Tutorial on Fourier and Wavelet Transformations in Geometric Algebra

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Acknowledgements

I do believe in God's creative power in having brought it all into being in the first place, I find that studying the natural world is an opportunity to observe the majesty, the elegance, the intricacy of God's creation.

Francis Collins, Director of the US National Human Genome Research Institute, in Time Magazine, 5 Nov. 2006.

- I thank my wife, my children, my parents.
- B. Mawardi, G. Sommer, D. Hildenbrand, H. Li, R. Ablamowicz
- Organizers of AGACSE 3, Leipzig 2008.

CONTENTS

Multivectors, blades, reverse, scalar product

- Multivector $M \in \mathbb{G}_{p,q} = Cl_{p,q}, p+q=n$, has k-vector parts $(0 \leq k \leq n)$ scalar, vector, bi-vector, ..., pseudoscalar

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n, \tag{1}$$

blade index $A \in \{0, 1, 2, 3, 12, 23, 31, 123, \ldots, 12\ldots n\}$, $M_A \in \mathbb{R}$.

- Reverse of $M \in \mathbb{G}_{p,q}$

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k. \tag{2}$$

replaces complex conjugation/quaternion conjugation

- The scalar product of two multivectors $M, \widetilde{N} \in \mathbb{G}_{p,q}$ is defined as

$$M * \widetilde{N} = \langle M\widetilde{N} \rangle = \langle M\widetilde{N} \rangle_0 \tag{3}$$

For $M, \widetilde{N} \in \mathbb{G}_n = \mathbb{G}_{n,0}$ we get $M * \widetilde{N} = \sum_A M_A N_A$.

Modulus, blade subspace, pseudoscalar

- The modulus $|M|$ of a multivector $M \in \mathbb{G}_n = \mathbb{G}_{n,0}$ is defined as

$$|M|^2 = M * \widetilde{M} = \sum_{A=1}^{n} M_A^2. \tag{4}$$

- For $n = 2 (\mathrm{mod}\ 4)$, $n = 3 (\mathrm{mod}\ 4)$ pseudoscalar $i_n = \boldsymbol{e}_1 \boldsymbol{e}_2 \ldots \boldsymbol{e}_n$

$$i_n^2 = -1. \tag{5}$$

- A blade B describes a vector subspace

$$V_B = \{\boldsymbol{x} \in \mathbb{R}^{p,q} | \boldsymbol{x} \wedge B = 0\}.$$

  Its dual blade

$$B^* = B i_n^{-1}$$

  describes the complimentary vector subspace $V_B^{\perp}$.
- pseudoscalar i_n commutes for $n = 3 (\mathrm{mod}\, 4)$

$$i_n\, M = M\, i_n, \qquad \forall M \in \mathbb{G}_{n,0}.$$

Multivector functions

- Multivector valued function $f : \mathbb{R}^{p,q} \to \mathbb{G}_{p,q} = Cl_{p,q},\ p+q=n$, has 2^n blade components

$$f(\boldsymbol{x}) = \sum_{A=1}^{n} f_A(\boldsymbol{x})\boldsymbol{e}_A. \tag{6}$$

- We define the inner product of $\mathbb{R}^n \to Cl_{n,0}$ functions f, g by

$$(f,g) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\ d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A\widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^n} f_A(\boldsymbol{x})g_B(\boldsymbol{x})\ d^n\boldsymbol{x}, \tag{7}$$

- and the $L^2(\mathbb{R}^n; Cl_{n,0})$-norm

$$\|f\|^2 = \langle (f,f) \rangle = \int_{\mathbb{R}^n} |f(\boldsymbol{x})|^2 d^n\boldsymbol{x}. \tag{8}$$

Vector Differential & Vector Derivative

- Vector differential of f defined (any const. $\boldsymbol{a} \in \mathbb{R}^{p,q}$)

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon}, \qquad (9)$$

  NB: $\boldsymbol{a} \cdot \nabla$ scalar.

- *Vector* derivative ∇ can be expanded as

$$\nabla = \sum_{k=1}^{n} \boldsymbol{e}_k \partial_k, \qquad \partial_k = \boldsymbol{e}_k \cdot \nabla = \frac{\partial}{\partial x_k}, \qquad (10)$$

- both coordinate independent!

Geometric Calculus Examples

MV Functions

$f_1 = \boldsymbol{x}$
$f_2 = \boldsymbol{x}^2$
$f_3 = |\boldsymbol{x}|$
$f_4 = \boldsymbol{x} \cdot \langle A \rangle_k \;, 0 \leq k \leq n$
$f_5 = \log r$
$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$
$r = |\boldsymbol{r}|$

V-Differentials

$\boldsymbol{a} \cdot \nabla f_1 = \boldsymbol{a}$
$\boldsymbol{a} \cdot \nabla f_2 = 2\boldsymbol{a} \cdot \boldsymbol{x}$
$\boldsymbol{a} \cdot \nabla f_3 = \frac{\boldsymbol{a} \cdot \boldsymbol{x}}{|\boldsymbol{x}|}$
$\boldsymbol{a} \cdot \nabla f_4 = \boldsymbol{a} \cdot \langle A \rangle_k$
$\boldsymbol{a} \cdot \nabla f_5 = \frac{\boldsymbol{a} \cdot \boldsymbol{r}}{r^2}$

V-Derivatives

$\nabla f_1 = 3, \quad (n = 3)$
$\nabla f_2 = 2\boldsymbol{x}$
$\nabla f_3 = \boldsymbol{x}/|\boldsymbol{x}|$
$\nabla f_4 = k\langle A \rangle_k$
$\nabla f_5 = \boldsymbol{r}^{-1} = \boldsymbol{r}/r^2.$

References

1) Hestenes & Sobczyk, Clifford Algebra to Geometric Calculus, 1984.
2) Hitzer, Vector Differential Calculus, 2002.

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Rules for Vector Differential & Derivative

Derivative from differential (regard $\boldsymbol{x}$ = const., $\boldsymbol{a}$ = variable)

$$\nabla f = \nabla_{\boldsymbol{a}} (\boldsymbol{a} \cdot \nabla f) \tag{11}$$

Sum rules, product rules exist. Modification by non-commutativity: $\dot{\nabla} f \dot{g} \neq f \dot{\nabla} \dot{g}$

$$\nabla(fg) = (\dot{\nabla}\dot{f})g + \dot{\nabla} f \dot{g} = (\dot{\nabla}\dot{f})g + \sum_{k=1}^{n} \boldsymbol{e}_k f (\partial_k g). \tag{12}$$

Vector differential / derivative of $f(\boldsymbol{x}) = g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x}) \in \mathbb{R}$:

$$\boldsymbol{a} \cdot \nabla f = \{\boldsymbol{a} \cdot \nabla \lambda(\boldsymbol{x})\} \frac{\partial g}{\partial \lambda}, \quad \nabla f = (\nabla \lambda) \frac{\partial g}{\partial \lambda}. \tag{13}$$

Example: For $\boldsymbol{a} = \boldsymbol{e}_k$ $(1 \leq k \leq n)$ $\qquad \boldsymbol{e}_k \cdot \nabla f = \partial_k f = (\partial_k \lambda) \frac{\partial f}{\partial \lambda}$.

Classical scalar complex Fourier Transformations

Definition (1D Cassical FT)

For an integrable function $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the Fourier transform of f is the function $\mathcal{F}\{f\} \colon \mathbb{R} \to \mathbb{C} = \mathbb{G}_{0,1}$

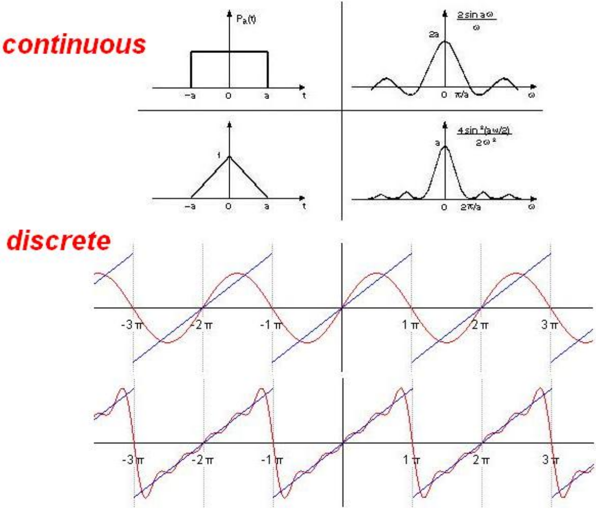
$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x}\, dx, \tag{14}$$

i imaginary unit: $i^2 = -1$.

Inverse FT

$$f(x) = \mathcal{F}^{-1}[\mathcal{F}\{f\}(\omega)] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}\{f\}(\omega)\, e^{i\omega x}\, d\omega. \tag{15}$$

Examples of continuous and discrete 1D FT [Images: Wikipedia]



Geometric Algebra Fourier Transformation (GA FT)

complex → geometric

- complex unit i → Geometric roots of -1, e.g. pseudoscalars i_n, $n = 2, 3(\text{mod}\, 4)$.
- complex f → multivector function $f \in L^2(\mathbb{R}^n, \mathbb{G}_n)$

Definition (with pseudoscalars in dims. 2,3,6,7,10,11, ...)

The GA Fourier transform $\mathcal{F}\{f\}\colon \mathbb{R}^n \to \mathbb{G}_n = Cl_{n,0}$, $n = 2, 3(\text{mod}\, 4)$ is given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}, \tag{16}$$

for multivector functions $f\colon \mathbb{R}^n \to \mathbb{G}_n$. NB: Also possible for $Cl_{0,n}$, $n = 1, 2(\text{mod}\, 4)$.

Possible applications: dimension specific transformations for actual data and signals, with desired subspace structure.

Inversion of these GA FTs

$$f(\boldsymbol{x}) = \mathcal{F}^{-1}[\mathcal{F}\{f\}(\boldsymbol{\omega})] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^3} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega}. \tag{17}$$

Overview of complex, Clifford, quaternion, spacetime FTs

- Complex Fourier Transformation (FT)

$$\mathcal{F}_{\mathbb{C}}\{f\}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x}\, dx \tag{18}$$

- Clifford Geometric Algebra (GA) FT in $\mathbb{G}_3$ $(i \to i_3)$

$$\mathcal{F}_{\mathbb{G}_3}\{f\}(\vec{\omega}) = \int_{\mathbb{R}^3} f(\vec{x})\, e^{-i_3\vec{\omega}\cdot\vec{x}}\, d^3\vec{x} \tag{19}$$

- GA FT in $\mathbb{G}_{n,0}, \mathbb{G}_{0,n'}$ $(i \to i_n)$

$$\mathcal{F}_{\mathbb{G}_n}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \tag{20}$$

- QFT in $\mathbb{H}$ $(i \to \boldsymbol{i}, \boldsymbol{j})$

$$\mathcal{F}_{\mathbb{H}}\{f\}(\boldsymbol{u}) = \hat{f}(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-\boldsymbol{i}xu} f(\boldsymbol{x})\, e^{-\boldsymbol{j}yv} d^2\boldsymbol{x} \tag{21}$$

- SFT in spacetime algebra $\mathbb{G}_{3,1}$ $(\boldsymbol{i} \to \boldsymbol{e}_t,\, \boldsymbol{j} \to i_3)$

$$\mathcal{F}_{STA}\{f\}(\boldsymbol{u}) = \overset{\diamond}{f}(\boldsymbol{u}) = \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t\, ts} f(\boldsymbol{x})\, e^{-i_3\vec{x}\cdot\vec{u}} d^3\vec{x}dt \tag{22}$$

Properties of the GA FT in $\mathbb{G}_n, n = 3 \,(\mathrm{mod}\, 4)$

General properties: Linearity, shift, modulation ($\boldsymbol{\omega}$-shift), scaling, convolution, Plancherel theorem and Parseval theorem

$$\|f\| = \frac{1}{(2\pi)^{n/2}} \|\mathcal{F}\{f\}\|. \tag{23}$$

Table: Unique properties of GA FT Multiv. Funct. $f \in L^2(\mathbb{R}^n, \mathbb{G}_n)$, $\boldsymbol{a} \in \mathbb{R}^n$, $m \in \mathbb{N}$.

Property	Multiv. Funct.	GA FT
Vec. diff.	$(\boldsymbol{a} \cdot \nabla)^m f(\boldsymbol{x})$	$i_n^m \, (\boldsymbol{a} \cdot \boldsymbol{\omega})^m \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$(\boldsymbol{a} \cdot \boldsymbol{x})^m \, f(\boldsymbol{x})$	$i_n^m \, (\boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}})^m \, \mathcal{F}\{f\}(\boldsymbol{\omega})$
Moments of $\boldsymbol{x}$	$\boldsymbol{x}^m f(\boldsymbol{x})$	$i_n^m \, \nabla_{\boldsymbol{\omega}}^m \, \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$f(\boldsymbol{x})\boldsymbol{x}^m$	$i_n^m \, \mathcal{F}\{f\}(\boldsymbol{\omega}) \, \nabla_{\boldsymbol{\omega}}^m$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$i_n^m \, \boldsymbol{\omega}^m \mathcal{F}\{f\}(\boldsymbol{\omega})$
dual moments	$f(\boldsymbol{x})\nabla^m$	$i_n^m \, \mathcal{F}\{f\}(\boldsymbol{\omega}) \, \boldsymbol{\omega}^m$

NB: Standard complex FT only gives coordinate moments and partial derivative properties.

Properties of the GA FT in $\mathbb{G}_n, n = 2 \,(\mathrm{mod}\, 4)$

Property	Multiv. Funct.	CFT
Left lin.	$\alpha f(\boldsymbol{x}) + \beta\, g(\boldsymbol{x})$	$\alpha\mathcal{F}\{f\}(\boldsymbol{\omega}) + \beta\mathcal{F}\{g\}(\boldsymbol{\omega})$
$\boldsymbol{x}$-Shift	$f(\boldsymbol{x} - \boldsymbol{a})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{a}}$
$\boldsymbol{\omega}$-Shift	$f(\boldsymbol{x})\, e^{i_n \boldsymbol{\omega}_0 \cdot \boldsymbol{x}}$	$\mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{a^n}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Convolution	$(f \star g)(\boldsymbol{x})$	$\mathcal{F}\{f\}(-\boldsymbol{\omega})\,\mathcal{F}\{g_{odd}\}(\boldsymbol{\omega})$ $+\mathcal{F}\{f\}(\boldsymbol{\omega})\,\mathcal{F}\{g_{even}\}(\boldsymbol{\omega})$
Vec. diff.	$(\boldsymbol{a}\cdot\nabla)^m f(\boldsymbol{x})$	$(\boldsymbol{a}\cdot\boldsymbol{\omega})^m \mathcal{F}\{f\}(\boldsymbol{\omega})\; i_n^m$
	$(\boldsymbol{a}\cdot\boldsymbol{x})^m\; f(\boldsymbol{x})$	$(\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}})^m\; \mathcal{F}\{f\}(\boldsymbol{\omega})\; i_n^m$
Moments of $\boldsymbol{x}$	$\boldsymbol{x}^m f(\boldsymbol{x})$	$\nabla_{\boldsymbol{\omega}}^m\; \mathcal{F}\{f\}(\boldsymbol{\omega})\; i_n^m$
	$f(\boldsymbol{x})\, \boldsymbol{x}^m$	$\mathcal{F}\{f\}((-1)^m\boldsymbol{\omega})\; \nabla_{\boldsymbol{\omega}}^m\; i_n^m$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$\boldsymbol{\omega}^m\; \mathcal{F}\{f\}(\boldsymbol{\omega})\; i_n^m$
	$f(\boldsymbol{x})\nabla^m$	$\mathcal{F}\{f\}((-1)^m\boldsymbol{\omega})\; \boldsymbol{\omega}^m\; i_n^m$
Plancherel	$\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\, d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\, d^n\boldsymbol{\omega}$
sc. Parseval	$\Vert f \Vert$	$\frac{1}{(2\pi)^{n/2}}\Vert\mathcal{F}\{f\}\Vert$

Applications

- Sampling
- Representation of shape
- Uncertainty
- LSI filters (smoothing, edge detection)
- Signal analysis
- Image processing
- Fast (multi)vector pattern matching
- Visual flow analysis
- (Multi)vector field analysis

Relation with complex FT, Discrete GA FT, Fast GA FT

- Example of multivector signal decomposition

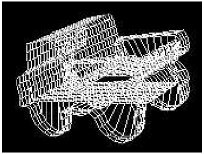
$$f = [f_0 + f_{123}i_3] + [f_1 + f_{23}i_3]\boldsymbol{e}_1 + [f_2 + f_{31}i_3]\boldsymbol{e}_2 + [f_3 + f_{12}i_3]\boldsymbol{e}_3 \tag{24}$$

  corresponds to 4 complex signals.
- Implementation of GA FT by 4 complex FTs

$$\mathcal{F}_{\mathbb{G}_3}\{f\} = \mathcal{F}[f_0 + f_{123}i_3] + \mathcal{F}[f_1 + f_{23}i_3]\boldsymbol{e}_1 + \mathcal{F}[f_2 + f_{31}i_3]\boldsymbol{e}_2 + \mathcal{F}[f_3 + f_{12}i_3]\boldsymbol{e}_3 \tag{25}$$

- This form of the GA FT leads to the discrete GA FT using 4 discrete complex FTs.
- 4 fast FTs (FFT) can be used to implement fast GA FT.

Application: Fourier Descriptor Repr. of Shape (B. Rosenhahn et al [6])



- Set of 3D sample points is projected along $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$: $f^l_{n_1,n_2}$, $l = 1, 2, 3.$
- Use of rotors for 2D discrete rotor FT ($-N_{1,2} \leq n_{1,2}, k_{1,2} \leq N_{1,2}$, $N' = 2N + 1$)

$$R^{k_1,n_1}_{1,l} = \exp(\frac{2\pi}{N'_1} k_1 n_1 \boldsymbol{e}_l i_3), \quad R^{k_2,n_2}_{2,l} = \dots \tag{26}$$

- Sample points $f^l_{n_1,n_2}$ give surface phase vectors

$$p^l_{k_1,k_2} = \frac{1}{N'_1 N'_2} \sum_{n_1,n2} f^l_{n_1,n_2} \widetilde{R^{k_1,n_1}_{1,l}}\, \widetilde{R^{k_2,n_2}_{2,l}} \boldsymbol{e}_l \tag{27}$$

- Surface estimation

$$F(\Phi_1, \Phi_2) = \sum_{l=1}^{3} f^l(\Phi_1, \Phi_2) \boldsymbol{e}_l = \sum_{l=1}^{3} \sum_{k_1,k_2} R^{k_1,\Phi_1}_{1,l} R^{k_2,\Phi_2}_{2,l} p^l_{k_1,k_2} \widetilde{R^{k_1,\Phi_1}_{1,l}}\, \widetilde{R^{k_2,\Phi_2}_{2,l}} \tag{28}$$

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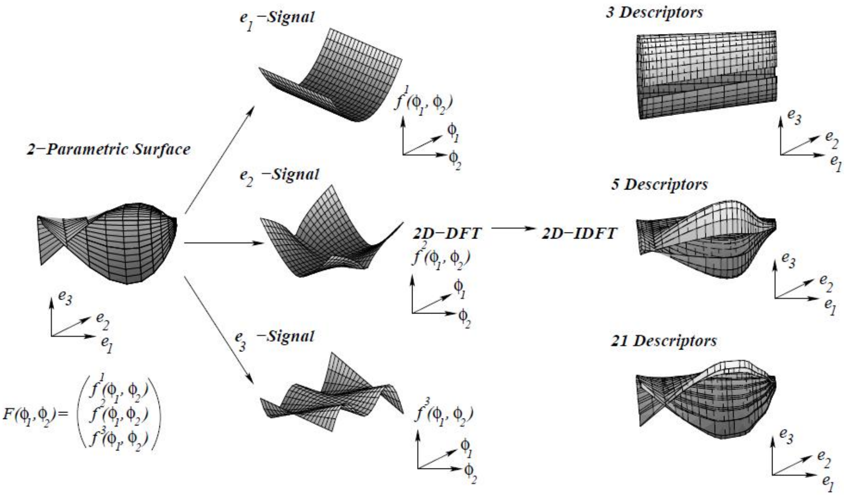

GA FT of Convolution

Convolution

The GA FT of the convolution of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$

$$(f{\star}g)(\boldsymbol{x}) = \int_{\mathbb{R}^3} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\,d^3\boldsymbol{y}, \tag{29}$$

equals for $n = 3(\bmod 4)$ the product of the GA FTs of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$:

$$\mathcal{F}\{f{\star}g\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega})\mathcal{F}\{g\}(\boldsymbol{\omega}). \tag{30}$$

Convolution for $n = 2(\bmod 4)$

For $n = 2(\bmod 4)$ we get due to non-commutativity of the pseudoscalar $i_n \in \mathbb{G}_n$

$$i_n\,M \neq M\,i_n, \qquad \text{for } M \in \mathbb{G}_n. \tag{31}$$

that

$$\mathcal{F}\{f \star g\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(-\boldsymbol{\omega})\mathcal{F}\{g_{odd}\}(\boldsymbol{\omega}) + \mathcal{F}\{f\}(\boldsymbol{\omega})\mathcal{F}\{g_{even}\}(\boldsymbol{\omega}) \tag{32}$$

GA FT of Correlation

Correlation

The GA FT of the correlation of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$

$$(f\bigstar g)(\boldsymbol{x}) = \int_{\mathbb{R}^3} f(\boldsymbol{y})g(\boldsymbol{x}+\boldsymbol{y})\, d^3\boldsymbol{y}, \tag{33}$$

equals for $n = 3(\mathrm{mod}\, 4)$ the product of the GA FTs of $f'(\boldsymbol{x}) = f(-\boldsymbol{x})$ and $g(\boldsymbol{x})$:

$$\mathcal{F}\{f\bigstar g\}(\boldsymbol{\omega}) = \mathcal{F}\{f'\}(\boldsymbol{\omega})\mathcal{F}\{g\}(\boldsymbol{\omega}). \tag{34}$$

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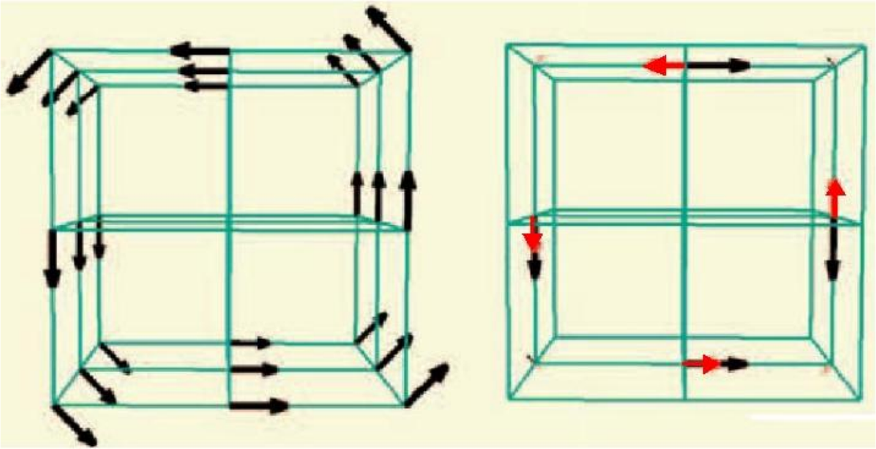
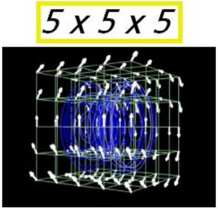
Application: Vector Pattern Analysis (J. Ebling, G. Scheuermann [7, 8])

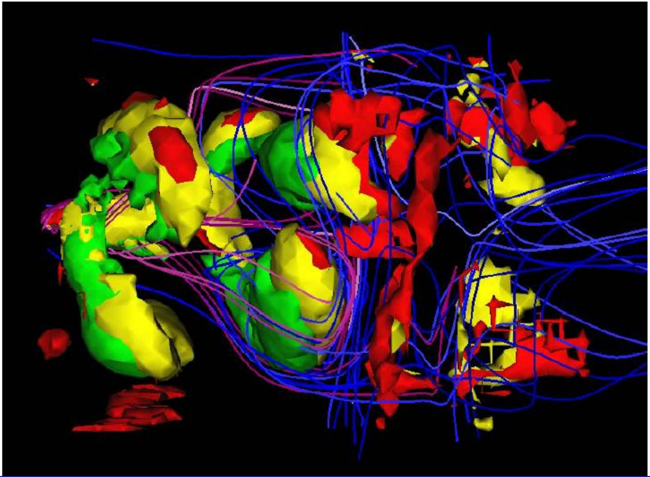
$3 \times 3 \times 3$ rotation pattern and discrete GA FT
black: vector components, red: bivector components (shown by normal vectors)



Application: Vector Pattern Matching (J. Ebling, G. Scheuermann [7, 8])

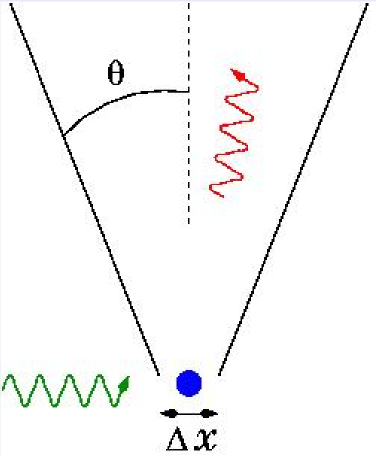
Convolution of $3 \times 3 \times 3 = 3^3$ (red), 5^3 (yellow), 8^3 (green) rotation patterns with gas furnace chamber flow field.





Uncertainty Principle in Physics (Image: Wikipedia)

Optical Measurement



Heisenberg Uncertainty Princ.

Standard deviations Δx, Δp of position x, momentum p measurements

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

f . . . wave function.

$$f \longrightarrow \Delta x,$$

$$\mathcal{F}\{f\} \longrightarrow \Delta p.$$

Application of GAFT: Uncertainty Principles

Directional Uncertainty Principle [Hitzer&Mawardi, Proc. ICCA7]

- Multivect. funct. $f \in L^2(\mathbb{R}^n, \mathbb{G}_n)$, $n = 2, 3 \pmod 4$
- GA FT $\mathcal{F}\{f\}(\boldsymbol{\omega})$. arbitrary const. vectors $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^{\boldsymbol{n}}$

$$\frac{1}{\|f\|^2}\int_{\mathbb{R}^n}(\boldsymbol{a}\cdot\boldsymbol{x})^2|f(\boldsymbol{x})|^2\,d^n\boldsymbol{x}\;\frac{1}{(2\pi)^n\|f\|^2}\int_{\mathbb{R}^n}(\boldsymbol{b}\cdot\boldsymbol{\omega})^2\;|\mathcal{F}\{f\}(\boldsymbol{\omega})|^2 d^n\boldsymbol{\omega} \geq \frac{(\boldsymbol{a}\cdot\boldsymbol{b})^2}{4}$$

Equality (minimal bound) for optimal Gaussian multivector functions

$$f(\boldsymbol{x}) = C_0\; e^{-k\;\boldsymbol{x}^2}, \tag{37}$$

$C_0 \in \mathbb{G}_n$ arbitrary const. multivector, $0 < k \in \mathbb{R}$.

Uncertainty principle (direction independent)

For $f \in L^2(\mathbb{R}^n, \mathbb{G}_n)$, $n = 2, 3 \pmod 4$ we obtain from the dir. UP that

$$\frac{1}{\|f\|^2}\int_{\mathbb{R}^n}\boldsymbol{x^2}\;|f(\boldsymbol{x})|^2\;d^n\boldsymbol{x}\;\frac{1}{(2\pi)^n\|f\|^2}\int_{\mathbb{R}^n}\boldsymbol{\omega^2}\;|\mathcal{F}\{f\}(\boldsymbol{\omega})|^2 d^n\boldsymbol{\omega} \geq \frac{n}{4}. \tag{38}$$

Quaternions

- Gauss, Rodrigues and Hamilton's 4D quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ with 3 imaginary units:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j},\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{39}$$

- Quaternion

$$q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R} \tag{40}$$

- has *quaternion conjugate* (reversion in $Cl^+_{3,0}$)

$$\tilde{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}, \tag{41}$$

- Leads to *norm* of $q \in \mathbb{H}$

$$|q| = \sqrt{q\tilde{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \qquad |pq| = |p||q|. \tag{42}$$

- Scalar part

$$Sc(q) = q_r = \frac{1}{2}(q + \tilde{q}). \tag{43}$$

$\pm$ Split of quaternions [Hitzer, AACA 2007]

Convenient *split* of quaternions

$$q = q_+ + q_-, \quad q_\pm = \frac{1}{2}(q \pm \boldsymbol{i} q \boldsymbol{j}). \tag{44}$$

Explicitly in real components $q_r, q_i, q_j, q_k \in \mathbb{R}$ using (39):

$$q_\pm = \{q_r \pm q_k + \boldsymbol{i}(q_i \mp q_j)\}\frac{1 \pm \boldsymbol{k}}{2} = \frac{1 \pm \boldsymbol{k}}{2}\{q_r \pm q_k + \boldsymbol{j}(q_j \mp q_i)\}. \tag{45}$$

Consequence: modulus identity

$$|q|^2 = |q_-|^2 + |q_+|^2. \tag{46}$$

Property: Scalar part of mixed split product

Given two quaternions p, q and applying the $\pm$ split we get zero for the scalar part of the mixed products

$$Sc(p_+ q_-) = 0, \qquad Sc(p_- q_+) = 0. \tag{47}$$

Definition of quaternion FT (QFT)

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\boldsymbol{i}x_1\omega_1} f(\boldsymbol{x})\, e^{-\boldsymbol{j}x_2\omega_2} d^2\boldsymbol{x}. \tag{48}$$

Linearity leads to

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f_- + f_+\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f_-\}(\boldsymbol{\omega}) + \mathcal{F}_q\{f_+\}(\boldsymbol{\omega}). \tag{49}$$

Remark: Other variations exist (see later).

Simple complex forms for QFT of $f_\pm$

The QFT of $f_\pm$ split parts of quaternion function $f \in L^2(\mathbb{R}^2, \mathbb{H})$ have simple complex forms

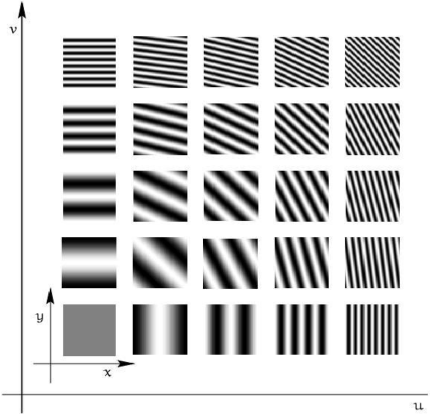
$$\hat{f}_\pm = \int_{\mathbb{R}^2} f_\pm e^{-\boldsymbol{j}(x_2\omega_2 \mp x_1\omega_1)} d^2x = \int_{\mathbb{R}^2} e^{-\boldsymbol{i}(x_1\omega_1 \mp x_2\omega_2)} f_\pm d^2x \ . \tag{50}$$

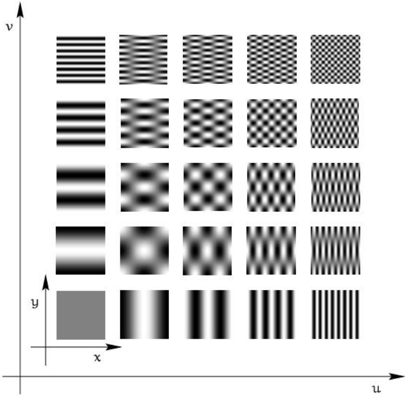
Free of coordinates:

$$\hat{f}_- = \int_{\mathbb{R}^2} f_- e^{-\boldsymbol{j}\, \boldsymbol{x}\cdot\boldsymbol{\omega}} d^2x, \quad \hat{f}_+ = \int_{\mathbb{R}^2} f_+ e^{-\boldsymbol{j}\, \boldsymbol{x}\cdot(\mathcal{U}_{\boldsymbol{e}_1}\boldsymbol{\omega})} d^2x, \tag{51}$$

the reflection $\mathcal{U}_{\boldsymbol{e}_1}\boldsymbol{\omega}$ changes component $\omega_1 \to -\omega_1$.

2D complex FT and QFT (Images: T. Bülow, thesis [4])





Applications of QFT, discrete and fast versions

- partial differential systems
- colour image processing
- filtering
- disparity estimation (two images differ by *local* translations)
- texture segmentation
- wide ranging higher dimensional generalizations

Discrete and fast QFT: Pei, Ding, Chang (2001) [13]

Discrete QFT

$$\mathcal{F}_{DQFT}\{f\}(\boldsymbol{\omega}) = \sum_{x_1=0}^{M-1} \sum_{x_2=0}^{N-1} e^{-\boldsymbol{i}x_1\omega_1/M} f(\boldsymbol{x})\, e^{-\boldsymbol{j}x_2\omega_2/N}. \tag{52}$$

Fast QFT

The simple complex forms

$$\hat{f} = \hat{f}_- + \hat{f}_+, \quad = \int_{\mathbb{R}^2} f_- e^{-\boldsymbol{j}\,\boldsymbol{x}\cdot\boldsymbol{\omega}} d^2x, + \int_{\mathbb{R}^2} f_+ e^{-\boldsymbol{j}\,\boldsymbol{x}\cdot(\mathcal{U}_{\boldsymbol{e}_1}\boldsymbol{\omega})} d^2x, \tag{53}$$

show that the QFT can be implemented with 2 complex $M \times N$ 2D discrete FTs. This requires

$$2MN\ \log_2 MN \tag{54}$$

real number multiplications.

Application to image processing

Application: Image transformations

The QFT of a quaternion function $f \in L^2(\mathbb{R}^2; \mathbb{H})$ with a $GL(\mathbb{R}^2)$ transformation $\mathcal{A}$ (stretches, reflections, rotations) of its vector argument $\boldsymbol{x}$ is

$$\widehat{f(\mathcal{A}\boldsymbol{x})}(\boldsymbol{u}) = |\det \mathcal{A}^{-1}| \, \{ \hat{f}_-(\overline{\mathcal{A}^{-1}}\boldsymbol{u}) + \hat{f}_+(\mathcal{U}_{\boldsymbol{e}_1}\overline{\mathcal{A}^{-1}}\mathcal{U}_{\boldsymbol{e}_1}\boldsymbol{u}) \} \, , \tag{55}$$

where $\overline{\mathcal{A}^{-1}}$ is the adjoint of the inverse of $\mathcal{A}$.

Properties of QFT

Standard properties: Linearity, shift, modulation, dilation, Plancherel theorem and *Parseval theorem* (signal energy)

$$\|f\| = \frac{1}{2\pi}\|\mathcal{F}_q\{f\}\|. \tag{56}$$

Table: Special QFT properties. Quat. Funct. $f \in L^2(\mathbb{R}^2;\mathbb{H})$, with $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^2$, $m, n \in \mathbb{N}_0$.

Property	**Quat. Funct.**	**QFT**
Part. deriv.	$\frac{\partial^{m+n}}{\partial x^m \partial y^n} f(\boldsymbol{x})$	$(\boldsymbol{i}u)^m \hat{f}(\boldsymbol{u})(\boldsymbol{j}v)^n$
Moments of x, y	$x^m y^n f(\boldsymbol{x})$	$\boldsymbol{i}^m \frac{\partial^{m+n}}{\partial u^m \partial v^n} \hat{f}(\boldsymbol{u})\, \boldsymbol{j}^n$
Powers* of $\boldsymbol{i}, \boldsymbol{j}$	$\boldsymbol{i}^m f(\boldsymbol{x})\, \boldsymbol{j}^n$	$\boldsymbol{i}^m \hat{f}(\boldsymbol{u})\, \boldsymbol{j}^n$

* Easy with correct order!

$\pm$ split and QFT commute:

$$\mathcal{F}_q\{f_\pm\} = \mathcal{F}_q\{f\}_\pm.$$

Properties of QFT

Integration of parts

With the vector differential $\boldsymbol{a}\cdot\nabla = a_1\partial_1 + a_2\partial_2$, with arbitrary constant $\boldsymbol{a}\in\mathbb{R}^2$, $g,h\in L^2(\mathbb{R}^2,\mathbb{H})$

$$\int_{R^2} g(\boldsymbol{x})[\boldsymbol{a}\cdot\nabla h(\boldsymbol{x})]d^2\boldsymbol{x} = \left[\int_R g(\boldsymbol{x})h(\boldsymbol{x})d\boldsymbol{x}\right]_{a\cdot x=-\infty}^{a\cdot x=\infty} - \int_{R^2}[\boldsymbol{a}\cdot\nabla g(\boldsymbol{x})]h(\boldsymbol{x})d^2\boldsymbol{x}. \quad (57)$$

Modulus identities

Due to $|q|^2 = |q_-|^2 + |q_+|^2$ we get for $f:\mathbb{R}^2\rightarrow\mathbb{H}$ the following identities

$$|f(\boldsymbol{x})|^2 = |f_-(\boldsymbol{x})|^2 + |f_+(\boldsymbol{x})|^2, \quad (58)$$

$$|\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 = |\mathcal{F}_q\{f_-\}(\boldsymbol{\omega})|^2 + |\mathcal{F}_q\{f_+\}(\boldsymbol{\omega})|^2. \quad (59)$$

Properties of QFT

QFT of vector differentials

Using the split $f = f_- + f_+$ we get the QFTs of the split parts. Let $\boldsymbol{b} \in \mathbb{R}^2$ be an arbitrary constant vector.

$$\mathcal{F}_q\{\boldsymbol{b}\cdot\nabla f_-\}(\boldsymbol{\omega}) = \boldsymbol{b}\cdot\boldsymbol{\omega}\mathcal{F}_q\{f_-\}(\boldsymbol{\omega})\,\boldsymbol{j} = \boldsymbol{i}\,\boldsymbol{b}\cdot\boldsymbol{\omega}\mathcal{F}_q\{f_-\}(\boldsymbol{\omega}), \tag{60}$$

$$\mathcal{F}_q\{\boldsymbol{b}\cdot\nabla f_+\}(\boldsymbol{\omega}) = (\boldsymbol{b}\cdot\mathcal{U}_{\boldsymbol{e}_1}\boldsymbol{\omega})\,\mathcal{F}_q\{f_+\}(\boldsymbol{\omega})\,\boldsymbol{j} = \boldsymbol{i}\,(\boldsymbol{b}\cdot\mathcal{U}_{\boldsymbol{e}_2}\boldsymbol{\omega})\,\mathcal{F}_q\{f_+\}(\boldsymbol{\omega}), \tag{61}$$

$$\mathcal{F}_q\{(\mathcal{U}_{\boldsymbol{e}_1}\boldsymbol{b}\cdot\nabla)f_+\}(\boldsymbol{\omega}) = \boldsymbol{b}\cdot\boldsymbol{\omega}\,\mathcal{F}_q\{f_+\}(\boldsymbol{\omega})\,\boldsymbol{j}, \tag{62}$$

$$\mathcal{F}_q\{(\mathcal{U}_{\boldsymbol{e}_2}\boldsymbol{b}\cdot\nabla)f_+\}(\boldsymbol{\omega}) = \boldsymbol{i}\,\boldsymbol{b}\cdot\boldsymbol{\omega}\,\mathcal{F}_q\{f_+\}(\boldsymbol{\omega}). \tag{63}$$

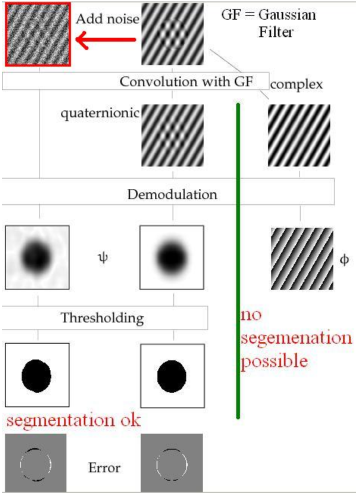
Gaussian quaternion filter (GF)

$$h(\boldsymbol{x}) \quad = \quad g(\boldsymbol{x})\, e^{\boldsymbol{i}2\pi x_1 \omega_{01}}\, e^{\boldsymbol{j}2\pi x_2 \omega_{02}}$$

2D Gaussian amplitude:

$$g(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_1\sigma_2}\, e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)}.$$

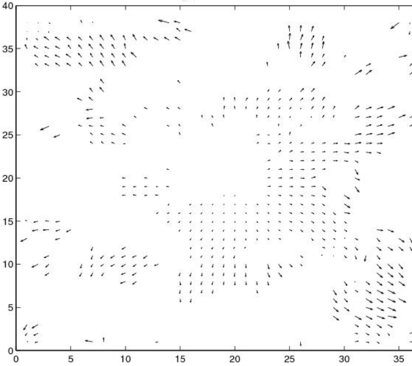
Application of QFT to texture segmentation (T. Bülow, thesis [4])



Application of QFT to disparity estimation (T. Bülow, thesis [4])

QFT method overcomes directional limitations of complex methods.

The first two frames of the tree-sequence.



The integrated displacement field

Uncertainty Principles for QFT (componentwise and directional)

Right sided quaternion Fourier transform (QFT) in $\mathbb{H}$

$$\mathcal{F}_r\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\, e^{-\boldsymbol{i}x_1\omega_1}\, e^{-\boldsymbol{j}x_2\omega_2} d^2\boldsymbol{x} \tag{64}$$

Componentwise ($k = 1, 2$) uncertainty [Mawardi&Hitzer, 2008]

$$\frac{1}{\|f\|^2}\int_{\mathbb{R}^2} x_k^2 |f(\boldsymbol{x})|^2 d^2\boldsymbol{x}\, \frac{1}{(2\pi)^2\|f\|^2}\int_{\mathbb{R}^2} \omega_k^2 |f(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega} \geq \frac{1}{4}. \tag{65}$$

Equality holds if and only if f is a Gaussian quaternion function.

Directional QFT Uncertainty Principle [Hitzer, 2008]

For two arbitrary constant vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$ (selecting two directions), and $f \in L^2(\mathbb{R}^2, \mathbb{H})$, $|\boldsymbol{x}|^{1/2} f \in L^2(\mathbb{R}^2, \mathbb{H})$, $\boldsymbol{b}' = -b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2$ we obtain

$$\int_{\mathbb{R}^2} (\boldsymbol{a}\cdot\boldsymbol{x})^2 |f(\boldsymbol{x})|^2 d^2\boldsymbol{x} \int_{\mathbb{R}^2} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega} \geq \frac{(2\pi)^2}{4}\left[(\boldsymbol{a}\cdot\boldsymbol{b})^2\|f_-\|^4 + (\boldsymbol{a}\cdot\boldsymbol{b}')^2\|f_+\|^4\right], \tag{66}$$

Motion in time: Video sequences, flow fields, ...

GA of spacetime (STA)

$$\{\boldsymbol{e}_t, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}, \quad \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_3^2 = 1. \tag{67}$$

$$\boldsymbol{e}_t^2 = -1, \quad i_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3, \quad i_3^2 = -1, \quad i_{st} = \boldsymbol{e}_t\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3, \quad i_{st}^2 = -1, \tag{68}$$

with time vector $\boldsymbol{e}_t$, 3D space volume i_3, hyper volume of spacetime i_{st}.

Isomorphism: Volume-time subalgebra to Quaternions

$$\{1, \boldsymbol{e}_t, i_3, i_{st}\} \longleftrightarrow \{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \tag{69}$$

NB: Does not work for $\mathbb{G}_{2,1}$ or $\mathbb{G}_{4,1}$, but works for conformal model of spacetime $\mathbb{G}_{5,2}$.

Split → Spacetime Split ($\boldsymbol{e}_t^* = \boldsymbol{e}_t i_{st}^{-1} = -\boldsymbol{e}_t i_{st} = -\boldsymbol{e}_t\boldsymbol{e}_t i_3 = i_3$)

$$f_\pm = \frac{1}{2}(f \pm \boldsymbol{e}_t f \boldsymbol{e}_t^*) \tag{70}$$

Time direction $\boldsymbol{e}_t$ determines complimentary 3D space i_3 as well!

QFT → Spacetime Fourier transform (SFT): $f \to \mathcal{F}_{SFT}\{f\}$

$$\mathcal{F}_{SFT}\{f\}(\boldsymbol{u}) = \overset{\diamond}{f}(\boldsymbol{u}) = \int_{\mathbb{R}^{3,1}} e^{-\boldsymbol{e}_t\, ts} f(\boldsymbol{x})\, e^{-i_3 \vec{x}\cdot\vec{u}} d^4\boldsymbol{x}\,. \tag{71}$$

with

- spacetime vectors $\boldsymbol{x} = t\boldsymbol{e}_t + \vec{x} \in \mathbb{R}^{3,1},\ \vec{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3 \in \mathbb{R}^3$
- spacetime volume $d^4\boldsymbol{x} = dtdxdydz$
- spacetime frequency vectors $\boldsymbol{u} = s\boldsymbol{e}_t + \vec{u} \in \mathbb{R}^{3,1},\ \vec{u} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + w\boldsymbol{e}_3 \in \mathbb{R}^3$
- maps 16D spacetime algebra functions $f : \mathbb{R}^{3,1} \to \mathbb{G}_{3,1} = Cl_{3,1}$
- to 16D spacetime spectrum functions $\overset{\diamond}{f} : \mathbb{R}^{3,1} \to Cl_{3,1}$
- Part $\int f(\boldsymbol{x})\, e^{-i_3\vec{x}\cdot\vec{u}} d^3\vec{x}$ ... GA FT of $\mathbb{G}_3$.
- Reversion replaced by principal involution (reversion and $\boldsymbol{e}_t \to -\boldsymbol{e}_t$).

Application: Multivector Wave Packets in Space and Time

SFT = Sum of Right/Left Propagating Multivector Wave Packets

$$\overset{\diamond}{f} = \overset{\diamond}{f}_+ + \overset{\diamond}{f}_- = \int_{\mathbb{R}^{3,1}} f_+ \, e^{-i_3(\vec{x}\cdot\vec{u} - ts)} d^4x + \int_{\mathbb{R}^{3,1}} f_- \, e^{-i_3(\vec{x}\cdot\vec{u} + ts)} d^4x. \qquad (72)$$

Directional 4D Spacetime Uncertainty for Multivector Wave Packets

For two arbitrary constant spacetime vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{3,1}$ (selecting two directions), and $f \in L^2(\mathbb{R}^{3,1}, \mathbb{G}_{3,1})$, $|\boldsymbol{x}|^{1/2} f \in L^2(\mathbb{R}^{3,1}, \mathbb{G}_{3,1})$ we obtain

$$\int_{\mathbb{R}^{3,1}} (a_t t - \vec{a}\cdot\vec{x})^2 |f(\boldsymbol{x})|^2 d^4\boldsymbol{x} \int_{\mathbb{R}^{3,1}} (b_t \omega_t - \vec{b}\cdot\vec{\omega})^2 |\mathcal{F}\{f\}(\boldsymbol{\omega})|^2 d^4\boldsymbol{\omega}$$
$$\geq \frac{(2\pi)^4}{4} \left[(a_t b_t - \vec{a}\cdot\vec{b})^2 \|f_-\|^4 + (a_t b_t + \vec{a}\cdot\vec{b})^2 \|f_+\|^4 \right],$$

Spacetime signal transformations

Application: Space & Time Signal Transformations

The SFT of a spacetime function $f \in L^2(\mathbb{R}^{3,1}; \mathbb{G}_{3,1})$ with a $GL(\mathbb{R}^{3,1})$ transformation $\mathcal{A}$ (stretches, reflections, rotations, acceleration, boost) of its spacetime vector argument $\boldsymbol{x}$ is

$$\mathcal{F}_{SFT}\{f(\mathcal{A}\boldsymbol{x})\}(\boldsymbol{u}) = |\det \mathcal{A}^{-1}|\,\{\,\overset{\diamond}{f}_{+}(\mathcal{U}_{\boldsymbol{e}_t}\overline{\mathcal{A}^{-1}}\,\mathcal{U}_{\boldsymbol{e}_t}\boldsymbol{u}) + \overset{\diamond}{f}_{-}(\overline{\mathcal{A}^{-1}}\boldsymbol{u})\,\}\;, \tag{73}$$

where $\overline{\mathcal{A}^{-1}}$ is the adjoint of the inverse of $\mathcal{A}$.

GA Wavelet Basics [Mawardi&Hitzer, 2007]

Transformation group $SIM(3)$

We apply 3D (elements of $\mathcal{G} = SIM(3)$) translations, scaling and rotations to a mother wavelet $\psi : \mathbb{R}^3 \to \mathbb{G}_3 = Cl_{3,0}$

$$\psi(\boldsymbol{x}) \longmapsto \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{a^{3/2}}\psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-\boldsymbol{b}}{a})) \tag{74}$$

Wavelet admissibility (condition on ψ)

$$C_\psi = \int_{\mathbb{R}^3} \frac{\widetilde{\widehat{\psi}}(\boldsymbol{\omega})\widehat{\psi}(\boldsymbol{\omega})}{|\boldsymbol{\omega}|^3}\, d^3\boldsymbol{\omega}. \tag{75}$$

is an invertible multivector constant and finite at a.e. $\boldsymbol{\omega} \in \mathbb{R}^3$, GA FT $\widehat{\psi} = \mathcal{F}\{\psi\}$.

GA Wavelet Transformation

Definition (GA wavelet transformation)

$$\begin{aligned} T_\psi : L^2(\mathbb{R}^3; Cl_{3,0}) &\rightarrow L^2(\mathcal{G}; Cl_{3,0}) \\ f &\rightarrow T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x}, \end{aligned} \tag{76}$$

$\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ ... GA mother wavelet, f ... image, multivector field, data, etc.

Properties: Locality (different from global FTs), left linearity; translation, dilation, rotation covariance.

Inversion of GA wavelet transform

Any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ can be decomposed as (C_ψ^{-1} ... admissibility)

$$f(\boldsymbol{x}) = \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}, \tag{77}$$

with left Haar measure on $\mathcal{G} = SIM(3)$: $\quad d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}) = d\mu(a, \boldsymbol{\theta}) d^3\boldsymbol{b}$

Applications (expected) of GA wavelets

- multi-dimensional image/signal processing
- JPEG
- local spherical harmonics (lighting)
- geological exploration
- seismology
- local multivector pattern matching
- and the like

Notation: Inner products and norms

We define the inner product of $f, g : \mathcal{G} \to Cl_{3,0}$ by

$$(f, g) = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b}) g\widetilde{(a, \boldsymbol{\theta}, \boldsymbol{b})}\, d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}), \qquad (78)$$

and the $L^2(\mathcal{G}; Cl_{3,0})$-norm

$$\|f\|^2 = \langle (f, f) \rangle = \int_{\mathcal{G}} |f(a, \boldsymbol{\theta}, \boldsymbol{b})|^2 d\lambda. \qquad (79)$$

GA Wavelet Uncertainty

Generalized GA wavelet uncertainty principle

Let ψ be a Clifford algebra wavelet that satisfies the admissibility condition. Then for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, the following inequality holds

$$\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \;\; C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \geq \frac{3(2\pi)^3}{4} \left[C_\psi * (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^2. \quad (80)$$

Uncertainty principle for scalar admissibility constant

For scalar admissibility constant C_ψ we get

Uncertainty principle for GA wavelet

Let ψ be a Clifford algebra wavelet with scalar admissibility constant. Then for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, the following inequality holds

$$\|\boldsymbol{b} T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \, \|\boldsymbol{\omega} \hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \geq 3C_\psi \frac{(2\pi)^3}{4} \|f\|^4_{L^2(\mathbb{R}^3;Cl_{3,0})}. \qquad (81)$$

NB1: This shows indeed, that the previous inequality (80) represents a multivector generalization of the uncertainty principle of inequality (81) for Clifford wavelets with scalar admissibility constant.
NB2: Compare with (direction independent) uncertainty principle for GA FT.

GA Gabor Wavelets

Example: GA Gabor Wavelets

$$\psi^c(\boldsymbol{x}) \quad = \quad g(\boldsymbol{x}) \left(e^{i_3 \boldsymbol{\omega}_0 \cdot \boldsymbol{x}} - \underbrace{e^{-\frac{1}{2}(\sigma_1^2 u_0^2 + \sigma_2^2 u_0^2 + \sigma_3^2 w_0^2)}}_{\text{constant}} \right)$$

3D Gaussian: $g(\boldsymbol{x}) = \dfrac{1}{(2\pi)^{\frac{3}{2}} \sigma_1 \sigma_2 \sigma_3} \, e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} + \frac{x_3^2}{\sigma_3^2}\right)}$.

Uncertainty principle for GA Gabor wavelet

Let ψ^c be an admissible GA Gabor wavelet. Assume $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, then the following inequality holds

$$\|\boldsymbol{b} T_{\psi^c} f(a, \boldsymbol{\theta}, \boldsymbol{b})\|^2_{L^2(\mathcal{G}; Cl_{3,0})} \, \|\boldsymbol{\omega} \hat{f}\|^2_{L^2(\mathbb{R}^3; Cl_{3,0})} \geq 3 C_{\psi^c} \frac{(2\pi)^3}{4} \|f\|^4. \qquad (82)$$

Conclusion

- Geometric calculus allows to construct a variety of GA Fourier transformations.
- Especially quaternion Fourier transformations are well studied.
- We studied the space time Fourier transformation giving rise to a multivector wave packet decomposition.
- Discrete and fast versions exist.
- Uncertainty limits established (principle bounds to accuracy).
- A diverse range of applications exists.
- For the future the application of local GA wavelets may overcome the limitations of global GA Fourier transformations.

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