Two Applicable Results in Conformal Geometric Algebra

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1. Topics and Results

Algebraic and Geometric Aspects of Conformal Geometric Algebra (CGA):

1. Polynomial parametrization of conformal transformations;
2. Incidence geometry of spheres and planes;
3. Geometry of Euclidean displacements;
4. Symbolic algebra of null vectors.

Parametrization of 3D conformal transformations

Vahlen matrices:

$$\mathbf{x} \longmapsto (\mathbf{A}\mathbf{x} + \mathbf{B})(\mathbf{C}\mathbf{x} + \mathbf{D})^{-1}, \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

where *Vahlen matrix* $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ is a 2×2 matrix over $CL(\mathbb{R}^3)$ such that

1. $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are either versors or zero;
2. $\mathbf{A}\mathbf{B}^\dagger, \mathbf{B}\mathbf{D}^\dagger, \mathbf{D}\mathbf{C}^\dagger, \mathbf{C}\mathbf{A}^\dagger$ are vectors;
3. $\Delta = \mathbf{A}\mathbf{D}^\dagger - \mathbf{B}\mathbf{C}^\dagger$ is a nonzero scalar.

Difficulty: parametrizing the versors.

Exponential map: Versor representation

$$\mathbf{x} \longmapsto \mathbf{V}\mathbf{x}\hat{\mathbf{V}}^{-1}, \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

and Lie algebra representation of rotors via the exponential map:

$$\mathbf{U} = \exp(\mathbf{u}) = 1 + \mathbf{u} + \frac{\mathbf{u}^2}{2!} + \cdots$$

Difficulty: evaluating the exponential map and its inverse.

Cayley transform: Versor representation and Lie algebra representation of rotors via rational linear map

$$\begin{aligned} \Lambda^2(\mathbb{R}^{4,1}) &\longrightarrow CL(\mathbb{R}^{4,1}) \\ \mathbf{B}_2 &\longmapsto (1+\mathbf{B}_2)(1-\mathbf{B}_2)^{-1}, \text{ where } 1-\mathbf{B}_2 \text{ is invertible.} \end{aligned}$$

Difficulty: computing the inverse of a multivector in $CL(\mathbb{R}^{4,1})$.

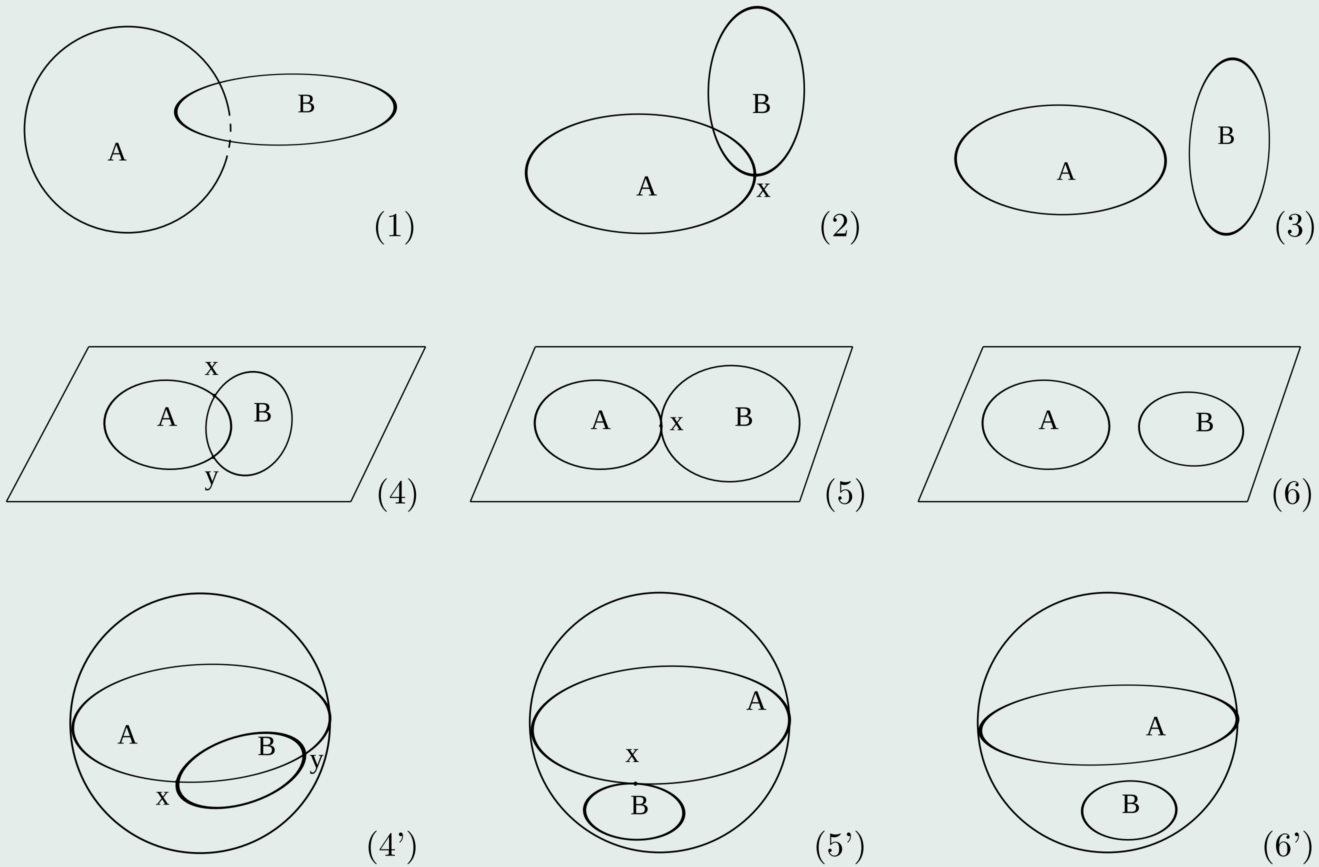
First applicable result

Polynomial Cayley transform: Versor representation and Lie algebra representation of rotors via a degree-4 polynomial map.

Inverse: square-root computing of real numbers.

Applicable to: motion planning, motion interpolating, etc.

Incidence geometry of planes and spheres of various dimensions



Second applicable result

A very simple algebraic operation called *total meet product*, that can be used as a classifier of all kinds of incidence relations of spheres and planes of various dimensions in $\mathbb{R}^n$.

The above two applicable results are included in

H. Li, *Invariant Algebras and Geometric Reasoning*, World Scientific, Singapore, 2008. (approx. 500 pages)

The geometry of null geometric algebra

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^{n+1,1}$ be null vectors. What is the nD Euclidean geometric meaning (via the conformal model) of

$$\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_r?$$

What about $\langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_r\rangle_{r-2k}$?

Definition.

Let $\mathcal{V}^n$ be an inner-product space spanned by null vectors. The *null Clifford space* over $\mathcal{V}^n$, still denoted by $\mathcal{G}(\mathcal{V}^n)$, is the set of $\mathbb{K}$-linear combinations of null monomials and single-graded null monomials. The *null Geometric Algebra* (NGA) over $\mathcal{V}^n$, still denoted by $\mathcal{G}(\mathcal{V}^n)$, refers to the null Clifford space equipped with the geometric product.

First fundamental theorem in NGA

Chained difference representations of null monomials:

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^{n+1,1}$ be null vectors such that $\mathbf{a}_i \cdot \mathbf{a}_1 \neq 0$ for $i \neq 0$. Then

$$\begin{aligned}\langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_r\rangle_{r-2l} = & -\langle \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\ \overrightarrow{\mathbf{a}_3\mathbf{a}_4}\cdots\overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r}\rangle_{r-2l} \\ & + \langle \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\ \overrightarrow{\mathbf{a}_3\mathbf{a}_4}\cdots\overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r}\rangle_{r-2l-2} \wedge \mathbf{a}_1 \wedge \mathbf{a}_r,\end{aligned}$$

$$\begin{aligned}\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_r = & \ \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\ \overrightarrow{\mathbf{a}_3\mathbf{a}_4}\cdots\overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r}\ \mathbf{a}_1\mathbf{a}_r \\ = & \ \mathbf{a}_1\mathbf{a}_r\ \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\ \overrightarrow{\mathbf{a}_3\mathbf{a}_4}\cdots\overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r},\end{aligned}$$

where

$$\overrightarrow{\mathbf{a}_i\mathbf{a}_j} = \mathbf{b}_j - \mathbf{b}_i,$$

and $\mathbf{b}_i$ is the the Euclidean point in $\mathbb{R}^n = (\mathbf{a}_1 \wedge \mathbf{a}_r)^\sim$ represented by null vector $\mathbf{a}_i \in \mathbb{R}^{n+1,1}$.

Second fundamental theorem in NGA

Modular chained difference representations of null monomials:

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in \mathbb{R}^{n+1,1}$ be null vectors such that $\mathbf{a}_i \cdot \mathbf{a}_1 \neq 0$ for $i \neq 0$. Then

$$\begin{aligned} \langle \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_r \rangle_{r-2l} = \quad & - \langle \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\, \overrightarrow{\mathbf{a}_3\mathbf{a}_4} \cdots \overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r} \rangle_{r-2l} \\ & + \langle \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\, \overrightarrow{\mathbf{a}_3\mathbf{a}_4} \cdots \overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r} \rangle_{r-2l-2} \wedge \mathbf{a}_1 \wedge \mathbf{w} \mod \mathbf{a}_1 \Lambda^{r-2l-1}(\mathbb{R}^n), \end{aligned}$$

$$\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_r = \overrightarrow{\mathbf{a}_2\mathbf{a}_3}\, \overrightarrow{\mathbf{a}_3\mathbf{a}_4} \cdots \overrightarrow{\mathbf{a}_{r-1}\mathbf{a}_r}\, \mathbf{a}_1\mathbf{w} \mod \mathbf{a}_1 \Lambda(\mathbb{R}^n),$$

where

$$\overrightarrow{\mathbf{a}_i\mathbf{a}_j} = \mathbf{b}_j - \mathbf{b}_i,$$

$\mathbf{b}_i$ is the the Euclidean point in $\mathbb{R}^n = (\mathbf{a}_1 \wedge \mathbf{w})^\sim$ represented by null vector $\mathbf{a}_i \in \mathbb{R}^{n+1,1}$, and $\mathbf{w}$ is a generic null vector in $\mathbb{R}^{n+1,1}$.

Clifford difference ring

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be atomic vectors of $\mathcal{V}^n$. Then all difference polynomials of the $\mathbf{a}_i$ form a ring of $CL(\mathcal{V}^n)$ under the addition and the geometric product, called the *Clifford difference ring* generated by the $\mathbf{a}_i$.

Why study this ring?

a single expansion leads to an exponential growth of the expression size:

monomial $(\mathbf{a}_1 - \mathbf{b}_1)(\mathbf{a}_2 - \mathbf{b}_2) \cdots (\mathbf{a}_r - \mathbf{b}_r) = 2^r$ terms when expanded multilinearly.

Third applicable result

Null geometric algebra: graph-theoretical method, PBD (permutational balanced difference) polynomial representation, etc.

Clifford difference ring: tabular representation, tensor product structure, etc.

Algebra of advanced invariants

Null Bracket Algebra (NBA): generated by the scalar parts and the dual of the pseudoscalar parts of null monomials.

Algebra of advanced covariants:
Null Grassmann-Cayley algebra (NGC), generated by single-graded null monomials, and equipped with the outer product and the meet product.

Fourth applicable result: NBA and NGC.

The above two results are included in:

H. Li, *Symbolic Computational Geometry with Advanced Invariant Algebras*, under revision. (approx. 450 pages)

2. Polynomial Parametrization of 3D Möbius Group

Terminology:

Versor: The geometric product of invertible vectors.

Rotor: The geometric product of an even number of invertible vectors.

Positive vector: a vector whose inner product with itself is positive.

Positive versor: The geometric product of positive vectors.

Positive rotor: The geometric product of an even number of positive vectors.

- Any conformal transformation in $\mathbb{R}^n$ is induced by a positive versor in $CL(\mathbb{R}^{n+1,1})$ that is unique up to scale.
- Any orientation-preserving conformal transformation in $\mathbb{R}^n$ is induced by a positive rotor in $CL(\mathbb{R}^{n+1,1})$ that is unique up to scale.

Theorem

For any $\mathbf{B}_2 \in \Lambda^2(\mathbb{R}^{4,1})$ such that $\mathbf{B}_2^2 \neq 1$, the following equality holds up to scale:

$$C(\mathbf{B}_2) = (1 + \mathbf{B}_2)^2(1 - \mathbf{B}_2 \cdot \mathbf{B}_2 + \mathbf{B}_2 \wedge \mathbf{B}_2).$$

Cayley transform is a polynomial map of degree 4 in $\mathbf{B}_2$, with values in the group of positive rotors of $\mathcal{G}(\mathbb{R}^{4,1})$.

Domain of definition: all bivectors in $\Lambda(\mathbb{R}^{4,1})$ except the Minkowski blades of unit magnitude.

Image space: all positive rotors (modulo scale) except those of the form $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4$, where the $\mathbf{b}_i$ are pairwise orthogonal positive vectors in $\mathbb{R}^{4,1}$.

Antipodal inversion

Algebraically: generated by $\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3\mathrm{b}_4$, where the b_i are pairwise orthogonal positive vectors in $\mathbb{R}^{4,1}$.

Geometrically: the composition of an inversion with respect to a sphere and the reflection with respect to the center of the sphere.

Inverse of Cayley transform

Definition.
A bivector is said to be *entangled*, or *coherent*, if in its completely orthogonal decomposition there are two components having equal square.

Property.
For a bivector $\mathbf{B}_2 \in \Lambda^2(\mathbb{R}^{4,1})$ to be entangled, it is both necessary and sufficient that

$$(\mathbf{B}_2 \cdot \mathbf{B}_2)^2 = (\mathbf{B}_2 \wedge \mathbf{B}_2)^2.$$

Theorem.
Let $\mathbf{A}$ be a positive rotor $\mathbf{A}$ that is in the range of Cayley transform up to scale. Then $\mathbf{A}$ has exactly one bivector preimage if and only if either

- it is in $\Lambda(\mathbf{C}_2)$ where $\mathbf{C}_2$ is a 2-blade of degenerate signature, or
- its bivector part is entangled.

Expression of inverse Cayley transform

If positive rotor $\mathbf{A}$ has a unique preimage, the preimage is

$$\frac{\langle\mathbf{A}\rangle_2}{\langle\mathbf{A}\rangle_4 + 2\langle\mathbf{A}\rangle + |\langle\mathbf{A}\rangle\langle\mathbf{A}\rangle_4|/\langle\mathbf{A}\rangle}.$$

If A has more than one preimage, then it has two, and they are inverse to each other:

$$\frac{\langle\mathbf{A}\rangle_2}{\langle\mathbf{A}\rangle_4 + \langle\mathbf{A}\rangle \pm |\mathbf{A}|}.$$

Examples

All orientation-preserving similarity transformations in $\mathbb{R}^3$ can be induced by bivectors in $\Lambda^2(\mathbb{R}^{4,1})$ through Cayley transform and adjoint action.

Any orientation-preserving similarity transformation which is not a translation is induced by the Cayley transform of exactly two bivectors. A translation is induced by a unique bivector.

Example 1. In $CL(\mathbb{R}^{4,1})$, let

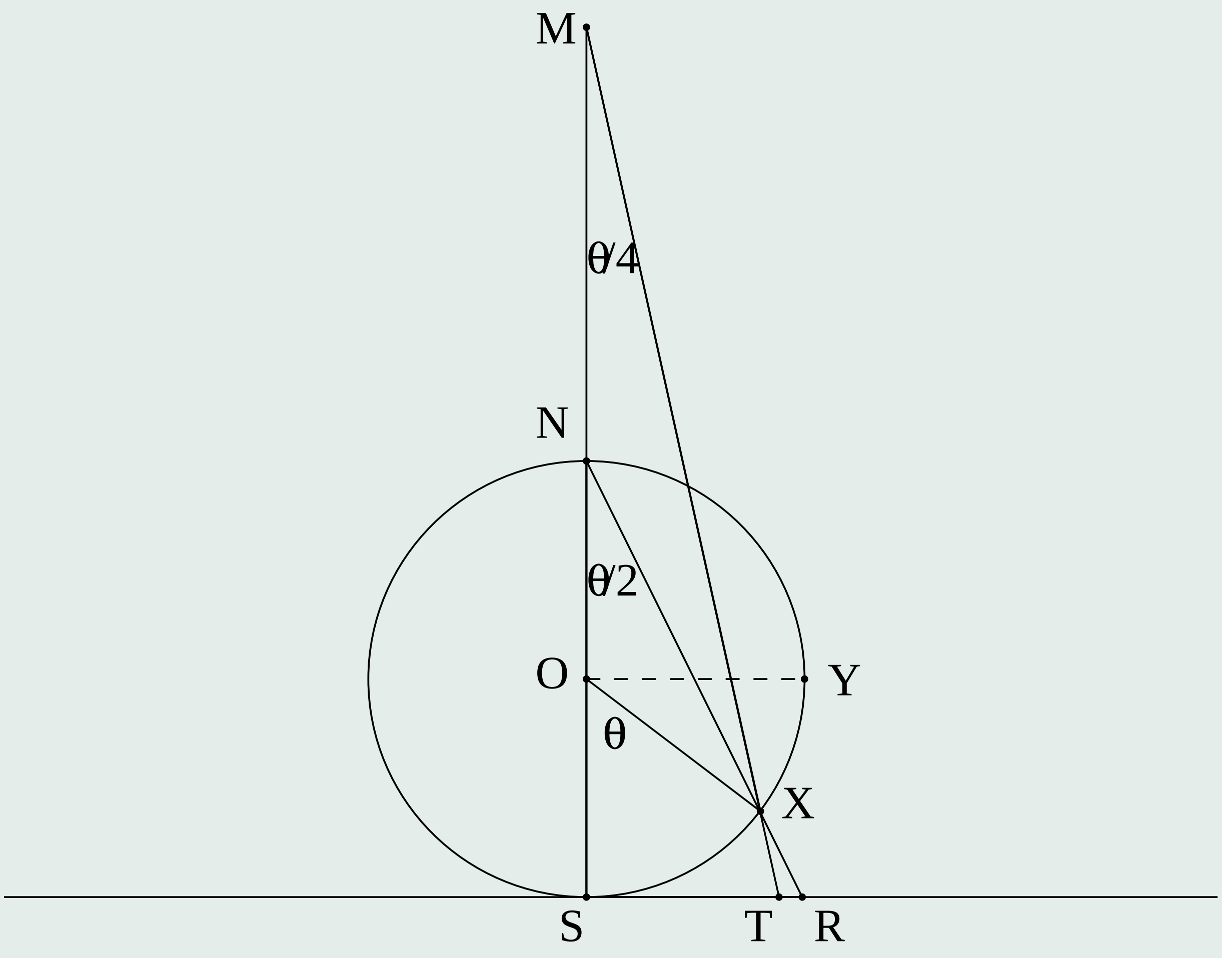
$$\mathbf{A} = e^{\mathbf{I}_2 \frac{\theta}{2}},$$

where $\mathbf{I}_2 \in \Lambda(\mathbf{e}^{\sim})$ is a Euclidean 2-blade of unit magnitude such that $\mathbf{I}_2^{\sim}$ represents the axis of rotation, and $-\theta$ is the angle of rotation. Then

$$\mathbf{B}_2 = \mathbf{I}_2 \tan\frac{\theta}{4}, \quad \mathbf{B}_2^{-1} = -\mathbf{I}_2 / \tan\frac{\theta}{4},$$

both generate $\mathbf{A}$ by Cayley transform.

The bivector representation of the rotation via the Cayley transform is a quarter-angle representation.


M
θ/4
N
θ/2
O
Y
θ
X
S
T
R

Example 2. let

$$\mathbf{A} = 1 + \frac{\mathbf{et}}{2},$$

where $\mathbf{t} \in \mathbf{e}^{\sim}$ is a positive vector. Then

$$\mathbf{B}_2 = \frac{\mathbf{et}}{4}$$

generates $\mathbf{A}$ by Cayley transform.

Example 3. Let

$$\mathbf{A} = e^{\frac{\theta}{2}\mathbf{e}\wedge\mathbf{a}},$$

where $\theta \in \mathbb{R}$, and $\mathbf{a} \in \mathcal{N}_{\mathbf{e}}$ represents a point. Rotor $\mathbf{A}$ generates the dilation centering at $\mathbf{a}$ and with scale $e^{-\theta}$.

Denote $\mathbf{I}_2 = \mathbf{e} \wedge \mathbf{a}$. Then

$$\mathbf{B}_2 = \mathbf{I}_2 \tanh\frac{\theta}{4}, \quad \mathbf{B}_2^{-1} = \mathbf{I}_2 / \tanh\frac{\theta}{4},$$

both generate $\mathbf{A}$ by Cayley transform.

Comparisons

Exponential map: transcendental, infinitely many inverses, but maps the Lie algebra $\Lambda^2(\mathbb{R}^{4,1})$ onto the group of positive rotors modulo scale.

Linear approximation of exp: has good performance only nearby the identity.

Quadratic approximation of exp: The *exterior exponential*

$$e^{\wedge \mathbf{B}_2} = 1 + \mathbf{B}_2 + \frac{\mathbf{B}_2 \wedge \mathbf{B}_2}{2!}.$$

Injective.

Domain of definition: a set $\mathbb{R}^{10} - V^9$, where V^9 is a 9D algebraic variety.

Image space modulo scale: the remainder of the special orthogonal group $SO(4,1)$, which is a 10D Lie group with two connected components, after removal of a 9D closed subset.

Polynomial Cayley transform: **Domain of definition:** a set $\mathbb{R}^{10} - V^5$, where V^5 is a 5D algebraic variety.

Image space modulo scale: the remainder of the Lorentz group of $\mathbb{R}^{4,1}$, which is a 10D connected Lie group, after removal of a 4D open disk.

3. Total Meet Product in 3D Conformal Incidence Geometry

Let there be two lines $\mathbf{12}$ and $\mathbf{1'2'}$ in space. There are the following four kinds of incidence relations between them:

- identical (collinear);
- parallel;
- intersecting;
- non-coplanar.

The total meet product between them is

$$
\begin{aligned}
(\mathbf{1}\wedge\mathbf{2})\bar{\vee}(\mathbf{1}'\wedge\mathbf{2}') = \quad & 1\otimes(\mathbf{1}\wedge\mathbf{2}\wedge\mathbf{1}'\wedge\mathbf{2}') \\
& + \mathbf{1}'\otimes(\mathbf{1}\wedge\mathbf{2}\wedge\mathbf{2}') - \mathbf{2}'\otimes(\mathbf{1}\wedge\mathbf{2}\wedge\mathbf{1}') \\
& + (\mathbf{1}'\wedge\mathbf{2}')\otimes(\mathbf{1}\wedge\mathbf{2}).
\end{aligned}
$$

Classification by the total meet product

- The two lines are non-coplanar if and only if the $(0,4)$-graded part is nonzero: $\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{1}' \wedge \mathbf{2}' \neq 0$.
- If the $(0,4)$-graded part is zero, the two lines are coplanar. If the $(1,3)$-graded part is also zero, the two lines are identical.
- If the $(0,4)$-graded part is zero but the $(1,3)$-graded part is nonzero, the supporting plane of the two lines is

$$[\mathbf{1}']\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{2}' - [\mathbf{2}']\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{1}',$$

  the intersection is

$$\mathbf{1}'[\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{2}'] - \mathbf{2}'[\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{1}'].$$

- The two lines are parallel if and only if the intersection is at infinity:

$$\partial(\mathbf{1}')[\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{2}'] = \partial(\mathbf{2}')[\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{1}'].$$

Definition of the total meet product

Let $\mathcal{V}^n$ be an nD vector space over a base field $\mathbb{K}$ of characteristic $\neq 2$. The *total meet product* of two multivectors in $\Lambda(\mathcal{V}^n)$ is a linear isomorphism in Grassmann algebra $\Lambda(\mathcal{V}^n) \otimes \Lambda(\mathcal{V}^n)$, defined for any r-blade $\mathbf{A}_r$ and s-blade $\mathbf{B}_s$ by

$$\mathbf{A}_r \,\bar{\vee}\, \mathbf{B}_s := \sum_{i=\max(0,r+s-n)}^{s} \; \sum_{(i,s-i)\vdash \mathbf{B}_s} \mathbf{B}_{s(1)} \otimes \big(\mathbf{A}_r \wedge \mathbf{B}_{s(2)}\big).$$

Applying to the conformal model for two circles in space

In $\Lambda(\mathbb{R}^{4,1})$, let $\mathbf{A}_3 = \mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}$ and $\mathrm{B}_3 = \mathbf{1}' \wedge \mathbf{2}' \wedge \mathbf{3}'$ be two circles each passing through three points. Then

$$\begin{aligned}\mathbf{A}_3 \bar{\nabla} (\mathbf{1}' \wedge \mathbf{2}' \wedge \mathbf{3}') = \quad & \mathbf{1}' \otimes \mathbf{A}_3 \wedge \mathbf{2}' \wedge \mathbf{3}' - \mathbf{2}' \otimes \mathbf{A}_3 \wedge \mathbf{1}' \wedge \mathbf{3}' + \mathbf{3}' \otimes \mathbf{A}_3 \wedge \mathbf{1}' \wedge \mathbf{2}' \\ & + \mathbf{1}' \wedge \mathbf{2}' \otimes \mathbf{A}_3 \wedge \mathbf{3}' - \mathbf{1}' \wedge \mathbf{3}' \otimes \mathbf{A}_3 \wedge \mathbf{2}' + \mathbf{2}' \wedge \mathbf{3}' \otimes \mathbf{A}_3 \wedge \mathbf{1}' \\ & + \mathbf{1}' \wedge \mathbf{2}' \wedge \mathbf{3}' \otimes \mathbf{A}_3,\end{aligned}$$

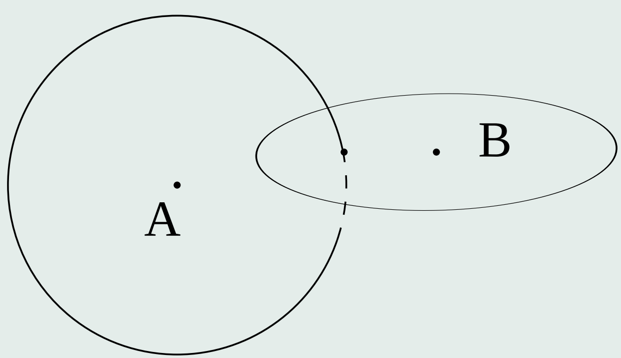
where the outer product precedes the tensor product.

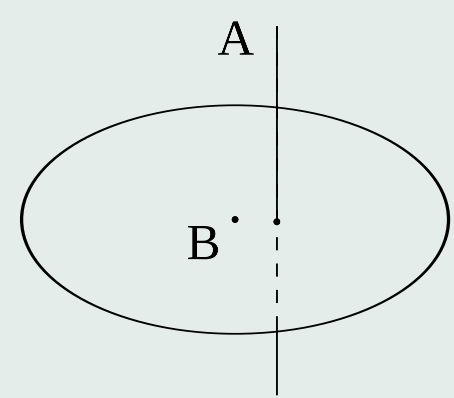
Knotting of two spheres or a sphere and a plane in $\mathbb{R}^n$

In $\mathbb{R}^n$, an rD sphere $\mathbf{A}$ and an sD sphere $\mathbf{B}$, where $0 \le r, s \le n-1$, are said to be *knotted*, if

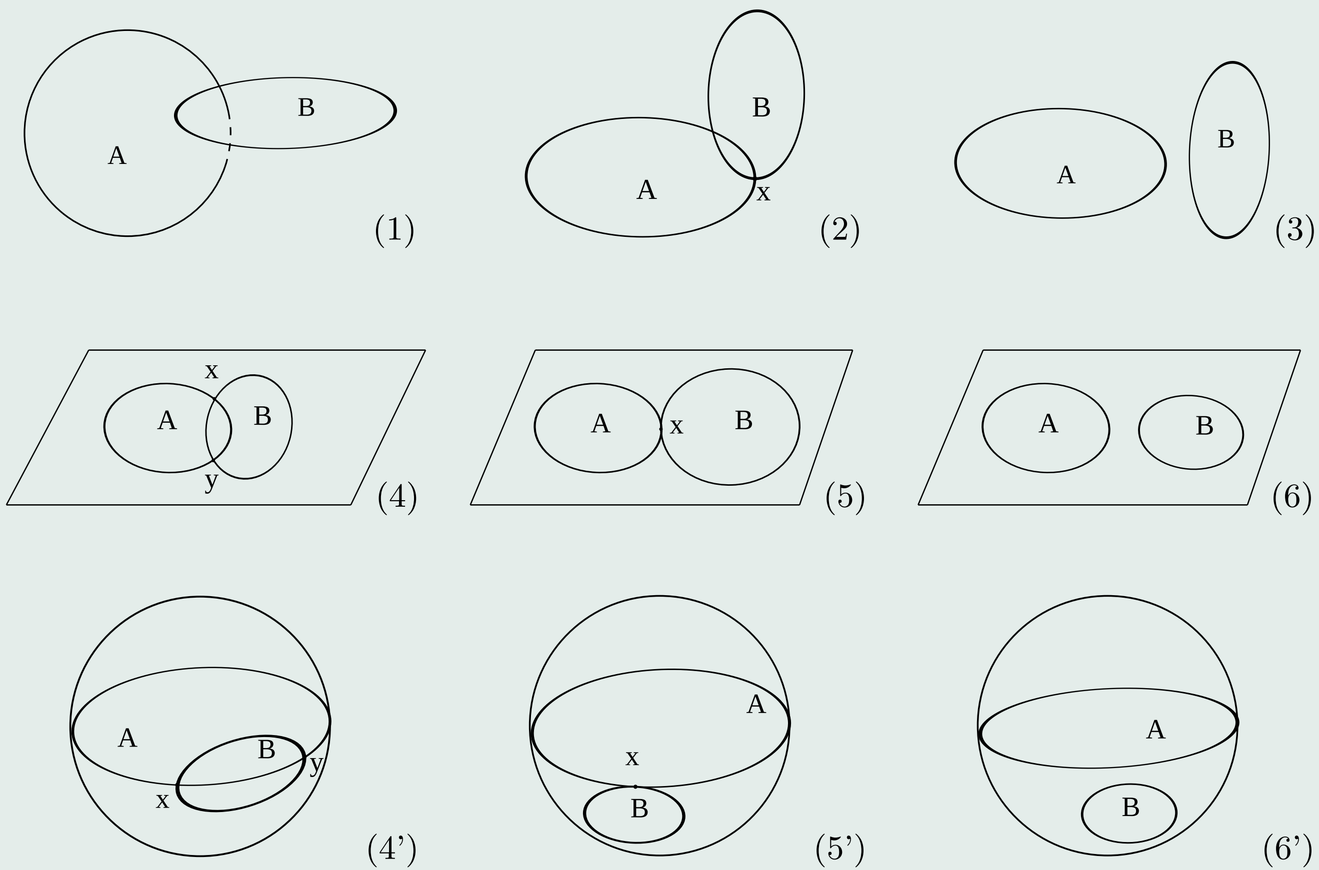
- they have no point in common,
- sphere $\mathbf{B}$ intersects the $(r+1)$D supporting plane $\mathbf{A}'$ of sphere $\mathbf{A}$ at a point inside sphere $\mathbf{A}$ and at the other point outside sphere $\mathbf{A}$,
- sphere $\mathbf{A}$ intersects the $(s+1)$D supporting plane $\mathbf{B}'$ of sphere $\mathbf{B}$ at a point inside sphere $\mathbf{B}$ and at the other point outside sphere $\mathbf{B}$.

An rD plane $\mathbf{A}$ and an sD sphere $\mathbf{B}$ are said to be *knotted*, if the intersection of plane $\mathbf{A}$ and the supporting plane $\mathbf{B}'$ of sphere $\mathbf{B}$ is a point inside sphere $\mathbf{B}$.





Incidence relations between two circles in space



Intersecting, tangent and separating of two spheres or a sphere and a plane

- Two spheres or planes of dimension between 0 and $n-1$ in $\mathbb{R}^n$ are said to be *separated*, if they neither have any point in common nor are knotted.
- The *extension* of two spheres or planes of dimension r, s respectively, refers to the plane or sphere of lowest dimension that contains both of them.
- For $0 \le r, s \le n-1$, an rD sphere and an sD sphere in $\mathbb{R}^n$ are said to be
  - *tD intersecting*, if their intersection is a tD sphere.
  - *tD tangent*, if they have a unique common point, called the *tangent point*, and they have a common tD tangent subspace at the tangent point.
  - *tD separated*, if they are separated, and their extension is a $(t+1)$D sphere or plane.

For an rD sphere and an sD plane, their tD intersecting, tangent and separated relations can be defined similarly.

Extension and intersection of two circles in space

From the total meet product of $\mathbf{A}_3 = \mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}$ and $\mathbf{B}_3 = \mathbf{1}' \wedge \mathbf{2}' \wedge \mathbf{3}'$, we get their $(k-2)$D extensions $\mathbf{E}_k$ where $k = 5, 4$, and $(l-2)$D intersections $\mathbf{I}_l$ where $l = 1, 2$:

$$\begin{aligned}
\mathbf{E}_5 &= [\mathbf{1}']\mathbf{A}_3 \wedge \mathbf{2}' \wedge \mathbf{3}' - [\mathbf{2}']\mathbf{A}_3 \wedge \mathbf{1}' \wedge \mathbf{3}' + [\mathbf{3}']\mathbf{A}_3 \wedge \mathbf{1}' \wedge \mathbf{2}', \\
\mathbf{I}_1 &= [\mathbf{A}_3 \wedge \mathbf{2}' \wedge \mathbf{3}']\mathbf{1}' - [\mathbf{A}_3 \wedge \mathbf{1}' \wedge \mathbf{3}']\mathbf{2}' + [\mathbf{A}_3 \wedge \mathbf{1}' \wedge \mathbf{2}']\mathbf{3}', \\
\mathbf{E}_4 &= [\mathbf{1}' \wedge \mathbf{2}']\mathbf{A}_3 \wedge \mathbf{3}' - [\mathbf{1}' \wedge \mathbf{3}']\mathbf{A}_3 \wedge \mathbf{2}' + [\mathbf{2}' \wedge \mathbf{3}']\mathbf{A}_3 \wedge \mathbf{1}', \\
\mathbf{I}_2 &= [\mathbf{A}_3 \wedge \mathbf{3}']\mathbf{1}' \wedge \mathbf{2}' - [\mathbf{A}_3 \wedge \mathbf{2}']\mathbf{1}' \wedge \mathbf{3}' + [\mathbf{A}_3 \wedge \mathbf{1}']\mathbf{2}' \wedge \mathbf{3}'.
\end{aligned}$$

1. The two circles are either knotted, or 0D tangent, or 2D planar separated, if and only if vector $\mathbf{I}_1$ is either negative, or null, or positive.
2. When $\mathbf{I}_1 = 0$, the two circles are coplanar or cospherical. They are coplanar and cospherical simultaneously if and only if they are identical, or equivalently, if and only if $\mathbf{I}_1 = \mathbf{I}_2 = 0$.
3. Assume that $\mathbf{I}_1 = 0$ but $\mathbf{I}_2 \neq 0$. Then Minkowski blade $\mathbf{E}_4$ represents the common supporting plane or sphere of the two circles, depending on whether or not $\mathbf{e} \in \mathbf{E}_4$.
4. The two circles are either 0D intersecting, or 1D tangent, or 1D separated, if and only if blade $\mathbf{I}_2$ is either Minkowski, or degenerate, or Euclidean.

The classifier may be useful in collision detection and neuron-based classification.

4. Conclusion

- The formulas on 3D Cayley transform and the total meet product, provide universal and compact representations of geometric transformations and configurations, and should prove to be useful in computer applications.
- New algebras are developed by investigating and applying the geometric algebra of null vectors, and have proved to be highly valuable in symbolic manipulations of geometries.