Defined Topological Relations in Description Logics

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Abstract: In this paper we extend the language $\mathcal{ALC}(\mathcal{D})$ by a new operator which allows one to define roles on behalf of properties of objects which are given by a concrete domain. This operator can be used to define topological relations. We show that the extension results in an undecidable language and ask for a discussion about the appropriate semantic level of representation for spatial knowledge.

1 Introduction

Spatial reasoning using description logics is a rarely addressed issue so far. In this paper we contribute to the discussion by examining the possibilities of modeling topological relations in descriptions logics. Topological relations have proved to be very useful for modeling spatial knowledge in various domains. Examples include GIS, visual programming languages and analysis of aerial images ([7], [10]).

In our examples we will use topological relations as definded in [4], but our investigations are not restricted to this special definition. In [4], Egenhofer identifies the eight atomic topological relations *disjunct*, *meet*, *equal*, *inside*, *coveredBy*, *contains*, *covers*, and *overlap*, which have different relational properties. For example, the relation *overlap* is irreflexive, symmetric and not transitive. Furthermore, to every relation there is an inverse (e.g. the inverse of *overlap* is *overlap*) and a composition table is given which identifies the topological relation resulting from the composition of two atomic relations (e.g. *overlap* composed with *equal* yields *overlap*). The composition of two relations may also yield a disjunction of atomic relations. A representation of topological relations must capture these properties and inferences. The goal of our approach is not to define a special-purpose spatial description logic but to use generic DL-modeling mechanisms. In addition to topological relations with their various properties we want to represent concrete geometric properties of objects and to keep these consistent with topological relations. This means, first, that the topological relation between two objects should be automatically inferred if their geometric properties are known and second, that inconsistencies between asserted relations and geometric properties must be detected.

Because of the need to reason with concrete objects we have chosen the description logic $\mathcal{ALC}(\mathcal{D})$ (see [3]) as a starting point for our investigations. $\mathcal{ALC}(\mathcal{D})$ provides a well-defined semantics for combining abstract reasoning with so-called concrete domains. It yet lacks a mechanism to define relations on behalf of properties of objects. In this paper we will extend $\mathcal{ALC}(\mathcal{D})$ by an operator that allows this. The properties of the relations defined with this new operator depend on the underlying concrete domain. We show that there exist concrete domains appropriate for modeling topological relations. Then we prove that subsumption is undecidable for the extended version of $\mathcal{ALC}(\mathcal{D})$.

2 Extending $\mathcal{ALC}(\mathcal{D})$

In our extended language, topological relations are represented as roles. Geometric properties of objects form the basis for the definition of these roles. In our formalism, these properties are represented using a concrete domain. As defined in [3], a concrete domain $\mathcal{D}$ consists of a set $\Delta_\mathcal{D}$, the domain of $\mathcal{D}$, and a set $\Phi_\mathcal{D}$, the predicate names of $\mathcal{D}$. Additionally, each predicate name is associated with an arity n, and an n-ary predicate $P^\mathcal{D} \subseteq \Delta_\mathcal{D}^n$. We represent the geometric properties of spatial objects as polygons and thus need a concrete domain $\mathcal{P}$ where $\Delta_\mathcal{P}$ is the set of polygons. Spatial objects are then represented as abstract objects which have a concrete object (a polygon describing its spatial extension) as filler of a special attribute *has-area*:

$$\textit{spatial-object} \doteq \textit{is-polygon}(\textit{has-area}).$$

In this definition the concept-forming predicate operator of $\mathcal{ALC}(\mathcal{D})$ is used, *has-area* is an attribute and *is-polygon* is a predicate from $\Phi_\mathcal{P}$ which is true for all elements of $\Delta_{\mathcal{P}}$. To represent topological relations and to ensure consistency with the geometric information, we need an operator to define roles on behalf of properties of objects. This operator is the role-forming equivalent to the concept-forming predicate operator used above. We extend $\mathcal{ALC}(\mathcal{D})$ by this new operator as follows. Let u_1, $\ldots$, u_n and v_1, $\ldots$, v_m be attribute chains. Then

$$P(u_1, \ldots, u_n)(v_1, \ldots, v_m)$$

is a role-forming operator which may be used in concept terms and assertional axioms wherever a primitive role is allowed. Let a and b be abstract objects and x_1, $\ldots$, x_n as well as y_1, $\ldots$, y_m be concrete objects. The semantics of the new operator is given as follows:

$$\begin{aligned}
&(P(u_1, \ldots, u_n)(v_1, \ldots, v_m))^{\mathcal{I}} = \\
&\quad \{(a, b) \in \Delta_{\mathcal{I}} \times \Delta_{\mathcal{I}} \mid \\
&\quad\quad \exists x_1, \ldots, x_n, y_1, \ldots, y_m \in \Delta_{\mathcal{D}}: \\
&\quad\quad\quad u_1^{\mathcal{I}}(a) = x_1, \ldots, u_n^{\mathcal{I}}(a) = x_n, \\
&\quad\quad\quad v_1^{\mathcal{I}}(b) = y_1, \ldots, v_m^{\mathcal{I}}(b) = y_m, \\
&\quad\quad\quad (x_1, \ldots, x_n, y_1, \ldots, y_m) \in P^{\mathcal{D}}\}.
\end{aligned}$$

This extended language is called $\mathcal{ALCRP}(\mathcal{D})$. Together with an appropriate set of predicates $\Phi_{\mathcal{P}}$, it may be used to define topological relations. For example, the *inside* relation could be represented using the role *inside-p*(*has-area*)(*has-area*) where *inside-p* is the binary predicate that is true iff the polygon given as first argument is inside the polygon given as second argument. Other topological relations like *overlap* and *disjoint* can be defined accordingly.

It remains to be shown that there is an adequate concrete domain $\mathcal{P}$ that is admissible. For $\mathcal{ALC}(\mathcal{D})$, a concrete domain $\mathcal{D}$ is called admissible (see [3]) iff its set of predicates $\Phi_{\mathcal{D}}$ is closed under negation and the satisfiability of any finite conjunction of predicates is decidable. An instance of $\mathcal{ALC}(\mathcal{D})$ is decidable iff the instantiating concrete domain is admissible. A concrete domain for $\mathcal{ALCRP}(\mathcal{D})$ will obviously have to fulfill at least the same requirements to be admissible. In our case, a minimal set of predicates $\Phi_{\mathcal{P}}$ consists of 8 predicates for defining the 8 roles, the 8 complementary predicates,

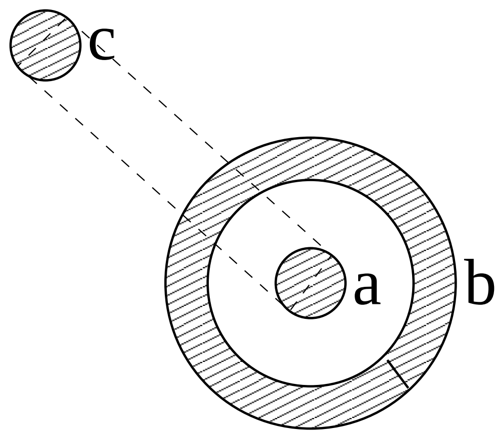

Figure 1: A realizability problem

the predicate *is-polygon* used above and its complement. The complement of a predicate used to define a certain topological relation may be used to define a relation which is the disjunction of the other 7 relations. This concrete domain does not allow concrete objects to be associated with concrete polygonal areas (for this additional unary predicates would be needed). Therefore, a standard algorithm for relational consistency and realizability of sets of topological constraints ([5], [6]) can be used to decide the satisfiability of finite conjunctions of predicates in our concrete domain. Relational consistency and realizability are different problems since there may be models which are relationally consistent but not realizable in the plane. In the case of this simple concrete domain, planarity algorithms from graph theory can be used to decide realizability ([6]). From this follows that this concrete domain is admissible.

A more complex set of predicates $\Phi_{\mathcal{P}}$ would contain a unary predicate $equals_p$ for any polygon p. With these predicates it is possible to identify concrete objects with polygonal regions. $\Phi_{\mathcal{P}}$ must also contain the complements $notequals_p$ of these additional predicates. In this extended concrete domain, relational consistency is also decidable. This can be seen as follows. For some pairs of objects the polygonal areas of both objects may be known. In these cases the relation that holds between the two objects can be computed and forms an additional constraint. After these additional constraints have been identified, a standard algorithm for relational consistency can be applied. With the extended set of predicates, realizability becomes much more important. Using only four polygons, Figure 1 shows a situation which is relationally consistent but not realizable in the plane. Assume that the polygonal areas of the objects **a**, **b** and **c** are given as depicted in the figure. Let there be another object for wich we know that it *overlaps* **a** and **c** and that it is *disjunct* to **b**. This situation obviously cannot be realized in the plane although the set of contraints is relationally consistent. Realizability for this extended concrete domain is also decidable. A sketch of the algorithm can not be given here because of space limitations.

In the following we will denote the topological roles as defined above by *inside*, *overlap* etc. instead of explicitly using the defining roleterms. These roles can be used to define concepts such as

$$\begin{aligned} \mathit{runway} \doteq & \mathit{concrete\text{-}area} \sqcap \exists \mathit{meets}.\mathit{taxiway} \sqcap \\ & \forall \mathit{meets}.\neg \mathit{terminal} \\ C \doteq & \exists \mathit{inside}.\exists \mathit{meet}.D \sqcap \forall \mathit{disjoint}.\neg D. \end{aligned}$$

A runway has at least one taxiway to which it is directly connected and there are no terminal buildings in its direct neighborhood which distinguishes it from an apron. C is an inconsistent concept (i.e. $C^{\mathcal{I}} = \emptyset$ for all $\mathcal{I}$) because the composition of *inside* and *meet* yields *disjoint*. This demonstrates that the inferences in Egenhofers composition table are computed correctly by our formalism. In fact, the definition of topological relations using the new operator fulfills all representational requirements formulated in Section 1.

The new operator is of course not restricted to this application domain. It may be used to define all kinds of relations that can be defined in terms of properties of the objects between which the relation does or does not hold. Other relations may for example be different spatial relations like *near*, *right-of*, *north*, etc. or temporal relations as defined by Allen (see [1]).

3 Decidability

The most important inference for description logics is deciding subsumption between two concepts. Subsumption can be reduced to the satisfiability problem for concepts, since a concept A subsumes a concept B iff $B \sqcap \neg A$ is not satisfiable. Unfortunately it can be proven that the satisfiability problem for $\mathcal{ALCRP}(\mathcal{D})$ is not decidable. This can be done by a reduction to the Post Correspondence Problem (PCP), which is defined as follows.

Given a nonempty finite set $S = \{(l_i, r_i); i = 1, \ldots, m\}$, where the l_i and r_i are words over an alphabet Σ, a *solution of* S is a sequence of indices $i_1, ..., i_k$ with $k \geq 1$ such that the concatenations $w_l = l_{i_1} \cdots l_{i_k}$ and $w_r = r_{i_1} \cdots r_{i_k}$ denote the same word. The PCP is known to be undecidable if Σ contains at least two symbols ([9]). The following reduction is inspired by the similar one for $\mathcal{ALC}(\mathcal{D})$ extended by a transitivity operator given in [3].

For the reduction, the elements of Σ are interpreted as digits of numbers at base B, where $B := |\Sigma| + 1$. $\overline{w}$ denotes the nonnegative integer at base 10 which the (nonempty) word w represents at base B. If vw is the concatenation of two words $v, w \in \Sigma^*$, then $\overline{vw} = \overline{v} * B^{|w|} + \overline{w}$, where $|w|$ is the length of the word w. $w \mapsto \overline{w}$ is a $1-1$-mapping from Σ^* into the set of nonnegative integers.

Let l, r, w_l, w_r and f_1,...,f_m be attributes, and R be a placeholder for the defined role $\mathit{trans\text{-}p}(wl, wr)(wl, wr)$, then for a given instance S of the PCP we define a concept *C(S)*:

$$\begin{aligned} \mathit{C(S)} \doteq & \mathit{null\text{-}p}(w_l) \sqcap \mathit{null\text{-}p}(w_r) \sqcap & (1) \\ & \sqcap_{i=1}^{m} \mathit{cnstr\text{-}p}_l^i(w_l, f_i w_l) \sqcap & (2) \\ & \sqcap_{i=1}^{m} \mathit{cnstr\text{-}p}_r^i(w_r, f_i w_r) \sqcap & (3) \\ & \forall R. \sqcap_{i=1}^{m} \mathit{cnstr\text{-}p}_l^i(w_l, f_i w_l) \sqcap & (4) \\ & \forall R. \sqcap_{i=1}^{m} \mathit{cnstr\text{-}p}_r^i(w_r, f_i w_r) \sqcap & (5) \\ & \forall R.\mathit{notequal\text{-}p}(w_l, w_r). & (6) \end{aligned}$$

Please note that the $f_i w_x$ terms denote composition of attributes. The predicates used are defined as follows:

$$\begin{aligned} \mathit{trans\text{-}p}(a, b, c, d) &:= a < c \wedge b < d \\ \mathit{null\text{-}p}(a) &:= a = 0 \\ \mathit{cnstr\text{-}p}_l^i(a, b) &:= b = \overline{l_i} + a * B^{|l_i|} \\ \mathit{cnstr\text{-}p}_r^i(a, b) &:= b = \overline{r_i} + a * B^{|r_i|} \\ \mathit{notequal\text{-}p}(a, b) &:= a \neq b \end{aligned}$$

It is proven below that the concept *C(S)* is satisfiable iff the PCP S has no solution. A satisfiability algorithm for $\mathcal{ALCRP}(\mathcal{D})$ could hence be used to decide whether a PCP has a solution. It has to be shown that (i) if *C(S)* is satisfiable then S has no solution and (ii) if S has no solution then *C(S)* is satisfiable.

The first point can be seen by examining the definition of *C(S)*. If *C(S)* is satisfiable then there exists an interpretation $\mathcal{I}$ with $C(S)^{\mathcal{I}} \neq \emptyset$. This interpretation encodes all possible sequences that could be a solution of the PCP S. It has the form of an infinite tree. Each node (abstract object) in the tree has m successors. The edges (attributes) to these successors are labeled with f_1 to f_m, respectively. The edges represent single indices and the nodes represent sequences of indices defined by the labels of the path from the root-node. The role R is used as a "universal role" to propagate concepts to every node in the tree: every node in the tree is an R-rolefiller of the root-node. In lines 4 and 5 predicate-operators which use the predicate *constr-p* are propagated to every node in the tree. These concepts enforce the successors of each node. The concepts in lines 2 and 3 do the same for the root node. Each node in the tree has concrete fillers of the wl and wr attributes. These fillers are enforced by the concept terms in lines 1-5. The fillers of these attributes encode the two concatenations that are defined by the index sequence which the node represents. Line 6 enforces that for all objects in the tree the fillers of wl and wr (and thus the concatenations for all possible sequences) differ. From this it is clear that the concept *C(S)* can only be satisfiable if the PCP S has no solution.

To prove the second point we give an interpretation with $C(S)^{\mathcal{I}} \neq \emptyset$ for a given PCP S for which it is known that no solution exists. This intepretation has the same structure as described above. This structure is depicted in figure 2. In the figure the role R has been omitted for clarity.

$$\Delta_{\mathcal{I}} = \{a_{ij}; i \geq 0, 0 \leq j < m^i\};$$
$$\forall i \geq 0, 0 \leq j < m^i :$$
$$f_1^{\mathcal{I}}(a_{ij}) = a_{i+1\ j*m},\ \ \ldots,\ \ f_m^{\mathcal{I}}(a_{ij}) = a_{i+1\ j*m+m-1},$$
$$w_l^{\mathcal{I}}(a_{ij}) = \overline{\phi_l(i,j)}, \quad w_r^{\mathcal{I}}(a_{ij}) = \overline{\phi_r(i,j)}$$

where ϕ_l and ϕ_r are two recursively defined con-

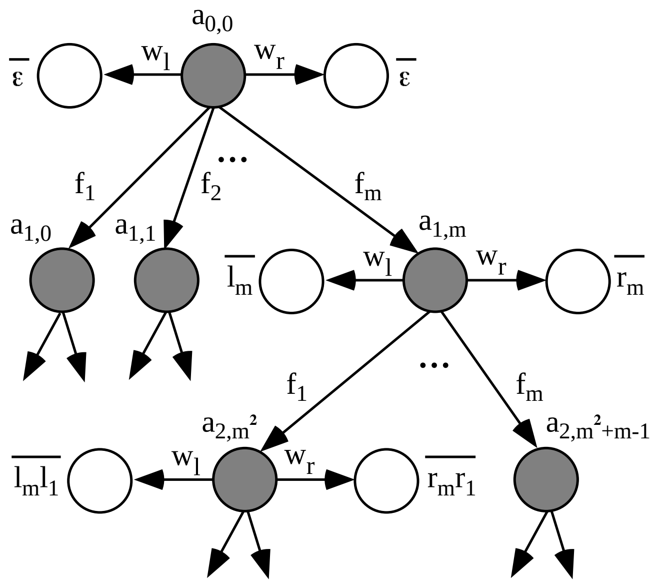

Figure 2: Structure of intepretations that satisfy *C(S)*.

catenation functions

$$\phi_l(0,0) = \epsilon$$
$$\phi_r(0,0) = \epsilon$$
$$\phi_l(i,j) = concat(\phi_l(i-1, \lfloor j/m \rfloor), l_{j+1-(m*\lfloor j/m \rfloor)})$$
$$\phi_r(i,j) = concat(\phi_r(i-1, \lfloor j/m \rfloor), r_{j+1-(m*\lfloor j/m \rfloor)}).$$

□

In the definition of *C(S)* the concept-forming predicate operator $P(u_1, \ldots, u_n)$ can be replaced by its role-forming pendant $\exists P'(u_1, \ldots, u_n)().\top$.

Please note that the emptiness of the second attribute list is not a restriction since one could introduce an additional attribute and extend the predicate to ignore this additional argument. Thus, even for the language consisting of conjunction, value-restriction, the new operator and the special concept $\top$ (with the semantics $\top^{\mathcal{I}}$ = abstract domain) subsumption is undecidable.

4 Discussion

In this paper we have shown that in description logics, a representation of spatial relations together with concrete geometric information is difficult when generic concept- and role-forming operators are employed. We have defined a role-forming operator that allows one to define roles on behalf of properties of objects. Even minimal languages that contain the new operator turned out to be undecidable. The proposed operator seems rather "natural" for this application domain and it seems hard to find an alternative one. From these results, the question of the appropriate semantic level of representation for spatial information in description logics arises. It remains to be examined if better results with respect to decidability can be obtained by designing a special-purpose topological description logic. One approach could be to combine the semantics of description logic and topology by providing a fixed set of roles that have the same properties as topological relations. It is important to note that the combination of description logic and topological relations itself bears a lot of complexity because of the various properties and interrelations of the relations. For example, in a language containing the *inside*-relation, value restriction and exists restriction, one has to deal with infinite models, even if no concrete geometric information is given. For example, the concept $C \doteq \exists inside \sqcap \forall inside.\exists inside$ has only infinite models. Note that this concept cannot be satisfied by a cyclic model because *inside* is not an ordinary transitive relation but has a spatial semantics which enforces that no object can be inside itself. This example only uses the properties of the *inside*-relation, but it is easy to construct other concepts which are only satisfiable by much more complex infinite models. For a description logic which models topological relations using concept-forming operators and remains decidable by restricting expressivity, see [8]. The general question is under which conditions a special-purpose description logic can provide more expressive power than a generic one while still remaining decidable. For an example of a special-purpose description logic see [2], where a temporal description logic is defined. The possible benefits from defining a special-purpose topological description logic still have to be investigated.

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