# Vector Field Processing with Clifford Convolution and Clifford Fourier Transform

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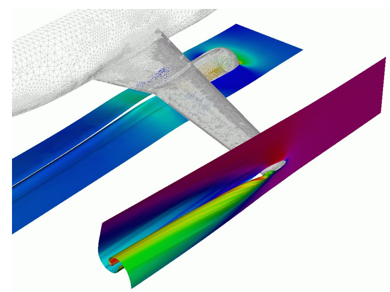
# Outline

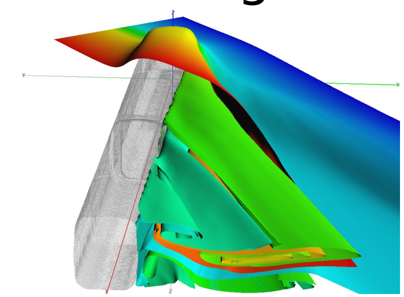
- **The problem**: CFD data and PIV measurements
- **The idea**: Transfer of image processing to vector fields
- **The solution**: Clifford Convolution
- **Practical results:** Vortex detection and separation lines
- **Theoretical results:** Clifford Fourier Transform
- **Other ideas**: IH-Fourier Series, Bi-quaternion Fourier Transform
- **Conclusion and open questions**

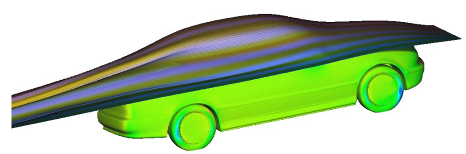
# 1. The Problem: CFD Data

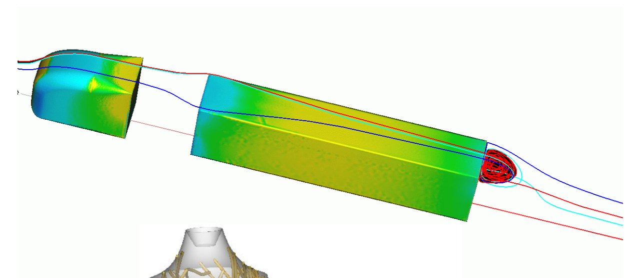
The analysis of flow is relevant for engineers in the aerospace, automotive and mechanical engineering industry, e. g. for design of

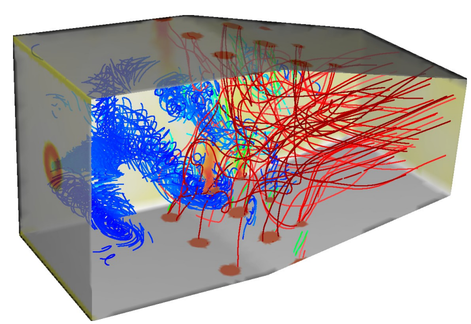
- airplanes
- helicopters
- trains
- cars
- trucks
- turbines
- combustion chambers

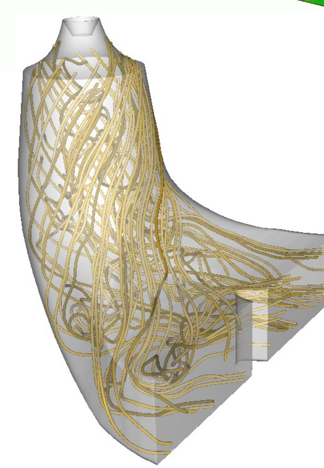






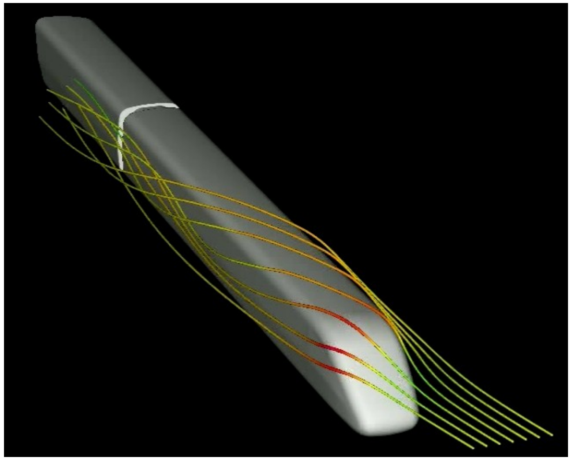


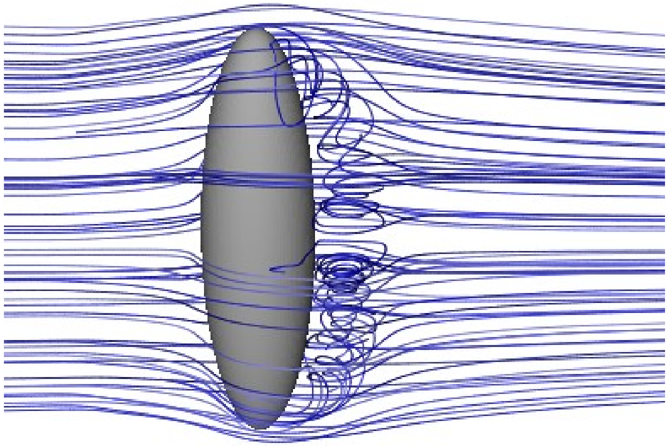


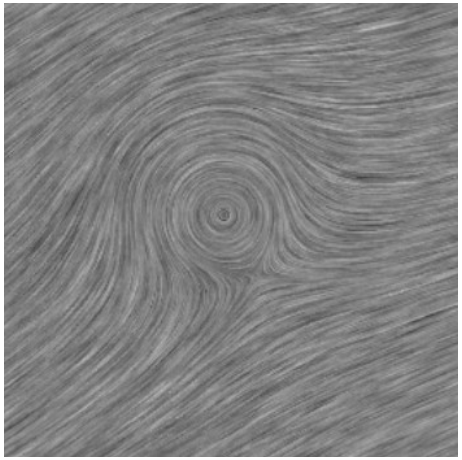


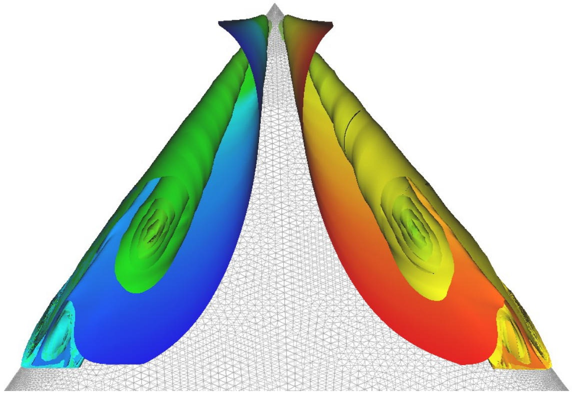
# The problem: CFD Post-processing

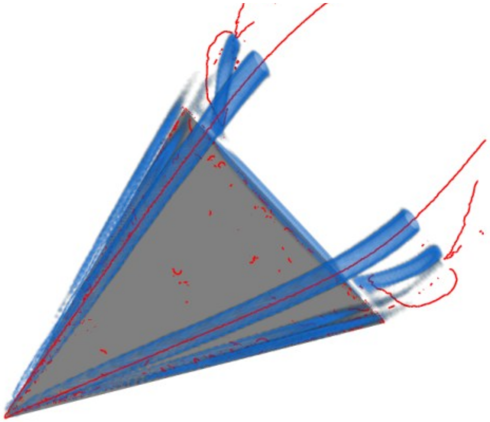
- Direct visualization shows arrows, streamlines, streamsurfaces
- Advanced visualization detects features like vortices, separation

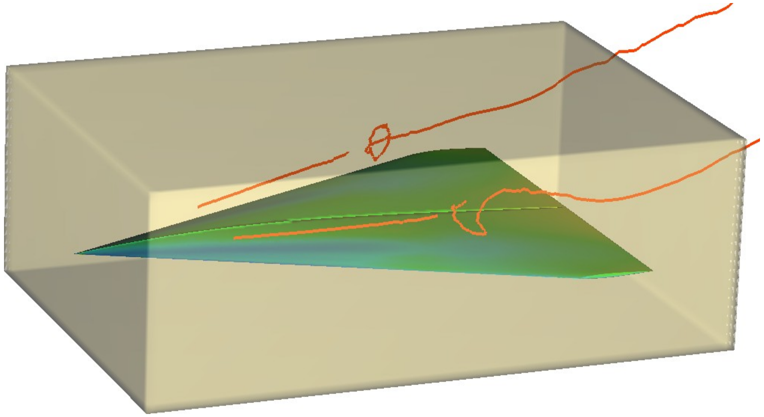






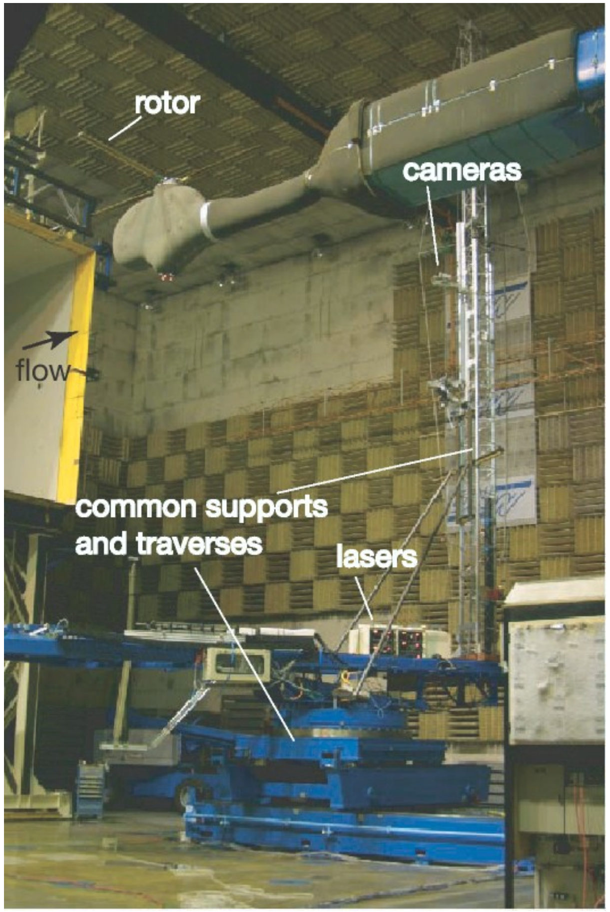


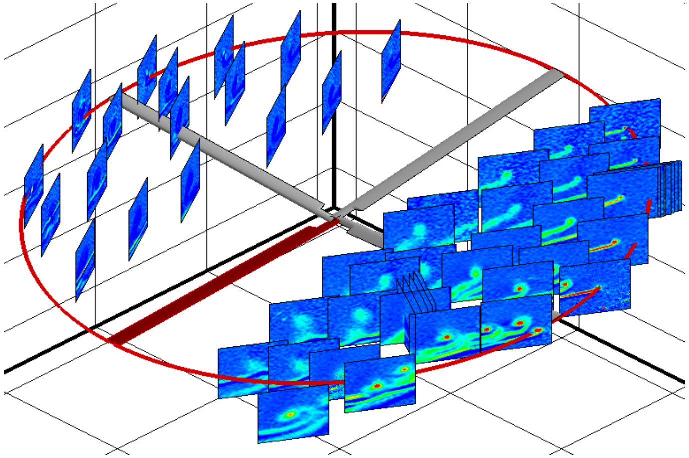


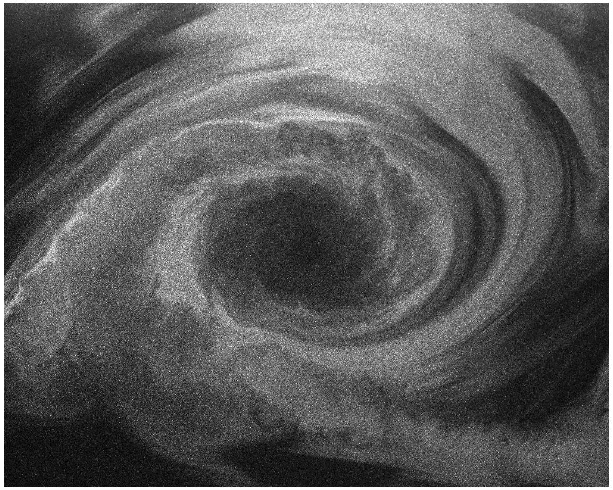


# The problem: PIV measurements

- 3D velocity field can be measured by particle image velocimetry
- Here: velocity measurement behind helicopter rotor

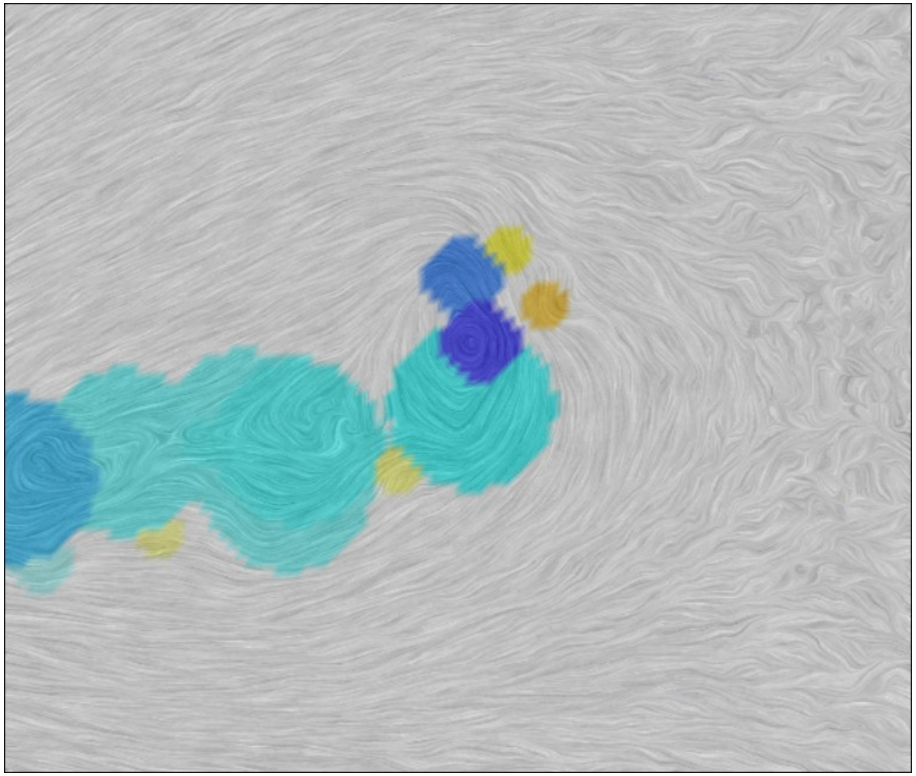


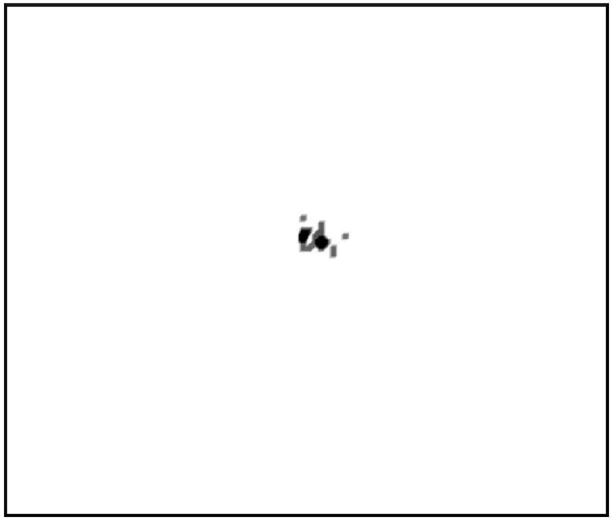


# The problem: PIV measurements

- Goal: Detect the vortices in the measurement planes.
- Determine vortex parameters like center, extend, strength



Marked vortices found by our method



Vortex centers of different measurements

# 2. The Idea: Image Processing on Vector Fields

- Robust feature detection in scalar fields uses image processing!
- Image processing uses filter operations.
- Filter operations for flow fields need vector field filter!
- Many filter in image processing are based on convolution.
- **Convolution for vector fields**?

# The idea: Convolution Filter

- Convolution integral

$$H * F(x) = \int_{IE^d} H(y) F(x-y) dy$$

- Multiplication for vector fields?

=> Geometric Algebra!

# Previous attempts

- Scalar fields H and F: ordinary convolution in image processing
- Scalar field H and vector field F: scalar multiplication.
- Vector fields H, F: scalar convolution in each coordinate [Granlund and Knutsson, 1995]
- Vector fields H, F: scalar product [Heiberg, 2003]

# 3. The solution: Clifford convolution

Let $I\!E^d$ be euclidean d-space.

Let $G_d$ be the Geometric algebra for euclidean d-space.

Let $H, F : I\!E^d \rightarrow G_d$ be two multivector fields.

We define

$$H * F(x) = \int_{I\!E^d} H(y) F(x-y) |dy|$$

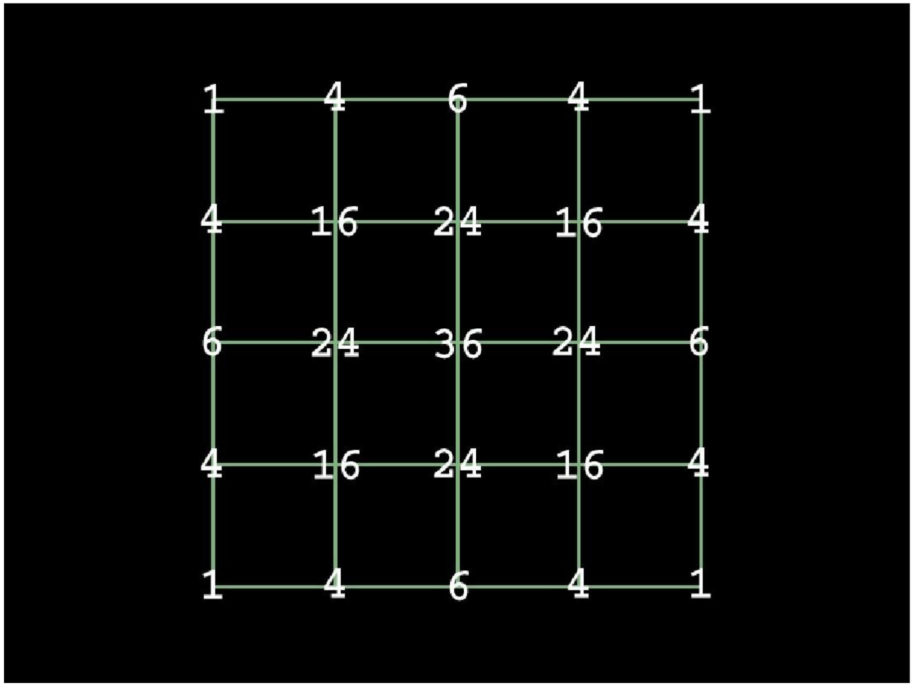
as (right) **Clifford convolution** [Ebling and S., 2003].

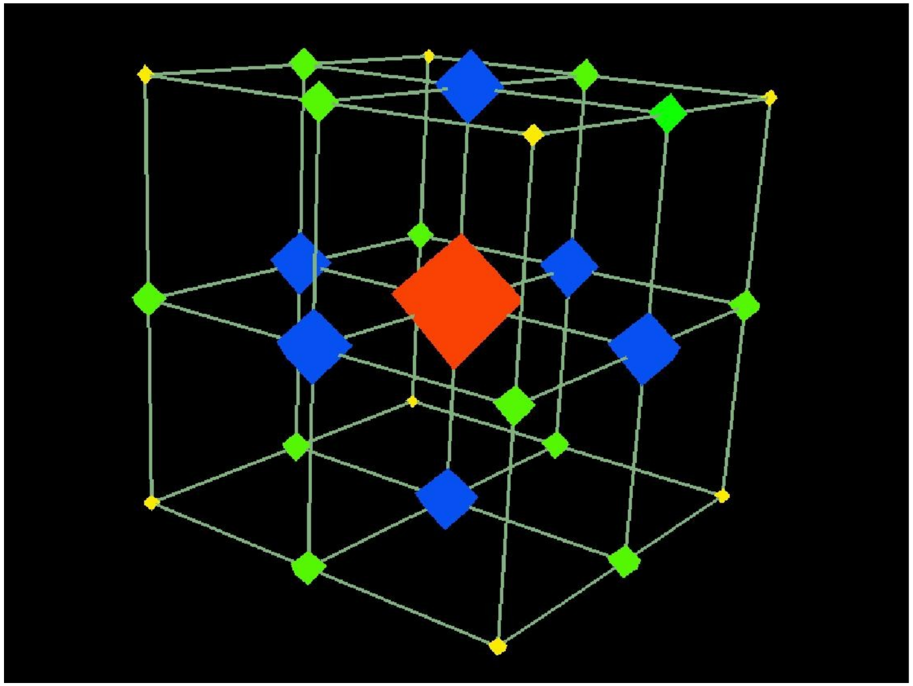
(There is a left Clifford convolution with commuted factors, too.)

# Special cases

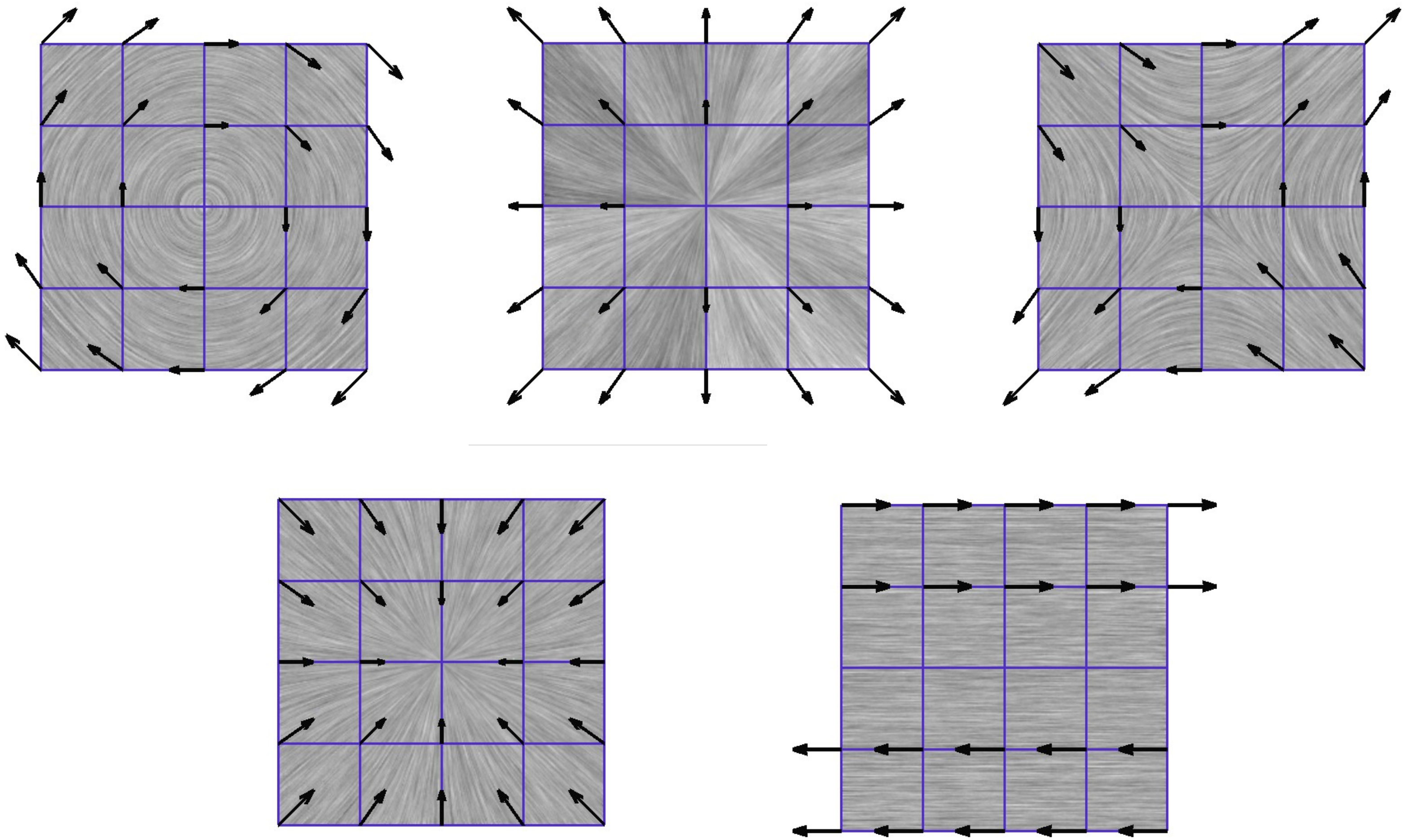
- H, F scalar fields: ordinary convolution of image processing
- H scalar field, F vector field: scalar multiplication.
- H, F vector fields: Scalar component is Heiberg's scalar product.

# Scalar Kernels

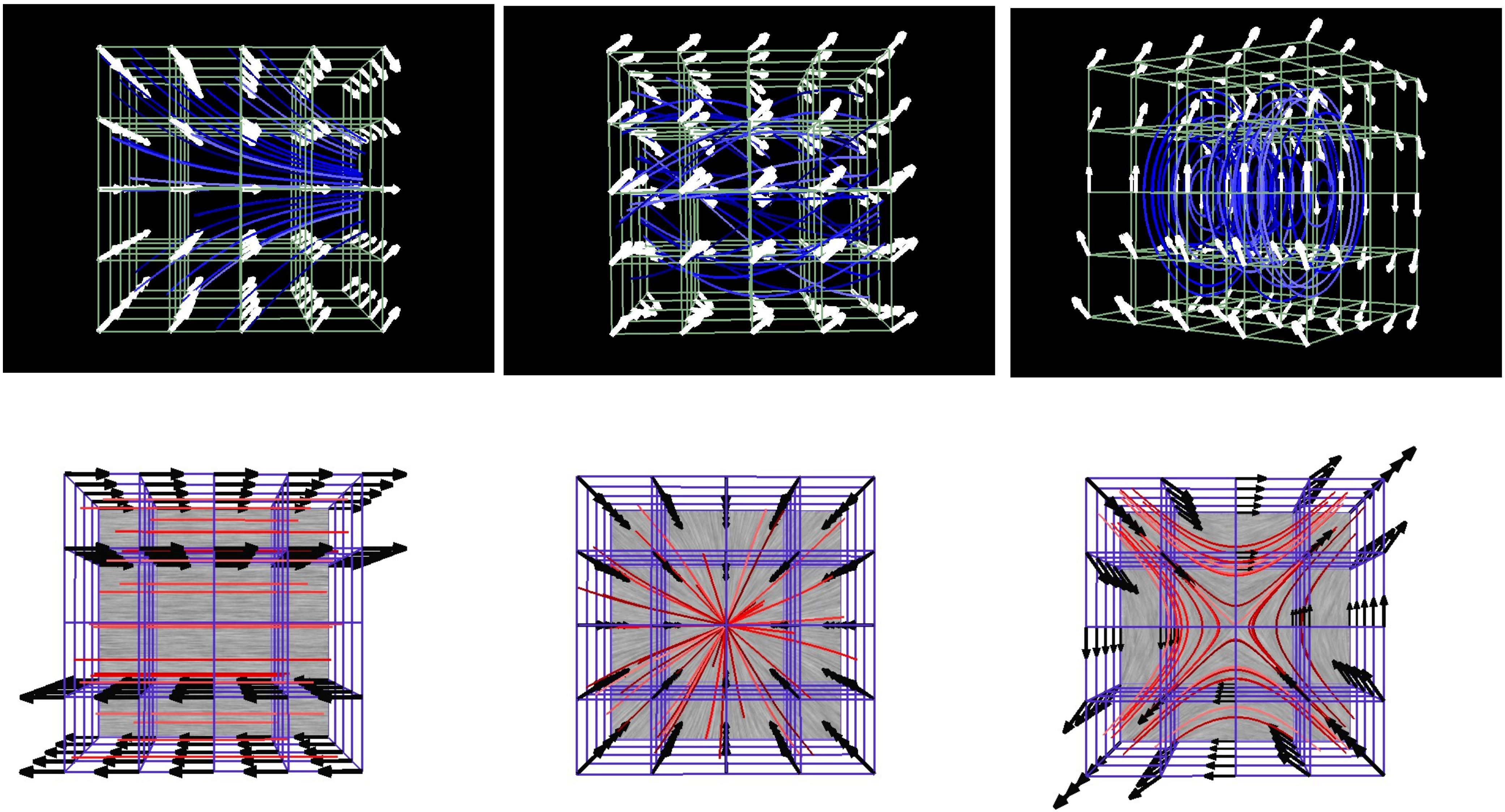


# 2D Vector Kernels



# 3D Vector Kernels



# Interpretation

- Instead of convolution, look at spatial correlation:

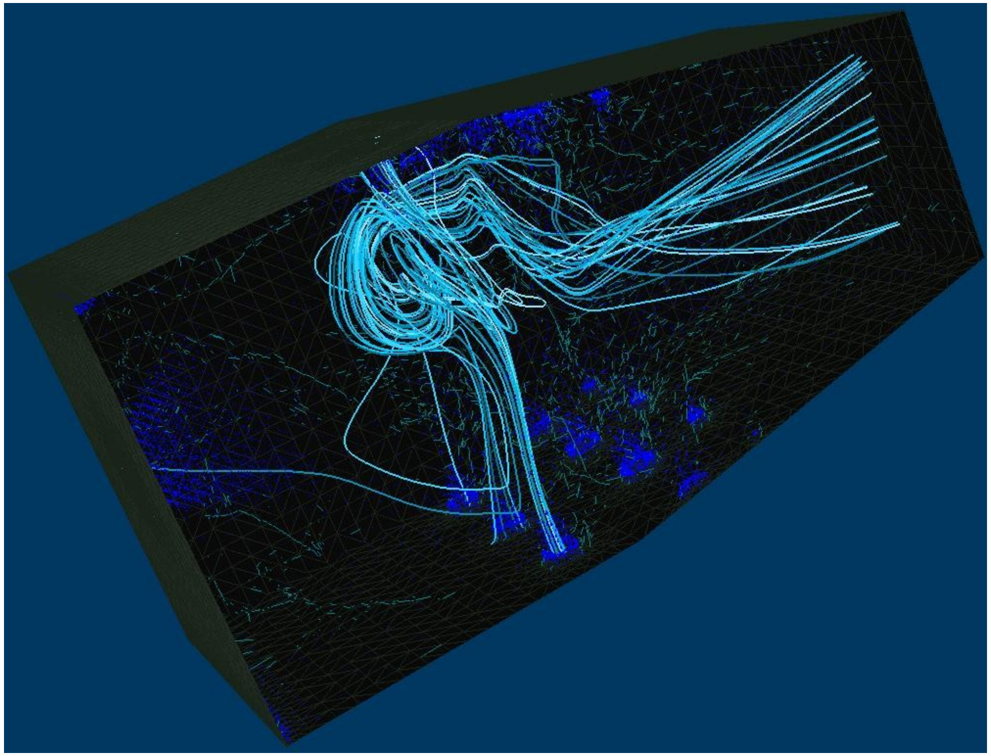
$$H \times F(x) = \int_{IE^d} H(y) F(x+y) |dy|$$

The kernel H describes the local structure!

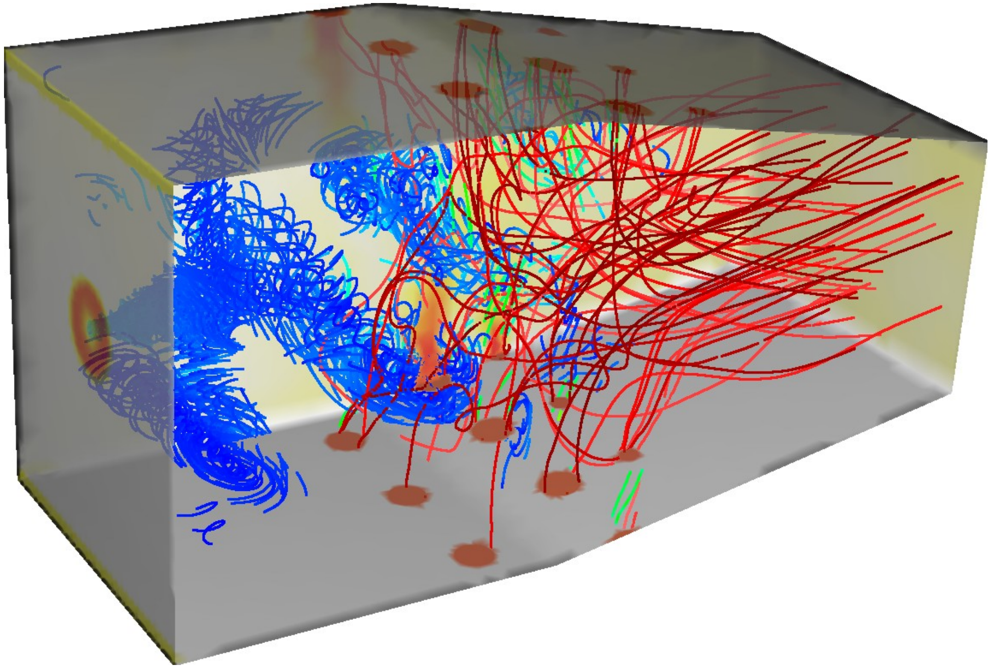
- Arbitrary combinations of convolutions possible:
  - Derivative operators (e.g. rotation, divergence)
  - Smoothing operators as regularization (e.g. Gaussians)
  - Correlation with any vector field of interest to the user

# 4. Practical results: Combustion chamber

- Gas combustion chamber of a heating

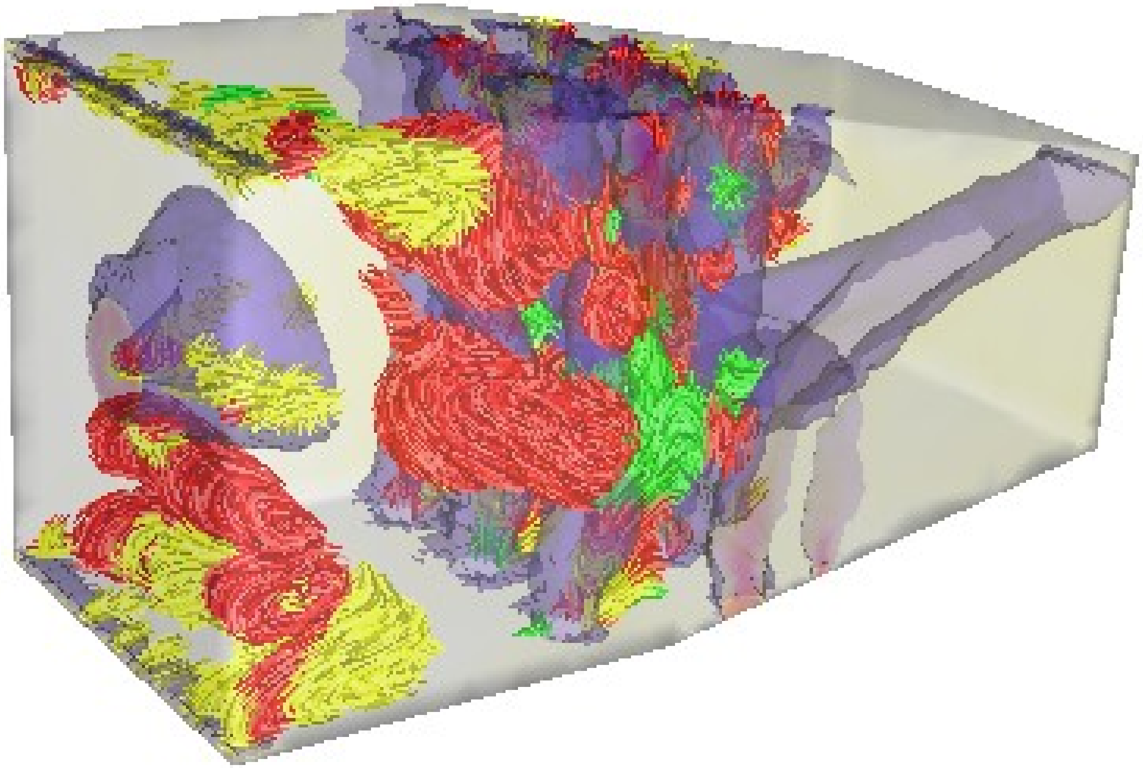


Inlets for air



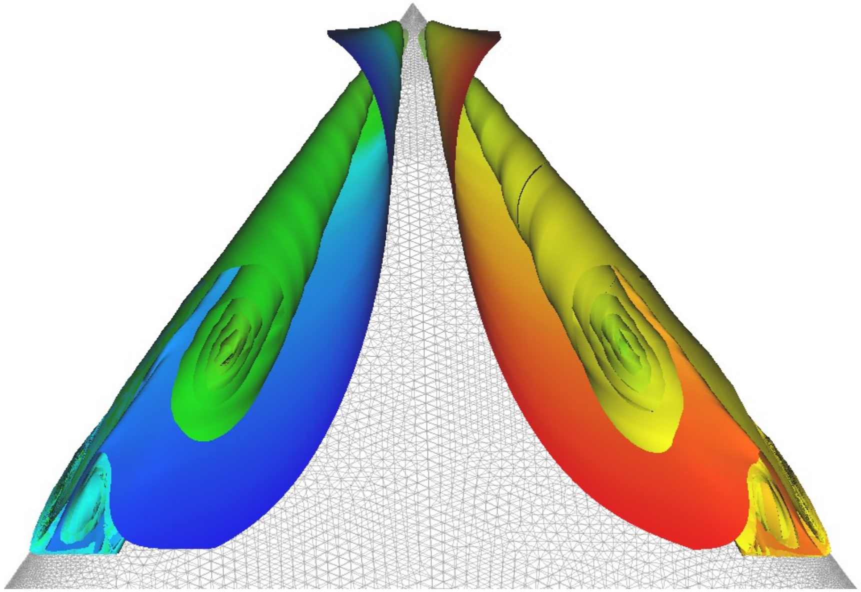
Blue: Streamlets with high similarity to rotation,
red: streamlines with high velocity

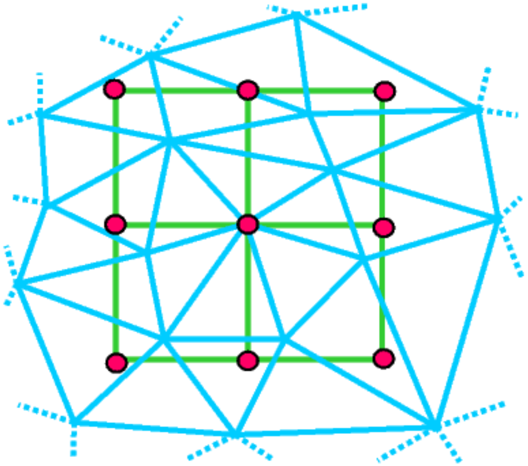
# Segmentation in Combustion chamber

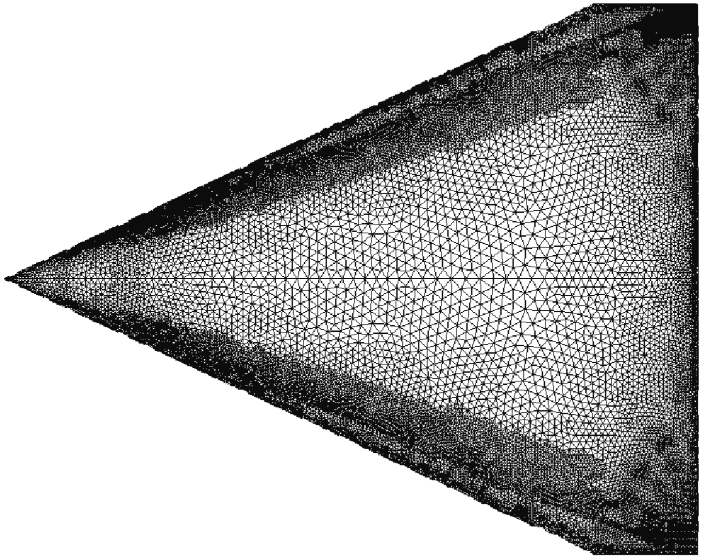


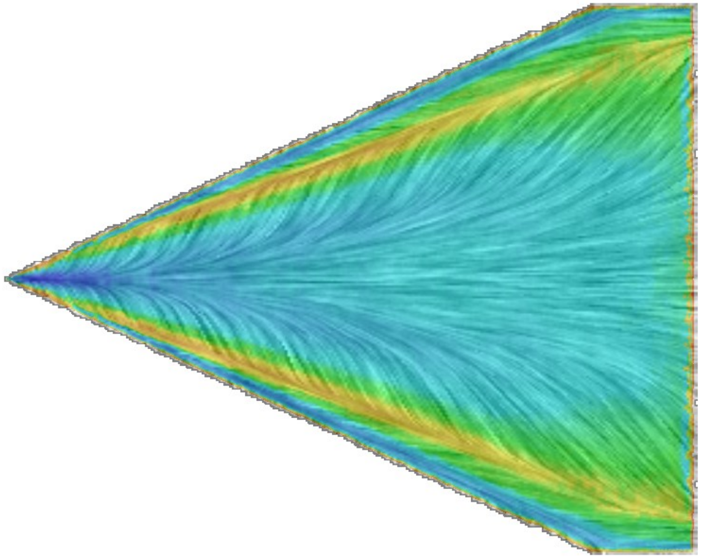
- Segmentation of the field:
- Red: strong rotation yellow: strong shear flow green: saddle regions
- Blue: Isosurface of high velocity regions.

# Separation lines on Delta wing









# 5. Theoretical results: Clifford Fourier Transform

- Generalized convolution => Generalized Fourier Transform?
- Convolution theorem?
- Derivative theorem?
- Fast Fourier Transform?

# Theoretical results: Clifford Fourier Transform

Let $F: IE^d \to G_d$ be two multivector fields, d=2,3.

We define as Clifford Fourier transform

$$\mathcal{F}\{F\}(u) := \int_{IE^d} F(x) \exp(-2\pi i_d x \cdot u) |dx|$$

and as inverse

$$\mathcal{F}^{-1}\{F\}(u) := \int_{IE^d} F(x) \exp(2\pi i_d x \cdot u) |dx|$$

# Theoretical results: 3D theorems

We get the well-known Fourier transform theorems in 3D:

**Shift theorem**:

$$\mathcal{F}\{F(x-x')\}(u)=\mathcal{F}\{F\}(u)\,e^{-2\pi I_3\langle x',u\rangle}$$

**Convolution theorem**:

$$\mathcal{F}\{H*F\}(u)=\mathcal{F}\{H\}(u)\mathcal{F}\{F\}(u)$$

**Derivative theorem**:

$$\mathcal{F}\{\nabla f\}(u)=2\pi I_3 u\,\mathcal{F}\{f\}(u)$$

$$\mathcal{F}\{\Delta f\}(u)=-4\pi^2u^2\mathcal{F}\{f\}(u)$$

# Theoretical results: 2D theorems

We get less elegant versions of the theorems in 2D, due to the missing commutation of $i_2$ with vectors in $G_2$.

Let $H : IE^2 \to G_2$ be a multivector field, $v : IE^2 \to IE^2 \subset G_2$ a vector field.

**Shift theorem**: $\mathcal{F}\{v(x - x')\}(u) = \mathcal{F}\{v\}(u) \exp(-2\pi i_2 x \cdot u)$

**Convolution theorem**: $\mathcal{F}\{H * v\}(u) = \mathcal{F}\{H\}(u)\ \mathcal{F}\{v\}(u)$

**Derivative theorem**: $\mathcal{F}\{\nabla v\}(u) = -2\pi i_2 u \mathcal{F}\{v\}(u)$

$$\mathcal{F}\{\Delta v\}(u) = 4\pi^2 u^2 \mathcal{F}\{v\}(u)$$

(For the spinor-valued part, one gets different signs.
The right convolution has similar theorems.
The Laplaceoperator has the same properties for spinors.)

# 3D-CFT = 4 FT

- In 3D, we are calculating four independent usual complex FT:

$$F = F_0 1 + F_1 e_1 + F_2 e_2 + F_3 e_3 + F_{12} e_1 e_2 + F_{23} e_2 e_3 + F_{31} e_3 e_1 + F_{123} i_3$$

$$\begin{aligned} \mathcal{F}\{F\}(u) = & [\mathcal{F}\{F_0 + F_{123} i_3\}(u)] 1 + \\ & [\mathcal{F}\{F_1 + F_{23} i_3\}(u)] e_1 + \\ & [\mathcal{F}\{F_2 + F_{31} i_3\}(u)] e_2 + \\ & [\mathcal{F}\{F_3 + F_{12} i_3\}(u)] e_3 \end{aligned}$$

- For a vector field $v : IE^3 \rightarrow IE^3 \subset G_3$, this means three FT of the components.

# 2D-CFT = 2 FT

- In 2D, we are calculating two independent usual complex FT:

$$F = F_0 + F_1 e_1 + F_2 e_2 + F_{12} i_2$$

$$\mathcal{F}\{F\}(u) = 1[\mathcal{F}\{F_0 + F_{12} i_2\}(u)] + e_1[\mathcal{F}\{F_1 + F_2 i_2\}(u)]$$

- For a vector field $v: IE^2 \rightarrow IE^2 \subset G_2$, this means one FT of a complex signal.

# 6. Other ideas: Bi-quaternion FT

Bi-quaternion Fourier transform [Sangwine et al. 2008]:

Let $\mathbb{H}_{\mathbb{C}} = \{q_0 + q_1 i + q_2 j + q_3 k \mid q_k = \Re(q_k) + I\,\Im(q_k) \in \mathbb{C}\}$ be the bi-quaternions.

Use the algebra isomorphism

$$\mathbb{H}_{\mathbb{C}} \to G_3 \quad , \quad i \to e_1 e_2 \quad , \quad j \to e_2 e_3 \quad , \quad k \to e_3 e_1 \quad , \quad I \to i_3$$

For an $\mu \in \mathbb{H}_{\mathbb{C}}$ with $\mu^2 = -1$ define the **right BiQFT:**

$$\mathcal{F}_r^{\mu}\{F\}(u) = \int_{\mathbb{E}^3} F(x) \exp(-2\pi\,\mu\, x \cdot u)\,|dx|$$

# BiQFT – Relation to CFT

For $\mu = I\,(=i_3)$, we get the 3D CFT:

$$\mathcal{F}_r^{\mu}\{F\}(u) = \int_{IE^3} F(x)\exp(-2\pi\,\mu\,x\cdot u)|dx| = \int_{IE^3} F(x)\exp(-2\pi\,i_3\,x\cdot u)|dx| = \mathcal{F}\{F(x)\}(u)$$

For a pure bivector $\mu = <\mu>_2$,

there are orthonormal bivectors $\mu, \nu, \xi$ and an orthonormal linear map T with $T(1)=1, T(i)=\mu, T(j)=\nu, T(k)=\xi$ such that

$$\mathcal{F}_r^{i} = T^{-1}\mathcal{F}_r^{\mu}\,T$$

- All BiQFT with pure bivector differ only by linear transformation!

# BiQFT – Relation to CFT

For a vector field $v: IE^3 \rightarrow IE^3 \subset G_3 \quad , \quad x \rightarrow \sum_{l=1}^{3} v_l(x) e_l$, we have

$$v(x) = -v_3(x) Ii - v_1(x) Ij + v_2(x) Ik = (-v_3(x) Ii) + (-v_1(x) I + v_2(x) Ii)\, j$$

and for $\mu = i\,(= e_1 e_2)$, we get

$$\mathcal{F}_r^i\{v\}(u) = \int_{IE^3} (-v_3(x) i) \exp(-2\pi i\, x \cdot u) |dx|\, I + \int_{IE^3} (-v_1(x) + v_2(x) i) \exp(-2\pi i\, x \cdot u) |dx|\, I$$

This means the first two components undergo a 2D-CFT and the third component is transformed as single real signal.

**BiQFT is a better generalization of 2D-CFT to 3D than 3D-CFT**.

# IH-holomorphic Fourier series

Let $IH := \{q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \mid q_l \in \mathbb{R}, e_3 := i_2\} \simeq G_2$ be the quarternions.

Let $F : IH \to IH$ be a function on the quarternions.

F is **left IH-holomorphic** [Gürlebeck, Harbetha, Sprößig, 2006], if

$$\bar{\partial} f = 0 \quad \text{with} \quad \bar{\partial} := \frac{\partial}{\partial q_0} + \sum_{k=1}^{3} \frac{\partial}{\partial q_k} e_k \quad .$$

Let $IB := \{q \in IH \mid |q| = 1\}$ be the unit sphere in IH.

There is an orthonormal basis of IH-holomorphic polynomials of

$$L^2(IB) \cap \ker(\bar{\partial}) = \oplus_{k=0}^{\infty} H_k^+$$

# IH-holomorphic Fourier series

For a vector field

$$v: IE^3 \rightarrow IE^3 \subset G_3$$

We have to find a related function on IB.

We tried:

$$v(x) = f_1(x) e_1 \overline{f_1(x)}$$

$$v(x) = f_2(x) e_1 \qquad v(x) = e_1 f_3(x)$$

$$f_4(x) := v_1(x) e_1 + v_2(x) e_2 + v_3(x) i_2$$

But in all cases, general linear vector fields are not IH-holomorphic, so we could not apply the Fourier series.

# 7. Conclusion

- Image processing of vector fields helps in flow data analysis.
- Geometric algebra allows a suitable convolution operator.
- Convolution, correlation, derivative operators are possible.
- There is a Clifford Fourier transform with the nice theorems.
- BiQFT can explain differences between 2D-CFT and 3D-CFT.

# Open questions

**Clifford convolution:**

- Good interpretation of bivector part in Clifford convolution? How can we use it?
- Convolution on unsteady vector fields? (Space-time GA?)
- Conformal Geometric Algebra extensions of the convolution?

**Fourier transform:**

- Good interpretation for the vector field frequencies?
- What is the right way for the nd-CFT? (Not all metrics work at the moment, $i_n^2=-1$ necessary.)
- Is there a way to use the IH-holormorphic function approach?
- Comparison with other approaches, e.g. Batard?

**Wavelet transforms? Wavelet-based filter?**

# References


- J. Ebling, G. Scheuermann. Clifford Convolution and Pattern Matching on Vector Fields. Proceedings of IEEE Visualization 2003, IEEE CS, Los Alamitos, CA, USA, 2003, 193-200.
- J. Ebling, G. Scheuermann, B.G. van der Wall. Analysis and Visualization of 3-C PIV images from HART II Using Image Processing Methods. In EUROGRAPHICS – IEEE VGTC Symposium on Visualization 2005 Proceedings, IEEE CS, Los Alamitos, CA, USA, 2005, 161-168.
- J. Ebling, G. Scheuermann. Clifford Fourier Transform on Vector Fields. IEEE Transactions on Visualization and Computer Graphics 11(4):469-479, 2005.
- Gürlebeck, Habetha, Sprößig. Funktionentheorie in der Ebene und im Raum. Birkhäuser Verlag, Stuttgart, 2006.
- Sangwine, Le Bihan, Said. Fast Complexified Quaternion Fourier Transform. IEEE Transactions on Signal Processing 56(4):1522-1531, 2008.