integer such that $1 \leq k \leq m-1$ and let $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$. The next formula encodes information about the relation that the $(k+1)$-ary access type $\mathcal{M}$ defines over a V^ψ-model.

$$Access_{\mathcal{M}}(x, y_1, ..., y_k) := \bigvee_{(\beta_1,...,\beta_k) \,=\, \overline{\beta} \,\in\, (\mathrm{TP}_\psi^{Md(\psi)-1})^k} \Big(P_{(\mathcal{M},\overline{\beta})}(x) \wedge P_{\beta_1}(y_1) \ \wedge ... \wedge P_{\beta_k}(y_k)\Big).$$

We then define formulae $\chi_\tau(x)$ that recursively force the interpretations of the predicate symbols P_τ to match the extensions of the types $\tau \in \mathrm{TP}_\psi$. The content of this assertion is reflected in (the proof of) Lemma 4.6. First, let $\tau \in \mathrm{TP}_\psi^0$. We define

$$\chi_\tau(x) := \bigwedge_{\substack{P \,\in\, V_1^\psi, \\ \tau \,\Vdash\, P}} P(x) \qquad \wedge \bigwedge_{\substack{Q \,\in\, V_1^\psi, \\ \tau \,\nVdash\, Q}} \neg Q(x).$$

Now let $\tau \in \mathrm{TP}_\psi^{n+1}$, where $0 \leq n \leq Md(\psi) - 1$. We define

$$\chi_\tau^+(x) := \bigwedge_{\substack{k \,\in\, \{1,...,m-1\}, \\ \mathcal{M} \,\in\, \mathrm{ATP}_\psi(k+1), \\ (\sigma_1,...,\sigma_k) \,\in\, (\mathrm{TP}_\psi^n)^k, \\ \tau \,\Vdash\, \langle \mathcal{M} \rangle(\sigma_1,...,\sigma_k)}} \exists y_1...y_k \Big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\Big),$$

$$\chi_\tau^-(x) := \bigwedge_{\substack{k \,\in\, \{1,...,m-1\}, \\ \mathcal{M} \,\in\, \mathrm{ATP}_\psi(k+1), \\ (\sigma_1,...,\sigma_k) \,\in\, (\mathrm{TP}_\psi^n)^k, \\ \tau \,\Vdash\, \neg\langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k)}} \neg\exists y_1...y_k\Big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\Big),$$

and

$$\chi_\tau(x) \;:=\; P_{\tau'}(x) \;\wedge\; \chi_\tau^+(x) \;\wedge\; \chi_\tau^-(x),$$

where τ' is the unique type in TP_ψ^n such that $\tau \Vdash \tau'$.

Let $k \in \{1, ..., m-1\}$ and $A \subseteq \mathrm{ATP}_\psi(k+1)$, where $A \neq \emptyset$. Let

$$(\beta_1, ..., \beta_k) \;=\; \overline{\beta} \;\in\; (\mathrm{TP}_\psi^{Md(\psi)-1})^k.$$

The next formula encodes information about the *set* of $(k+1)$-ary access types that connect an element of the domain of a V^ψ-model to k-tuples of elements $(u_1, ..., u_k)$ such that for all i, the element u_i satisfies the type β_i. We define

$$\psi_{(A,\overline{\beta})}(x) := \bigwedge_{\mathcal{M} \,\in\, A} \exists y_1...y_k\Big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\beta_1}(y_1) \wedge ... \wedge P_{\beta_k}(y_k)\Big).$$

Our next aim is to define formulae ψ_{cons} and ψ'_{cons} that ensure that information about extensions of the access types over V^ψ is always consistent with interpretation of the access types over $V^\psi \setminus Q^\psi$, i.e., the access types describing non-quantified accessibility relations.

Let k be an integer such that $1 \leq k \leq m-1$. Fix a linear order on $\mathrm{ATP}_\psi(k+1)$. For each set $S \subseteq \mathrm{ATP}_\psi(k+1)$, let $S(i)$ denote the i-th member of the set S with respect to the linear order. Let $A \subseteq \mathrm{ATP}_\psi(k+1)$ be a nonempty set of access types. For each $i \in \{1, ..., |A|\}$, define a k-tuple $\overline{y}_i = (y_{i_1}, ..., y_{i_k})$ of variable symbols. Fix the collection of tuples so that no variable symbol is used twice. Let $\overline{y}_j \neq \overline{y}_l$ denote the formula

$$\bigvee_{n \,\in\, \{1,...,k\}} (\neg\, y_{j_n} = y_{l_n}).$$

Let $\chi_{A(i)}(x, \overline{y}_i)$ denote a first-order formula stating that the $(k+1)$-tuple $(x, \overline{y}_i)$ is connected according to the unique $(k+1)$-ary access type over $V^\psi \setminus Q^\psi$ that is consistent with the access type $A(i) \in A$. Let $\overline{\beta} = (\beta_1, ..., \beta_k)$ be a k-tuple of types in $\mathrm{TP}_\psi^{Md(\psi)-1}$. We let

$$\chi_{(A,\overline{\beta})}(x) \;:=\; \exists \overline{y}_1 ... \overline{y}_{|A|} \Big(\bigwedge_{\substack{j,l \,\in\, \{1 ,..., |A|\},\\ j \neq l}} \overline{y}_j \neq \overline{y}_l \quad \wedge$$

$$\bigwedge_{i \,\in\, \{1 ,..., |A|\}} \big(\chi_{A(i)}(x, \overline{y}_i) \wedge P_{\beta_1}(y_{i_1}) \wedge ... \wedge P_{\beta_k}(y_{i_k}) \big) \Big).$$

We define

$$\psi_{cons} := \forall x \Big(\bigwedge_{\substack{k \,\in\, \{1,...,m-1\},\\ A \,\subseteq\, \mathrm{ATP}_\psi(k+1),\ A \,\neq\, \emptyset,\\ \overline{\beta} \,\in\, (\mathrm{TP}_\psi^{Md(\psi)-1})^k}} \big(\psi_{(A,\overline{\beta})}(x) \;\rightarrow\; \chi_{(A,\overline{\beta})}(x) \big) \Big).$$

Let $\mathcal{R} \in \mathrm{ATP}_{V^\psi \setminus Q^\psi}(k+1)$, i.e., $\mathcal{R}$ is a $(k+1)$-ary access type over $V^\psi \setminus Q^\psi$. We let $C(\mathcal{R})$ denote the set of $(k+1)$-ary access types over V^ψ that are consistent with $\mathcal{R}$. Let $\chi_{\mathcal{R}}(x, y_1, ..., y_k)$ denote a first-order formula stating that the $(k+1)$-tuple $(x, y_1, ..., y_k)$ is connected according to the access type $\mathcal{R}$. Let $\overline{\beta} = (\beta_1, ..., \beta_k)$ be a k-tuple of types in $\mathrm{TP}_\psi^{Md(\psi)-1}$. Let $\mathcal{M}$ be $(k+1)$-ary access type over V^ψ. We let

$$\chi_{(\mathcal{R},\overline{\beta})}(x) \;:=\; \exists y_1 ... y_k \big(\chi_{\mathcal{R}}(x, y_1, ..., y_k) \wedge P_{\beta_1}(y_1) \wedge ... \wedge P_{\beta_k}(y_k) \big)$$

and

$$\psi_{(\mathcal{M},\overline{\beta})}(x) \;:=\; \exists z_1 ... z_k \big(Access_{\mathcal{M}}(x, z_1, ..., z_k) \wedge P_{\beta_1}(z_1) \wedge ... \wedge P_{\beta_k}(z_k) \big).$$

We define

$$\psi'_{cons} := \forall x \Big(\bigwedge_{\substack{k \,\in\, \{1,...,m-1\},\\ \mathcal{R} \,\in\, \mathrm{ATP}_{V^\psi \setminus Q^\psi}(k+1),\\ \overline{\beta} \,\in\, (\mathrm{TP}_\psi^{Md(\psi)-1})^k}} \Big(\chi_{(\mathcal{R},\overline{\beta})}(x) \;\rightarrow\; \bigvee_{\mathcal{M} \,\in\, C(\mathcal{R})} \psi_{(\mathcal{M},\overline{\beta})}(x) \Big) \Big).$$

Finally, we define

$$\delta_\psi := \psi_{uniq} \wedge \psi_{pack} \wedge \psi_{cons} \wedge \psi'_{cons} \wedge \bigwedge_{\tau \,\in\, \mathrm{TP}_\psi} \forall x \Big(P_\tau(x) \leftrightarrow \chi_\tau(x) \Big)$$

and

$$\varphi^*(x) := \overline{Q}^* \Big(\quad \delta_\psi \quad \wedge \bigvee_{\substack{\alpha \,\in\, \mathrm{TP}_\psi^{Md(\psi)}, \\ \alpha \,\Vdash\, \psi}} P_\alpha(x) \quad \Big).$$

We then fix an arbitrary pointed model (M, w) of the vocabulary $V^\psi \backslash Q^\psi$. In the next two subsections we establish that

$$(M, w) \Vdash \varphi \;\Leftrightarrow\; M, \frac{w}{x} \models \varphi^*(x).$$

4.2 Proving that $\Sigma^1_1(\mathrm{PBML}^=) \leq \exists\mathrm{MSO}$: part one

In this subsection we show that $(M, w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$. Thus we assume that $(M, w) \Vdash \varphi$. Therefore there exists some expansion M_h of M by interpretations of the symbols in Q^ψ such that $(M_h, w) \Vdash \psi$. The subscript "h" in M_h stands for the word "higher" and indicates that M_h is an expansion of M by interpretations of symbols of arity one *and higher arities.*

We then define an expansion M_1 of M by interpreting the unary symbols in Q^ψ_1 and also the unary symbols of the type P_τ and $P_{(\mathcal{M},\, \overline{\beta})}$, where τ is a type in TP_ψ, and where $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and $\overline{\beta} \in (\mathrm{TP}_\psi^{Md(\psi)-1})^k$ for some $k \in \{1, ..., m-1\}$.

For each $P \in Q^\psi_1$, we define $P^{M_1} = P^{M_h}$. For each $\tau \in \mathrm{TP}_\psi$, we let $P^{M_1}_\tau = \|\ \tau\ \|^{M_h}$. Let $k \in \{1, ..., m-1\}$. Let $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and

$$(\beta_1, ..., \beta_k) \;=\; \overline{\beta} \;\in\; (\mathrm{TP}_\psi^{Md(\psi)-1})^k.$$

We define $P^{M_1}_{(\mathcal{M}, \overline{\beta})}$ to be exactly the set of elements $v \in Dom(M)$ such that for some tuple $(u_1, ..., u_k) \in (Dom(M))^k$, we have $(v, u_1, ..., u_k) \in \mathcal{M}^{M_h}$ and $u_i \in \|\ \beta_i\ \|^{M_h}$ for all $i \in \{1, ..., k\}$. In other words, we define

$$P^{M_1}_{(\mathcal{M}, \overline{\beta})} \;=\; \|\ \ \langle \mathcal{M} \rangle (\beta_1, ..., \beta_k)\ \ \|^{M_h}\ .$$

Next we discuss a number of auxiliary lemmata, and then establish that $M_1, \frac{w}{x} \models \psi^*(x)$. Recall that $\psi^*(x)$ is the first-order part of the translation $\varphi^*(x)$ of φ.

The following lemma establishes how the formula $Access_{\mathcal{M}}(x, y_1, ..., y_k)$ encodes information about the action of the diamond operator $\langle \mathcal{M} \rangle$ on M_h.

Lemma 4.1. *Let n be an integer such that we have $0 \leq n < Md(\psi)$. Let $k \in \{1, ..., m-1\}$, and let $(\tau_1, ..., \tau_k)$ be a tuple of types in TP_{ψ}^{n}. Let $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$ and $v \in Dom(M)$. We have*

$$(M_h, v) \Vdash \langle \mathcal{M} \rangle (\tau_1, ..., \tau_k)$$
$$\Leftrightarrow$$
$$M_1, \tfrac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k) \big).$$

Proof. Assume that $(M_h, v) \Vdash \langle \mathcal{M} \rangle (\tau_1, ..., \tau_k)$. Thus there exists some tuple

$$(u_1, ..., u_k) \in \parallel \tau_1 \parallel^{M_h} \times ... \times \parallel \tau_k \parallel^{M_h}$$

such that $(v, u_1, ... u_k) \in \mathcal{M}^{M_h}$. Let

$$(\beta_1, ..., \beta_k) \ = \ \overline{\beta} \ \in (\mathrm{TP}_{\psi}^{Md(\psi)-1})^k$$

be the k-tuple of types in $\mathrm{TP}_{\psi}^{Md(\psi)-1}$ such that we have $u_i \in \parallel \beta_i \parallel^{M_h}$ for all $i \in \{1, ..., k\}$. Thus $v \in P^{M_1}_{(\mathcal{M}, \overline{\beta})}$, and therefore

$$M_1, \frac{v}{x} \frac{u_1}{y_1} ... \frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

As $u_i \in \parallel \tau_i \parallel^{M_h} = P^{M_1}_{\tau_i}$ for all $i \in \{1, ..., k\}$, we have

$$M_1, \frac{v}{x} \frac{u_1}{y_1} ... \frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k).$$

Therefore

$$M_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k) \big),$$

as desired.

In order to deal with the converse direction, assume that

$$M_1, \frac{v}{x} \ \models \ \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k) \big).$$

Therefore, for some tuple

$$(u_1, ..., u_k) \in P^{M_1}_{\tau_1} \times ... \times P^{M_1}_{\tau_k},$$

we have

$$M_1, \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

Therefore $v \in P^{M_1}_{(\mathcal{M},\overline{\beta})}$ for some tuple

$$(\beta_1, ..., \beta_k) = \overline{\beta} \in (\mathrm{TP}^{Md(\psi)-1}_{\psi})^k$$

such that $u_i \in P^{M_1}_{\beta_i}$ for all $i \in \{1, ..., k\}$. We have $Md(\tau_i) \leq Md(\beta_i)$ for all $i \in \{1, ..., k\}$. Also, by the definition of the model M_1, we have $P^{M_1}_{\sigma} = \parallel \sigma \parallel^{M_h}$ for all $\sigma \in \mathrm{TP}_{\psi}$, so each set $P^{M_1}_{\sigma}$ is the extension of the type σ. Therefore, as $u_i \in P^{M_1}_{\beta_i} \cap P^{M_1}_{\tau_i}$ for all $i \in \{1, ..., k\}$, we conclude that $\parallel \beta_i \parallel^{M_h} \subseteq \parallel \tau_i \parallel^{M_h}$ for all $i \in \{1, ..., k\}$. Hence

$$\parallel \beta_1 \parallel^{M_h} \times ... \times \parallel \beta_k \parallel^{M_h} \subseteq \parallel \tau_1 \parallel^{M_h} \times ... \times \parallel \tau_k \parallel^{M_h}.$$

Also, as $v \in P^{M_1}_{(\mathcal{M},\overline{\beta})}$, we have $(v, u'_1, ..., u'_k) \in \mathcal{M}^{M_h}$ for some tuple

$$(u'_1, ..., u'_k) \in \parallel \beta_1 \parallel^{M_h} \times ... \times \parallel \beta_k \parallel^{M_h}.$$

Therefore we conclude that $(M_h, v) \Vdash \langle \mathcal{M} \rangle(\tau_1, ..., \tau_k)$, as desired. □

We then establish a link between interpretations of the formulae $\chi_{\tau}(x)$ and interpretations of the predicate symbols P_{τ} in the model M_1.

Lemma 4.2. *Let $v \in Dom(M)$ and $\tau \in \mathrm{TP}_{\psi}$. We have $M_1, \frac{v}{x} \models P_{\tau}(x)$ iff $M_1, \frac{v}{x} \models \chi_{\tau}(x)$.*

Proof. As $\parallel P \parallel^{M_h} = P^{M_1}$ for all $P \in V^{\psi}_1$, the claim follows directly for all $\tau \in \mathrm{TP}^0_{\psi}$. Therefore we may assume that $\tau \in \mathrm{TP}^{\geq 1}_{\psi}$. Throughout the proof, we let τ' denote the unique type in $\mathrm{TP}^{Md(\tau)-1}_{\psi}$ such that $\tau \Vdash \tau'$.

Assume that $M_1, \frac{v}{x} \models P_{\tau}(x)$. As $P^{M_1}_{\tau} = \parallel \tau \parallel^{M_h}$, we have $(M_h, v) \Vdash \tau$. As $\tau \Vdash \tau'$, we have $(M_h, v) \Vdash \tau'$. Since $P^{M_1}_{\tau'} = \parallel \tau' \parallel^{M_h}$, we conclude that $M_1, \frac{v}{x} \models P_{\tau'}(x)$.

We then establish that $M_1, \frac{v}{x} \models \chi^+_{\tau}(x) \wedge \chi^-_{\tau}(x)$. Let $k \in \{1, ..., m-1\}$ and assume that $\tau \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, where we have $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$ and

$\sigma_i \in \mathrm{TP}_\psi^{Md(\tau)-1}$ for all $i \in \{1, ..., k\}$. As we have $(M_h, v) \Vdash \tau$, we also have $(M_h, v) \Vdash \langle\mathcal{M}\rangle(\sigma_1, ..., \sigma_k)$. Therefore, by Lemma 4.1,

$$M_1, \frac{v}{x} \;\models\; \exists y_1 ... y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big).$$

Similarly, if $\tau \Vdash \neg\langle\mathcal{M}\rangle(\sigma_1, ..., \sigma_k)$, we conclude by Lemma 4.1 that

$$M_1, \frac{v}{x} \;\models\; \neg\exists y\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big).$$

Thus $M_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, as desired.

For the converse, assume that $M_1, \frac{v}{x} \models \chi_\tau(x)$. In order to show that $M_1, \frac{v}{x} \models P_\tau(x)$, we will establish that $(M_h, v) \Vdash \tau$. As $P_\tau^{M_1} = \parallel \tau \parallel^{M_h}$, this suffices.

As $M_1, \frac{v}{x} \models P_{\tau'}(x)$ and $P_{\tau'}^{M_1} = \parallel \tau' \parallel^{M_h}$, we immediately observe that $(M_h, v) \Vdash \tau'$.

Let $\tau \Vdash \langle\mathcal{M}\rangle(\sigma_1, ..., \sigma_k)$, where $\mathcal{M} \in \mathrm{ATP}_\psi$ and $\sigma_i \in \mathrm{TP}_\psi^{Md(\tau)-1}$ for all $i \in \{1, ..., k\}$. As $M_1, \frac{v}{x} \models \chi_\tau^+(x)$, we have

$$M_1, \frac{v}{x} \;\models\; \exists y_1 ... y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big),$$

and therefore $(M_h, v) \Vdash \langle\mathcal{M}\rangle(\sigma_1, ..., \sigma_k)$ by Lemma 4.1. Similarly, if we have $\tau \Vdash \neg\langle\mathcal{M}\rangle(\sigma_1, ..., \sigma_k)$, then, as $M_1, \frac{v}{x} \models \chi_\tau^-(x)$, we conclude that

$$M_1, \frac{v}{x} \;\models\; \neg\exists y_1 ... y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big),$$

and therefore $(M_h, v) \Vdash \neg\langle\mathcal{M}\rangle(\sigma_1, ..., \sigma_k)$ by Lemma 4.1. Thus $(M_h, v) \Vdash \tau$, and hence $M_1, \frac{v}{x} \models P_\tau(x)$, as desired. $\square$

We then conclude the first direction of the proof of the claim that $(M, w) \Vdash \varphi$ iff $M, \frac{w}{x} \models \varphi^*(x)$.

Lemma 4.3. *Under the assumption $(M, w) \Vdash \varphi$, we have $M, \frac{w}{x} \models \varphi^*(x)$.*

Proof. We have assumed that $(M, w) \Vdash \varphi$ and thereby concluded that there exists a model M_h such that $(M_h, w) \Vdash \psi$. We have then defined the model M_1, and we now establish the claim of the current lemma by proving that $M_1, \frac{w}{x} \models \psi^*(x)$. Recall that $\psi^*(x)$ is the formula

$$\delta_\psi \quad \wedge \bigvee_{\substack{\alpha \,\in\, \mathrm{TP}_\psi^{Md(\psi)}, \\ \alpha \,\Vdash\, \psi}} P_\alpha(x) \ ,$$

where δ_ψ denotes the formula

$$\psi_{uniq} \wedge \psi_{pack} \wedge \psi_{cons} \wedge \psi'_{cons} \wedge \bigwedge_{\tau \,\in\, \mathrm{TP}_\psi} \forall x \Big(P_\tau(x) \leftrightarrow \chi_\tau(x) \Big).$$

Let ψ' denote a disjunction of exactly all the types $\alpha \in \mathrm{TP}_\psi^{Md(\psi)}$ such that $\alpha \Vdash \psi$. As ψ and ψ' are V^ψ-equivalent (and in fact uniformly equivalent), we have $(M_h, w) \Vdash \psi'$. Therefore $(M_h, w) \Vdash \alpha$ for some $\alpha \in \mathrm{TP}_\psi^{Md(\psi)}$ occurring in the disjunction. Hence, as $\parallel \alpha \parallel^{M_h} = P_\alpha^{M_1}$, we conclude that $M_1, \frac{w}{x} \models P_\alpha(x)$.

We then show that $M_1 \models \psi_{cons}$. Let $v \in Dom(M)$ and assume that $M_1, \frac{v}{x} \models \psi_{(A, \overline{\beta})}(x)$ for some nonempty $A \subseteq \mathrm{ATP}_\psi(k+1)$ and some tuple of types

$$(\beta_1, ..., \beta_k) \;=\; \overline{\beta} \;\in\; (\mathrm{TP}_\psi^{Md(\psi)-1})^k.$$

Recall that $A(i)$ denotes the i-th access type in A with respect to the linear ordering of $\mathrm{ATP}_\psi(k+1)$ we fixed. As $M_1, \frac{v}{x} \models \psi_{(A, \overline{\beta})}(x)$, we conclude by Lemma 4.1 that $(M_h, v) \Vdash \langle A(i) \rangle (\beta_1, ..., \beta_k)$ for each $i \in \{1, ..., |A|\}$. Thus there must exist $|A|$ distinct k-tuples

$$\overline{u}_1, \,...\,, \overline{u}_{|A|} \;\in\; \parallel \beta_1 \parallel^{M_h} \times ... \times \parallel \beta_k \parallel^{M_h} \;=\; P_{\beta_1}^{M_1} \times ... \times P_{\beta_k}^{M_1}$$

such that $(v, \overline{u}_i) \in (A(i))^{M_h}$ for each i. Let $\mathcal{R}_i$ denote the access type over $V^\psi \setminus Q^\psi$ consistent with $A(i)$. Recall that $\chi_{A(i)}(x, \overline{y}_i)$ is a first-order formula stating that the tuple $(x, \overline{y}_i)$ is connected according to the access type $\mathcal{R}_i$. We have $(v, \overline{u}_i) \in \mathcal{R}_i^{M_h} = \mathcal{R}_i^{M_1}$ for each i, and thus

$$M_1, \frac{v}{x}\frac{u_{i_1}}{y_{i_1}}...\frac{u_{i_k}}{y_{i_k}} \;\models\; \chi_{A(i)}(x, y_{i_1}, ..., y_{i_k}) \wedge P_{\beta_1}(y_{i_1}) \wedge ... \wedge P_{\beta_k}(y_{i_k})$$

for each i.

We then establish that $M_1 \models \psi'_{cons}$. Let $k \in \{1, ..., m-1\}$ and let $\mathcal{R}$ be a $(k+1)$-ary access type over $V^\psi \setminus Q^\psi$. Let $v \in Dom(M)$ and assume that

$$M_1, \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models \chi_{\mathcal{R}}(x, y_1, ..., y_k) \wedge P_{\beta_1}(y_1) \wedge ... \wedge P_{\beta_k}(y_k)$$

for some $u_1, ..., u_k \in Dom(M)$. Let $\mathcal{M}$ be the $(k+1)$-ary access type such that $(v, u_1, ..., u_k) \in \mathcal{M}^{M_h}$. Thus $(M_h, v) \Vdash \langle \mathcal{M} \rangle (\beta_1, ..., \beta_k)$, whence by Lemma 4.1, we have

$$M_1, \frac{v}{x} \;\models\; \exists z_1 ... z_k \big(Access_{\mathcal{M}}(x, z_1, ..., z_k) \wedge P_{\beta_1}(z_1) \wedge ... \wedge P_{\beta_k}(z_k) \big).$$

Clearly $\mathcal{M}$ is consistent with $\mathcal{R}$ and hence we have $\mathcal{M} \in C(\mathcal{R})$. Therefore $M_1 \models \psi'_{cons}$.

We have $M_1 \models \psi_{uniq} \wedge \psi_{pack}$ directly by properties of types. Therefore, in order to conclude the proof, we only need to establish that for each type $\tau \in \mathrm{TP}_{\psi}$ and each $v \in Dom(M)$, $M_1, \frac{v}{x} \models P_\tau(x) \leftrightarrow \chi_\tau(x)$. This follows directly by Lemma 4.2. $\square$

4.3 Proving that $\Sigma^1_1(\mathrm{PBML}^=) \leq \exists\mathrm{MSO}$: part two

In this subsection we show that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. Thus we assume that $M, \frac{w}{x} \models \varphi^*(x)$. Therefore there exists an expansion M'_1 of M by interpretations of the unary symbols P_τ and $P_{(\mathcal{M},\overline{\beta})}$, and also the symbols $P \in Q^{\psi}_1$, such that $M'_1, \frac{w}{x} \models \psi^*(x)$.

We define an expansion of M by interpreting all the relation symbols in Q^{ψ}. We call the resulting expansion M'_h. For each $P \in Q^{\psi}_1$, we define $P^{M'_h} = P^{M'_1}$. Let $v \in Dom(M)$ and $k \in \{1, ..., m-1\}$. Let

$$\overline{\beta} = (\beta_1, ..., \beta_k) \in (\mathrm{TP}^{Md(\psi)-1}_{\psi})^k.$$

Let $A_{k+1} \subseteq \mathrm{ATP}_{\psi}(k+1)$ be the set of access types $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$ such that for some tuple

$$(u_1, ..., u_k) \in P^{M'_1}_{\beta_1} \times ... \times P^{M'_1}_{\beta_k},$$

we have

$$M'_1, \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

As M'_1 satisfies the formula ψ_{cons}, we see that there exists a bijection f from the set A_{k+1} to a set

$$B \subseteq P^{M'_1}_{\beta_1} \times ... \times P^{M'_1}_{\beta_k}$$

such that for all $\mathcal{M} \in A_{k+1}$, we have $\big(v, f(\mathcal{M})\big) \in \mathcal{R}^{M'_1}_{\mathcal{M}}$, where $\mathcal{R}_{\mathcal{M}}$ is the access type in $\mathrm{ATP}_{V^{\psi}\setminus Q^{\psi}}(k+1)$ consistent with $\mathcal{M}$. Let $S \in Q^{\psi}_h$ be a relation symbol of the arity $k+1$. We define, for each $\mathcal{M} \in A_{k+1}$,

$$\big(v, f(\mathcal{M})\big) \ \in \ S^{M'_h} \ \text{ iff } \ S \in \mathcal{M}.$$

Recall that we write $S \in \mathcal{M}$ if S occurs in the type $\mathcal{M}$ (i.e., $\neg S$ does not occur in $\mathcal{M}$). We then consider the k-tuples in the set

$$(P^{M'_1}_{\beta_1} \times ... \times P^{M'_1}_{\beta_k}) \ \setminus \ B.$$

Let the tuple $(u_1, ..., u_k)$ belong to this set. Let $\mathcal{R}$ be the access type in $\mathrm{ATP}_{V^\psi \setminus Q^\psi}(k+1)$ such that $(v, u_1, ..., u_k) \in \mathcal{R}^{M_1'}$. As M_1' satisfies ψ'_{cons}, we observe that there exists some $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ consistent with $\mathcal{R}$ and some tuple

$$(u_1', ..., u_k') \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$$

such that

$$M_1', \frac{v}{x}\frac{u_1'}{y_1}...\frac{u_k'}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

Again let $S \in Q_h^\psi$ be a relation symbol of the arity $k+1$. We define

$$(v, u_1, ..., u_k) \ \in \ S^{M_h'} \ \text{ iff } \ S \in \mathcal{M}.$$

For each $v \in Dom(M)$ and $k \in \{1, ..., m-1\}$, we go through each tuple $\overline{\beta} \in (\mathrm{TP}_\psi^{Md(\psi)-1})^k$, and construct the extensions $S^{M_h'}$ of the $(k+1)$-ary symbols $S \in Q_h^\psi$ in the described way. This procedure defines the expansion M_h' of M. As the model M_1' satisfies ψ_{uniq}, the model M_h' is well defined.

Next we discuss a number of auxiliary lemmata and then establish that $(M_h', w) \Vdash \psi$. The following lemma is a direct consequence of the way we define the extensions $S^{M_h'}$ of the relation symbols $S \in Q_h^\psi$.

Lemma 4.4. *Let $v \in Dom(M)$. Let $k \in \{1, ..., m-1\}$, $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and $(\beta_1, ..., \beta_k) \in (\mathrm{TP}_\psi^{Md(\psi)-1})^k$. Then*

$$(v, u_1, ..., u_k) \in \mathcal{M}^{M_h'}$$

for some $(u_1, ..., u_k) \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$ if and only if we have

$$M_1', \frac{v}{x}\frac{u_1'}{y_1}...\frac{u_k'}{y_k} \models Access_{\mathcal{M}}(x, y_1, ...y_k)$$

for some $(u_1', ..., u_k') \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$.

The diamond $\langle \mathcal{M} \rangle$ encodes information about the relation that the formula $Access_{\mathcal{M}}(x, y_1, ..., y_k)$ defines over M_1'. The next lemma establishes this link.

Lemma 4.5. *Let n be an integer such that $0 \leq n < Md(\psi)$, and let $k \in \{1, ..., m-1\}$. Let $(\tau_1, ..., \tau_k) \in (\mathrm{TP}_\psi^n)^k$ and $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$. Assume that $\| \tau_i \|^{M_h'} = P_{\tau_i}^{M_1'}$ for all $i \in \{1, ..., k\}$. Let $v \in Dom(M)$. Then*

$$(M_h', v) \Vdash \langle \mathcal{M} \rangle (\tau_1, ..., \tau_k)$$

if and only if

$$M_1', \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k) \big).$$

Proof. Assume that $(M_h', v) \Vdash \langle \mathcal{M} \rangle (\tau_1, ..., \tau_k)$. Thus $(v, u_1, ..., u_k) \in \mathcal{M}^{M_h'}$ for some tuple

$$(u_1, ..., u_k) \in \parallel \tau_1 \parallel^{M_h'} \times ... \times \parallel \tau_k \parallel^{M_h'} = P_{\tau_1}^{M_1'} \times ... \times P_{\tau_k}^{M_1'}.$$

As $M_1' \models \psi_{uniq}$, we observe that for each $i \in \{1, ..., k\}$, there exists exactly one type $\beta_i \in \mathrm{TP}_{\psi}^{Md(\psi)-1}$ such that $u_i \in P_{\beta_i}^{M_1'}$. Therefore, by Lemma 4.4, we have

$$M_1', \frac{v}{x} \frac{u_1'}{y_1} ... \frac{u_k'}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k)$$

for some $(u_1', ..., u_k') \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$. Pick an arbitrary $j \in \{1, ..., k\}$.

1. If $n = Md(\psi) - 1$, then, as $M_1' \models \psi_{uniq}$ and $u_j \in P_{\beta_j}^{M_1'} \cap P_{\tau_j}^{M_1'}$, we have $\beta_j = \tau_j$, and thus $u_j' \in P_{\tau_j}^{M_1'}$.

2. If $n < Md(\psi) - 1$, then, since $M_1' \models \psi_{pack}$ and as $u_j \in P_{\tau_j}^{M_1'} \cap P_{\beta_j}^{M_1'}$ and $u_j' \in P_{\beta_j}^{M_1'}$, we again have $u_j' \in P_{\tau_j}^{M_1'}$.

Therefore

$$M_1', \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k) \big),$$

as required.

For the converse, assume that

$$M_1', \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k) \big).$$

Therefore

$$M_1', \frac{v}{x} \frac{u_1}{y_1} ... \frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k)$$

for some tuple

$$(u_1, ..., u_k) \in P_{\tau_1}^{M_1'} \times ... \times P_{\tau_k}^{M_1'} = \parallel \tau_1 \parallel^{M_h'} \times ... \times \parallel \tau_k \parallel^{M_h'}.$$

As $M_1' \models \psi_{uniq}$, we infer that for each u_i, there exists a type $\beta_i \in \mathrm{TP}_{\psi}^{Md(\psi)-1}$ such that $u_i \in P_{\beta_i}^{M_1'}$. By Lemma 4.4, we therefore have

$$(v, u_1', ..., u_k') \in \mathcal{M}^{M_h'}$$

for some tuple

$$(u_1', ..., u_k') \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}.$$

Pick an arbitrary $j \in \{1, ..., k\}$. As above, we have the following cases.

1. If $n = Md(\psi) - 1$, then, as $M_1' \models \psi_{uniq}$ and $u_j \in P_{\beta_j}^{M_1'} \cap P_{\tau_j}^{M_1'}$, we have $\beta_j = \tau_j$, and thus $u_j' \in P_{\tau_j}^{M_1'}$.
2. If $n < Md(\psi) - 1$, then, since $M_1' \models \psi_{pack}$ and as $u_j \in P_{\tau_j}^{M_1'} \cap P_{\beta_j}^{M_1'}$ and $u_j' \in P_{\beta_j}^{M_1'}$, we again have $u_j' \in P_{\tau_j}^{M_1'}$.

Therefore, as we have assumed that $P_{\tau_i}^{M_1'} = \parallel \tau_i \parallel^{M_h'}$ for all $i \in \{1, ..., k\}$, we conclude that $(M_h', v) \models \langle \mathcal{M} \rangle (\tau_1, ..., \tau_k)$, as desired. □

The next lemma establishes that extensions of the types $\tau \in \mathrm{TP}_\psi$ and interpretations of the predicate symbols P_τ coincide.

Lemma 4.6. *Let $\tau \in \mathrm{TP}_\psi$ and $v \in Dom(M)$. Then $(M_h', v) \Vdash \tau$ if and only if $M_1', \frac{v}{x} \models P_\tau(x)$.*

Proof. We prove the claim by induction on the modal depth of τ. If $\tau \in \mathrm{TP}_\psi^0$, then, as $M_1' \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$, the claim follows immediately.

Assume that $(M_h', v) \Vdash \tau$ for some $\tau \in \mathrm{TP}_\psi^{n+1}$, where $0 \leq n < Md(\psi)$. We will show that

$$M_1', \frac{v}{x} \models P_{\tau'}(x) \wedge \chi_\tau^+(x) \wedge \chi_\tau^-(x),$$

where τ' is the type of the modal depth n such that $\tau \Vdash \tau'$. This directly implies that $M_1', \frac{v}{x} \models P_\tau(x)$, since $M_1' \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$.

As $\tau \Vdash \tau'$, we have $(M_h', v) \Vdash \tau'$. Therefore $M_1', \frac{v}{x} \models P_{\tau'}(x)$ by the induction hypothesis. In order to establish that $M_1', \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, let $\tau \Vdash \langle \mathcal{M} \rangle (\sigma_1, ..., \sigma_k)$, where $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$, $k \in \{1, ..., m-1\}$ and $\sigma_i \in \mathrm{TP}_\psi^n$ for all $i \in \{1, ..., k\}$. Therefore $(M_h', v) \Vdash \langle \mathcal{M} \rangle (\sigma_1, ..., \sigma_k)$. Since by the induction hypothesis we have $\| \sigma_i \|^{M'_h} = P_{\sigma_i}^{M'_1}$ for all $i \in \{1, ..., k\}$, we conclude by Lemma 4.5 that

$$M'_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big).$$

Similarly, if $\tau \Vdash \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, then we have

$$M'_1, \frac{v}{x} \models \neg\exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big)$$

by the induction hypothesis and Lemma 4.5. Thus $M'_1, \frac{v}{x} \models \chi^+_\tau(x) \wedge \chi^-_\tau(x)$, and hence $M'_1, \frac{v}{x} \models P_\tau(x)$, as desired.

For the converse, assume that $M'_1, \frac{v}{x} \models P_\tau(x)$, where $\tau \in \mathrm{TP}^{n+1}_\psi$. Now, since $M'_1 \models \forall x (P_\tau(x) \leftrightarrow \chi_\tau(x))$, we conclude that $M'_1, \frac{v}{x} \models \chi_\tau(x)$. Therefore $M'_1, \frac{v}{x} \models P_{\tau'}(x)$, where τ' is the type of the modal depth n such that $\tau \Vdash \tau'$. Thus $(M'_h, v) \Vdash \tau'$ by the induction hypothesis.

Let $k \in \{1, ..., m-1\}$ and $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$. Assume that we have $\tau \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ for some $\sigma_1, ..., \sigma_k \in \mathrm{TP}^n_\psi$. As $M'_1, \frac{v}{x} \models \chi_\tau(x)$, we have $M'_1, \frac{v}{x} \models \chi^+_\tau(x)$, and therefore

$$M'_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big).$$

Hence, as we have $\| \sigma_i \|^{M'_h} = P_{\sigma_i}^{M'_1}$ for all $i \in \{1, ..., k\}$ by the induction hypothesis, we conclude that $(M'_h, v) \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ by Lemma 4.5. Similarly, if $\tau \Vdash \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, we conclude that $(M'_h, v) \Vdash \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ by the induction hypothesis and Lemma 4.5. We have therefore established that $(M'_h, v) \Vdash \tau$, as required. $\square$

We then finally conclude the proof of the claim that $M, \frac{w}{x} \models \varphi^*(x)$ if and only if $(M, w) \Vdash \varphi$.

Lemma 4.7. *Under the assumption $M, \frac{w}{x} \models \varphi^*(x)$, we have $(M, w) \Vdash \varphi$.*

Proof. We have assumed that $M, \frac{w}{x} \models \varphi^*(x)$ and thereby concluded that there exists a model M'_1 such that $M'_1, \frac{w}{x} \models \psi^*(x)$. We have then defined the model M'_h, and we now establish the claim of current the lemma by showing that $(M'_h, w) \Vdash \psi$.

As $M'_1, \frac{w}{x} \models \psi^*(x)$, we have $M'_1, \frac{w}{x} \models P_\alpha(x)$ for some type $\alpha \in \mathrm{TP}^{Md(\psi)}$ such that $\alpha \Vdash \psi$. Therefore $(M'_h, w) \Vdash \alpha$ by Lemma 4.6. As $\alpha \Vdash \psi$, we have $(M'_h, w) \Vdash \psi$, as desired. $\square$

The following theorem now follows directly by virtue of Lemmata 4.3 and 4.7.

Theorem 4.8. *Each formula of* $\Sigma^1_1(\mathrm{PBML}^=)$ *translates to a uniformly equivalent formula of* $\exists\mathrm{MSO}$*. The translation is effective.*

The following corollary is immediate.

Corollary 4.9. *Each formula of* $\Pi^1_1(\mathrm{PBML}^=)$ *translates to a uniformly equivalent formula of* $\forall\mathrm{MSO}$*. The translation is effective.*

Theorem 4.8 implies a range of decidability results.

Theorem 4.10. *Let* V *and* $U \subseteq V$ *be sets of indices. Let* $\mathcal{D}$ *be a class of Kripke frames* $(W, \{R_j\}_{j\in U})$ *with binary relations* R_i*. Define the class*

$$\mathcal{C} = \{\ (W, \{R_i\}_{i\in V}) \mid R_i \subseteq W \times W,\ (W, \{R_j\}_{j\in U}) \in \mathcal{D}\ \}$$

of Kripke frames. Now, if the $\forall\mathrm{MSO}$ *theory of* $\mathcal{D}$ *is decidable, then the satisfiability problem for* $\mathrm{BML}^=$ *w.r.t.* $\mathcal{C}$ *is decidable.*

Proof. Given a formula ψ of $\mathrm{BML}^=$, we existentially quantify all the relation symbols (unary and binary) occurring ψ, except for those in $\{R_j\}_{j\in U}$. We end up with a $\Sigma^1_1(\mathrm{BML}^=)$ formula φ, which we then effectively translate to a uniformly equivalent $\exists\mathrm{MSO}$ formula $\varphi^*(x)$, applying our translation above. We then modify this formula to an $\exists\mathrm{MSO}$ *sentence* χ, which is uniformly equivalent to the sentence $\exists x\, \varphi^*(x)$. Let χ' denote a sentence of $\forall\mathrm{MSO}$ uniformly equivalent to $\neg\chi$. Using the decision procedure for the $\forall\mathrm{MSO}$ theory of $\mathcal{D}$, we then check whether the sentence χ' is valid over $\mathcal{D}$. If it is, then ψ is not satisfiable w.r.t. $\mathcal{C}$, and if χ' is not valid over $\mathcal{D}$, then ψ is satisfiable w.r.t. $\mathcal{C}$. $\square$

We describe one possible application of Theorem 4.10. Let $\mathcal{C}$ be the class of countably infinite multimodal frames $(W, \{R_i\}_{i\in\mathbb{N}})$, where R_0 is a built-in dense linear ordering of W without endpoints. In other words,

$$\mathcal{C} = \{\ (W, \{R_i\}_{i\in\mathbb{N}}) \mid R_i \subseteq W \times W,\ (W, R_0) \in \mathcal{D}\ \},$$

where $\mathcal{D}$ is the class of countably infinite Kripke frames (W, R_0) such that R_0 is a dense linear ordering of W without endpoints. Assume we would like to know whether the satisfiability problem of multimodal logic—perhaps extended with, say, the difference modality—is decidable with respect to $\mathcal{C}$. By Theorem 4.10, we *directly* see that, indeed, it is decidable due to the following immediate observation. The MSO theory of $(\mathbb{Q}, <^{\mathbb{Q}})$ is known to be decidable [27], and therefore the $\forall$MSO theory of $\mathcal{D}$ is decidable.

Theorem 4.10 implies a wide range of decidability results for multimodal logic. There exists a large body of knowledge concerning structures and classes of structures with a decidable MSO (and therefore $\forall$MSO) theory, see [29] for example.

5 Conclusions

In this article we have investigated the expressive power of modal logics with existential prenex quantification of accessibility relations. We have shown that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into $\exists$MSO, and also that $\Sigma^1_1(\mathrm{ML})$ translates into monadic $\Sigma^1_1(\mathrm{MLE})$. These results directly imply that $\Pi^1_1(\mathrm{PBML}^=)$ translates into $\forall$MSO and $\Pi^1_1(\mathrm{ML})$ into monadic $\Pi^1_1(\mathrm{MLE})$. As corollaries of the translations, we have obtained results that can be used in order to establish decidability results for (extensions of) multimodal logics with respect to classes of frames with built-in relations.

In the future we expect to strengthen the obtained results. The main objective is to try to understand for what kinds fragments L of first-order logic the extension $\Sigma^1_1(\mathrm{L})$ collapses into $\exists$MSO. The next planned step involves considering graded (polyadic) modalities. While directly interesting, investigations along these kinds of lines could elucidate the role the *arities* of existentially quantified relations play in making the expressive power of (existential) second-order logic.

It also remains to be seen whether our investigations provide a stepping stone towards answering the question about existence of a class of finite directed graphs definable in $\Sigma^1_1(\mathrm{FO}^2)$ but not definable in $\exists$MSO. To show that $\Sigma^1_1(\mathrm{FO}^2)$ is contained in $\exists$MSO, one would have to extend the translation from $\Sigma^1_1(\mathrm{BML}^=)$ into $\exists$MSO such that it takes into account the possibility of using the converse operation.

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