Vector Field Processing with Clifford Convolution and Clifford Fourier Transform

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Outline

- **The problem**: CFD data and PIV measurements
- **The idea**: Transfer of image processing to vector fields
- **The solution**: Clifford Convolution
- **Practical results**: Vortex detection and separation lines
- **Theoretical results**: Clifford Fourier Transform
- **Other ideas**: IH-Fourier Series, Bi-quaternion Fourier Transform
- **Conclusion and open questions**
1. The Problem: CFD Data

The analysis of flow is relevant for engineers in the aerospace, automotive and mechanical engineering industry, e.g. for design of

- airplanes
- helicopters
- trains
- cars
- trucks
- turbines
- combustion chambers
The problem: CFD Post-processing

- Direct visualization shows arrows, streamlines, streamsurfaces
- Advanced visualization detects features like vortices, separation
The problem: PIV measurements

- 3D velocity field can be measured by particle image velocimetry
- Here: velocity measurement behind helicopter rotor
The problem: PIV measurements

- Goal: Detect the vortices in the measurement planes.
- Determine vortex parameters like center, extend, strength

Marked vortices found by our method

Vortex centers of different measurements
2. The Idea: Image Processing on Vector Fields

- Robust feature detection in scalar fields uses image processing!
- Image processing uses filter operations.
- Filter operations for flow fields need vector field filter!
- Many filter in image processing are based on convolution.
- **Convolution for vector fields?**
The idea: Convolution Filter

- Convolution integral

\[ H \ast F(x) = \int_{IE^d} H(y) F(x - y) \, dy \]

- Multiplication for vector fields?

=> Geometric Algebra!
Previous attempts

- Scalar fields $H$ and $F$: ordinary convolution in image processing
- Scalar field $H$ and vector field $F$: scalar multiplication.
- Vector fields $H$, $F$: scalar convolution in each coordinate [Granlund and Knutsson, 1995]
- Vector fields $H$, $F$: scalar product [Heiberg, 2003]
3. The solution: Clifford convolution

Let $IE^d$ be euclidean $d$-space.

Let $G_d$ be the Geometric algebra for euclidean $d$-space.

Let $H, F: IE^d \rightarrow G_d$ be two multivector fields.

We define

$$H \ast F(x) = \int_{IE^d} H(y) F(x - y)|dy|$$

as (right) **Clifford convolution** [Ebbling and S., 2003].

(There is a left Clifford convolution with commuted factors, too.)
Special cases

- H, F scalar fields: ordinary convolution of image processing
- H scalar field, F vector field: scalar multiplication.
- H, F vector fields: Scalar component is Heiberg's scalar product.
Scalar Kernels

[Diagram showing scalar kernels on a 2D grid and a 3D cube]
2D Vector Kernels
3D Vector Kernels
Interpretation

- Instead of convolution, look at spatial correlation:

\[ H \times F(x) = \int_{IE^d} H(y) F(x+y)|dy| \]

The kernel H describes the local structure!

- Arbitrary combinations of convolutions possible:
  - Derivative operators (e.g. rotation, divergence)
  - Smoothing operators as regularization (e.g. Gaussians)
  - Correlation with any vector field of interest to the user
4. Practical results: Combustion chamber

- Gas combustion chamber of a heating

Inlets for air

**Blue**: Streamlets with high similarity to rotation,
**red**: streamlines with high velocity
Segmentation in Combustion chamber

- Segmentation of the field:
- **Red**: strong rotation  **yellow**: strong shear flow  **green**: saddle regions
- **Blue**: Isosurface of high velocity regions.
Separation lines on Delta wing
5. Theoretical results: Clifford Fourier Transform

- Generalized convolution $\Rightarrow$ Generalized Fourier Transform?
- Convolution theorem?
- Derivative theorem?
- Fast Fourier Transform?
Theoretical results: Clifford Fourier Transform

Let \( F: IE^d \to G_d \) be two multivector fields, \( d=2,3 \).

We define as Clifford Fourier transform

\[
\mathcal{F}\{F\}(u) := \int_{IE^d} F(x) \exp(-2\pi i d x \cdot u) |dx|
\]

and as inverse

\[
\mathcal{F}^{-1}\{F\}(u) := \int_{IE^d} F(x) \exp(2\pi i d x \cdot u) |dx|
\]
We get the well-known Fourier transform theorems in 3D:

**Shift theorem:**\[ \mathcal{F} \{ F(x-x') \}(u) = \mathcal{F} \{ F \}(u) e^{-2\pi I_3 \langle x', u \rangle} \]

**Convolution theorem:**\[ \mathcal{F} \{ H \ast F \}(u) = \mathcal{F} \{ H \}(u) \mathcal{F} \{ F \}(u) \]

**Derivative theorem:**\[ \mathcal{F} \{ \nabla f \}(u) = 2\pi I_3 u \mathcal{F} \{ f \}(u) \]
\[ \mathcal{F} \{ \Delta f \}(u) = -4\pi^2 u^2 \mathcal{F} \{ f \}(u) \]
Theoretical results: 2D theorems

We get less elegant versions of the theorems in 2D, due to the missing commutation of $i_2$ with vectors in $G_2$.

Let $H : IE^2 \rightarrow G_2$ be a multivector field, $v : IE^2 \rightarrow IE^2 \subset G_2$ a vector field.

**Shift theorem:**
\[
\mathcal{F} \{v(x-x')\}(u) = \mathcal{F} \{v\}(u) \exp(-2\pi i_2 x \cdot u)
\]

**Convolution theorem:**
\[
\mathcal{F} \{H*v\}(u) = \mathcal{F} \{H\}(u) \cdot \mathcal{F} \{v\}(u)
\]

**Derivative theorem:**
\[
\mathcal{F} \{\nabla v\}(u) = -2\pi i_2 u \cdot \mathcal{F} \{v\}(u)
\]
\[
\mathcal{F} \{\Delta v\}(u) = 4\pi^2 u^2 \cdot \mathcal{F} \{v\}(u)
\]

(For the spinor-valued part, one gets different signs. The right convolution has similar theorems. The Laplace operator has the same properties for spinors.)
3D-CFT = 4 FT

- In 3D, we are calculating four independent usual complex FT:

\[ F = F_0 + F_1 e_1 + F_2 e_2 + F_3 e_3 + F_{12} e_1 e_2 + F_{23} e_2 e_3 + F_{31} e_3 e_1 + F_{123} i_3 \]

\[ \mathcal{F} \{ F \}(u) = [\mathcal{F} \{ F_0 + F_{123} i_3 \}(u)] 1 + \]

\[ [\mathcal{F} \{ F_1 + F_{23} i_3 \}(u)] e_1 + \]

\[ [\mathcal{F} \{ F_2 + F_{31} i_3 \}(u)] e_2 + \]

\[ [\mathcal{F} \{ F_3 + F_{12} i_3 \}(u)] e_3 \]

- For a vector field \( v: \mathbb{I} \mathbb{E}^3 \to \mathbb{I} \mathbb{E}^3 \subset G_3 \), this means three FT of the components.
In 2D, we are calculating two independent usual complex FT:

\[ F = F_0 + F_1 e_1 + F_2 e_2 + F_{12} i_2 \]

\[ \mathcal{F} \{ F \}(u) = 1 \mathcal{F} \{ F_0 + F_{12} i_2 \}(u) + e_1 \mathcal{F} \{ F_1 + F_{2} i_2 \}(u) \]

For a vector field \( v : IE^2 \to IE^2 \subset G_2 \), this means one FT of a complex signal.
6. Other ideas: Bi-quaternion FT

Bi-quaternion Fourier transform [Sangwine et al. 2008]:

Let \( IH_\mathbb{C} = \{ q_0 + q_1 i + q_2 j + q_3 k \mid q_k = \Re(q_k) + I \Im(q_k) \in \mathbb{C} \} \) be the bi-quaternions.

Use the algebra isomorphism

\[
IH_\mathbb{C} \rightarrow G_3, \quad i \rightarrow e_1 e_2, \quad j \rightarrow e_2 e_3, \quad k \rightarrow e_3 e_1, \quad I \rightarrow i_3
\]

For an \( \mu \in IH_\mathbb{C} \) with \( \mu^2 = -1 \) define the **right BiQFT**: 

\[
F^\mu_r[F](u) = \int_{IE^3} F(x) \exp(-2\pi \mu x \cdot u) |dx|
\]
BiQFT – Relation to CFT

For $\mu = I (= i_3)$, we get the 3D CFT:

$$\mathcal{F}_{x}^{\mu \{ F \}}(u) = \int_{\mathbb{R}^3} F(x) \exp(-2\pi \mu x \cdot u)|dx| = \int_{\mathbb{R}^3} F(x) \exp(-2\pi i_3 x \cdot u)|dx| = \mathcal{F}\{ F(x) \}(u)$$

For a pure bivector $\mu = <\mu>_2$, there are orthonormal bivectors $\mu, \nu, \xi$ and an orthonormal linear map $T$ with $T(1)=1, T(i)=\mu, T(j)=\nu, T(k)=\xi$ such that

$$\mathcal{F}_{x}^{i} = T^{-1} \mathcal{F}_{x}^{\mu} T$$

- All BiQFT with pure bivector differ only by linear transformation!
BiQFT – Relation to CFT

For a vector field \( \mathbf{v} : IE^3 \to IE^3 \subseteq G_3 \), \( x \to \sum_{l=1}^{3} v_l(x)e_l \), we have

\[
\mathbf{v}(x) = -v_3(x)i - v_1(x)j + v_2(x)k = (-v_3(x)i) + (-v_1(x)I + v_2(x)I)j
\]

and for \( \mu = i( = e_1e_2) \), we get

\[
\mathcal{F}_r^i[v](u) = \int_{IE^3} (-v_3(x)i)\exp(-2\pi i x \cdot u)dx|I + \int_{IE^3} (-v_1(x) + v_2(x)i)\exp(-2\pi i x \cdot u)dx|I
\]

This means the first two components undergo a 2D-CFT and the third component is transformed as single real signal.

**BiQFT is a better generalization of 2D-CFT to 3D than 3D-CFT.**
IH-holomorphic Fourier series

Let $IH := \{q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 | q_i \in \mathbb{R}, e_3 := i_2 \} \simeq G_2$ be the quaternions.

Let $F : IH \to IH$ be a function on the quaternions.

$F$ is **left IH-holomorphic** [Gürlebeck, Harbetha, Sprößig, 2006], if

$$\bar{\partial} f = 0 \quad \text{with} \quad \bar{\partial} := \frac{\partial}{\partial q_0} + \sum_{k=1}^{3} \frac{\partial}{\partial q_k} e_k.$$

Let $IB := \{q \in IH | |q| = 1 \}$ be the unit sphere in $IH$.

There is an orthonormal basis of IH-holomorphic polynomials of

$$L^2(IB) \cap \ker(\bar{\partial}) = \bigoplus_{k=0}^{\infty} H_k^+$$
IH-holomorphic Fourier series

For a vector field

\[ v: IE^3 \rightarrow IE^3 \subset G_3 \]

We have to find a related function on IB.

We tried:

\[ v(x) = f_1(x) e_1 f_1(x) \]

\[ v(x) = f_2(x) e_1 \quad v(x) = e_1 f_3(x) \]

\[ f_4(x) := v_1(x) e_1 + v_2(x) e_2 + v_3(x) i_2 \]

But in all cases, general linear vector fields are not IH-holomorphic, so we could not apply the Fourier series.
7. Conclusion

- Image processing of vector fields helps in flow data analysis.
- Geometric algebra allows a suitable convolution operator.
- Convolution, correlation, derivative operators are possible.
- There is a Clifford Fourier transform with the nice theorems.
- BiQFT can explain differences between 2D-CFT and 3D-CFT.
Open questions

Clifford convolution:
- Good interpretation of bivector part in Clifford convolution? How can we use it?
- Convolution on unsteady vector fields? (Space-time GA?)
- Conformal Geometric Algebra extensions of the convolution?

Fourier transform:
- Good interpretation for the vector field frequencies?
- What is the right way for the nd-CFT? (Not all metrics work at the moment, $i^2_n=-1$ necessary.)
- Is there a way to use the IH-holomorphic function approach?
- Comparison with other approaches, e.g. Batard?

Wavelet transforms? Wavelet-based filter?
References


