Representation of Crystallographic Subperiodic Groups by Geometric Algebra

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Acknowledgements

You answer us with awesome deeds of righteousness, O God our Savior, the hope of all the ends of the earth and of the farthest seas ...

Psalm 65:5 [8]

- I thank my wife, my children, my parents.
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- IUCr
- Organizers of AGACSE 3, Leipzig 2008.

Contents

Introduction

- The 3D crystallographic space groups represented in GA (Hestenes 2002 [2], Hestenes & Holt 2007 [3]).
- Interactive 3D Space Group Visualizer created [5].
- For crystallographers the subperiodic space groups in 2D and 3D with only 1 or 2 degrees of freedom for translations are also of great interest [7].

- Clifford's associative geometric product

$$ab = a \cdot b + a \wedge b. \tag{1}$$

- Reflection of vector x at hyperplane (through origin) with normal a

$$x' = -a^{-1} x a, \qquad a^{-1} = \frac{a}{a^2}. \tag{2}$$

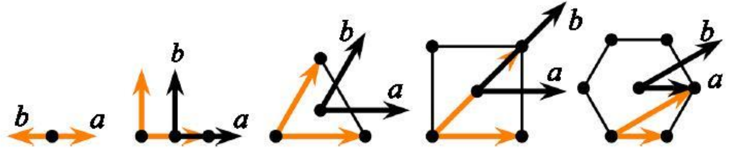
- The composition of two reflections yields a rotation

$$x' = (ab)^{-1} x ab, \qquad (ab)^{-1} = b^{-1} a^{-1}. \tag{3}$$

- Geometric product of k normal vectors (versor S) corresponds to all symmetry transformations of 2D, 3D crystal cell point groups

$$x' = (-1)^k S^{-1} x S. \tag{4}$$

- 2D point groups are generated (cf. Table 1) by vectors subtending angles π/p



- **Example:** Hexagonal point group

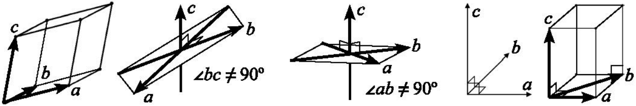
$$6 = \{a, b, R = ab, R^2, R^3, R^4, R^5, R^6 = -1, aR^2, bR^2, aR^4, bR^4\}. \quad (5)$$

- Rotation subgroup (denoted with bar): $\bar{6}$.

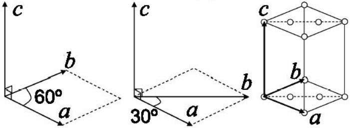
Table: Geometric and international notation for 2D point groups.

Crystal	Oblique		Rectangular		Trigonal		Square		Hexagonal	
geometric	$\bar{1}$	$\bar{2}$	1	2	3	$\bar{3}$	4	$\bar{4}$	6	$\bar{6}$
international	1	2	m	mm	3m	3	4m	4	6m	6

Selection of three vectors a, b, c from each crystal cell for generating 3D Point Groups



Triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, and tetragonal cell vectors a, b, c



trigonal (left), hexagonal (center) and hexagonally centered cell vectors a, b, c

Table: Geometric 3D point group symbols and generators with $\theta_{a,b} = \pi/p$, $\theta_{b,c} = \pi/q$, $\theta_{a,c} = \pi/2$, $p, q \in \{1, 2, 3, 4, 6\}$.

Symbol	1	$p \neq 1$	$\bar{p}$	pq	$\bar{p}q$	$p\bar{q}$	$\bar{p}\bar{q}$	$\overline{pq}$
Generators	a	a, b	ab	a, b, c	ab, c	a, bc	ab, bc	abc

Space Groups

- Composition with translations best in conformal model of Euclidean space (in the GA of $\mathbb{R}^{4,1}$).
- A plane can be described by the vector

$$m = p - d\, e_\infty, \tag{6}$$

  p is a unit normal to the plane, d its signed scalar distance from origin.
- Reflecting at 2 parallel planes m, m' with distance $t/2$ we get the *translation operator* (by $t \in \mathbb{R}^3$)

$$X' = m' m X m m' = T_{\vec{t}}^{-1} X T_t, \quad T_t = 1 + \frac{1}{2} t e_\infty. \tag{7}$$

- Reflection at 2 non-parallel planes m, m': rotation around m, m'-intersection by twice the angle subtended by m, m'.

Subperiodic groups – subgroups of space groups

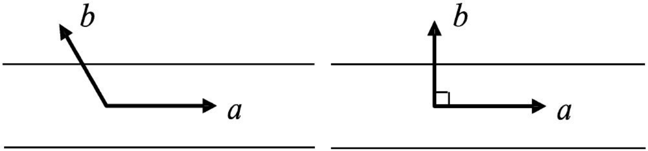
- Subperiodic groups are subgroups of full 2D and 3D space groups.
- Frieze groups (7) are all subgroups of 2D space groups with only one DOF for translation.
- Rod groups (75) are all subgroups of 3D space groups with only one DOF for translation.
- Layer groups (80) are all subgroups of 3D space groups with only two DOF for translation.

Notation

- One dot between the Bravais symbol ($\not{p}$, p, c) and index 1: b is generator.
- Two dots between the Bravais symbol ($\not{p}$, p, c) and index 1: c is generator .
- One dot between the Bravais symbol and index 2 (without or with bar): vectors b, c are generators .
- Two dots between the Bravais symbol and index 2 (without or with bar): vectors a, c are generators .
- Indexes a, b, c, n (and g for frieze groups) in 1st, 2nd, 3rd position after the Bravais symbol: reflections a, b, c (order!) change to glide reflections.
- Index n for diagonal glide.
- Dots also position indicators.
  Rod group 5: $\not{p}_c 1$ has glide reflection $aT_c^{1/2}$.
  Rod group 19: $\not{p}._c 2$ has $bT_c^{1/2}$.
  Layer group 39: $p_b 2_a 2_n$ has $aT_b{}^{\frac{1}{2}}$, $bT_a^{1/2}$, and $cT_{a+b}{}^{1/2}$.
- $\bar{n}_p$: right handed screw rotation of $2\pi/n$ around $\bar{n}$-axis, with pitch $T_t^{p/n}$. t is shortest lattice translation vector parallel to axis, in screw direction.
  Layer group 21: $p\bar{2}\bar{2}_1\bar{2}_1$ has screw generators $bcT_a^{1/2}$ and $acT_b^{1/2}$.

Frieze Groups

Generating vectors a, b of oblique and rectangular cells for 2D Frieze groups.



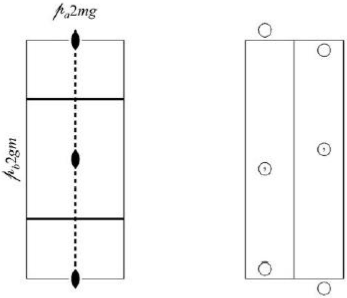
Single translation direction a.

Tables of Crystallography (IUCr)

Depiction of Frieze group No. 7 in Int. Tables of Cryst. Vol. E. [7]

International Tables for Crystallography (2006). Vol. E, Frieze group 7, p. 36.

$p2mg$	$2mm$	Rectangular
No. 7	$p2mg$	Patterson symmetry $p2mm$



Geometric symbols for Frieze groups

The seven Frieze groups with new geometric symbols and generators. SG = space group, SGN = space group numbers, T_a is omitted.

Frieze Group #	Intern. Notat.	3D SG#	Intern. 3D SGN	Geom. 3D SGN	2D SG#	Intern. 2D SGN	Geom. 2D SGN	Geom. Notat.	Frieze Group Generators
Oblique									
F_1	$\not{p}1$	1	$P1$	$P\bar{1}$	1	$p1$	$p\bar{1}$	$\not{p}\bar{1}$	
F_2	$\not{p}211$	3	$P2$	$P\bar{2}$	2	$p2$	$p\bar{2}$	$\not{p}\bar{2}$	$a \wedge b$
Rectangular									
F_3	$\not{p}1m1$	6	Pm	$P1$	3	$pm(p1m1)$	$p1$	$\not{p}1$	a
F_4	$\not{p}11m$	6	Pm	$P1$	3	$pm(p11m)$	$p1$	$\not{p}.1$	b
F_5	$\not{p}11g$	7	Pc	P_a1	4	$pg(p11g)$	p_g1	$\not{p}._g1$	$bT_a{}^{1/2}$
F_6	$\not{p}2mm$	25	$Pmm2$	$P2$	6	$p2mm$	$p2$	$\not{p}2$	a, b
F_7	$\not{p}2mg$	28	$Pma2$	$P2_a$	7	$p2mg$	$p2_g$	$\not{p}2_g$	$a, bT_a{}^{1/2}$

Rod Groups

Generating vectors a, b, c of triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, tetragonal, trigonal, hexagonal and hexagonally centered (Bravais symbol: H or h) cells for 3D rod (and layer) groups. Single translation direction c.

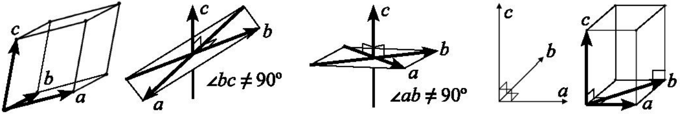


Fig. 3 From left to right: Triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, and tetragonal cell vectors a, b, c for rod and layer groups.

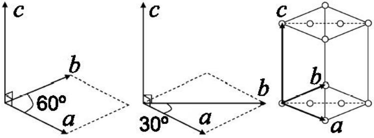


Fig. 4 Generating vectors a, b, c of trigonal (left), hexagonal (center) and hexagonally centered (right, Bravais symbol: H or h) cells for 3D rod and layer groups.

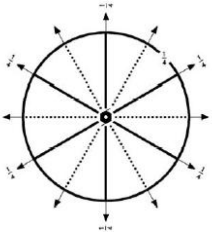
Tables of Crystallography (IUCr)

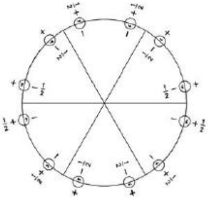
Depiction of rod group No. 75 in Int. Tables of Cryst. Vol. E. [7]

International Tables for Crystallography (2006). Vol. E, Rod group 75, pp. 214–217.

$p6_3/mmc$	$6/mmm$	Hexagonal
No. 75	$p6_3/m2/m2/c$	Patterson symmetry $p6/mmm$

FIRST SETTING





Triclinic and monoclinic rod groups. T_c is omitted.

Rod Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Rod Group Generators
Triclinic						
R_1	$\not{p}1$	1	$P1$	$P\bar{1}$	$\not{p}\bar{1}$	
R_2	$\not{p}2$	2	$P\bar{1}$	$P\overline{22}$	$\not{p}\overline{22}$	$a \wedge b \wedge c$
Monoclinic/inclined						
R_3	$\not{p}211$	3	$P112$	$P\bar{2}$	$\not{p}.\bar{2}$	$b \wedge c$
R_4	$\not{p}m11$	6	Pm	$P1$	$\not{p}1$	a
R_5	$\not{p}c11$	7	Pc	P_c1	$\not{p}_c1$	$aT_c^{1/2}$
R_6	$\not{p}2/m11$	10	$P2/m$	$P2\bar{2}$	$\not{p}2\bar{2}$	$a, b \wedge c$
R_7	$\not{p}2/c11$	13	$P2/c$	$P_a2\bar{2}$	$\not{p}_c2\bar{2}$	$aT_c^{1/2}, b \wedge c$
Monoclinic/orthogonal						
R_8	$\not{p}112$	3	$P112$	$P\bar{2}$	$\not{p}\bar{2}$	$a \wedge b$
R_9	$\not{p}112_1$	4	$P2_1$	$P\bar{2}_1$	$\not{p}\bar{2}_1$	$(a \wedge b)T_c^{1/2}$
R_{10}	$\not{p}11m$	6	Pm	$P1$	$\not{p}..1$	c
R_{11}	$\not{p}112/m$	10	$P2/m$	$P\bar{2}2$	$\not{p}\bar{2}2$	$a \wedge b, c$
R_{12}	$\not{p}112_1/m$	11	$P2_1/m$	$P\bar{2}_12$	$\not{p}\bar{2}_12$	$(a \wedge b)T_c^{1/2}, c$

Table of rod groups

Orthorhombic rod groups. T_c is omitted.

Rod Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Rod Group Generators
Orthorhombic						
R_{13}	$\not p222$	16	$P222$	$P\bar{2}\bar{2}\bar{2}$	$\not p\bar{2}\bar{2}\bar{2}$	ab, bc
R_{14}	$\not p222_1$	17	$P222_1$	$P\bar{2}_1\bar{2}\bar{2}$	$\not p\bar{2}_1\bar{2}\bar{2}$	$abT_c^{1/2}, bc$
R_{15}	$\not pmm2$	25	$Pmm2$	$P2$	$\not p2$	a, b
R_{16}	$\not pcc2$	27	$Pcc2$	P_c2_c	$\not p_c2_c$	$aT_c^{1/2}, bT_c^{1/2}$
R_{17}	$\not pmc2_1$	26	$Pmc2_1$	$P2_c$	$\not p2_c$	$a, bT_c^{1/2}$
R_{18}	$\not p2mm$	25	$Pmm2$	$P2$	$\not p.2$	b, c
R_{19}	$\not p2cm$	28	$Pma2$	$P2_a$	$\not p._c2$	$bT_c^{1/2}, c$
R_{20}	$\not pmmm$	47	$Pmmm$	$P22$	$\not p22$	a, b, c
R_{21}	$\not pccm$	49	$Pccm$	P_c2_c2	$\not p_c2_c2$	$aT_c^{1/2}, bT_c^{1/2}, c$
R_{22}	$\not pmcm$	51	$Pmma$	$P22_a$	$\not p2_c2$	$a, bT_c^{1/2}, c$

Table of rod groups

Tetragonal rod groups. T_c is omitted.

Rod Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Rod Group Generators
Tetragonal						
R_{23}	$\not{p}4$	75	$P4$	$P\bar{4}$	$\not{p}\bar{4}$	ab
R_{24}	$\not{p}4_1$	76	$P4_1$	$P\bar{4}_1$	$\not{p}\bar{4}_1$	$abT_c^{\frac{1}{4}}$
R_{25}	$\not{p}4_2$	77	$P4_2$	$P\bar{4}_2$	$\not{p}\bar{4}_2$	$abT_c^{\frac{1}{2}}$
R_{26}	$\not{p}4_3$	78	$P4_3$	$P\bar{4}_3$	$\not{p}\bar{4}_3$	$abT_c^{\frac{3}{4}}$
R_{27}	$\not{p}\bar{4}$	81	$P\bar{4}$	$P\overline{42}$	$\not{p}\overline{42}$	abc
R_{28}	$\not{p}4/m$	83	$P4/m$	$P\bar{4}2$	$\not{p}\bar{4}2$	ab, c
R_{29}	$\not{p}4_2/m$	84	$P4_2/m$	$P\bar{4}_22$	$\not{p}\bar{4}_22$	$abT_c^{\frac{1}{2}}, c$
R_{30}	$\not{p}422$	89	$P422$	$P\bar{4}\bar{2}\bar{2}$	$\not{p}\bar{4}\bar{2}\bar{2}$	ab, bc
R_{31}	$\not{p}4_122$	91	$P4_122$	$P\bar{4}_1\bar{2}\bar{2}$	$\not{p}\bar{4}_1\bar{2}\bar{2}$	$abT_c^{\frac{1}{4}}, bc$
R_{32}	$\not{p}4_222$	93	$P4_222$	$P\bar{4}_2\bar{2}\bar{2}$	$\not{p}\bar{4}_2\bar{2}\bar{2}$	$abT_c^{\frac{1}{2}}, bc$
R_{33}	$\not{p}4_322$	95	$P4_322$	$P\bar{4}_3\bar{2}\bar{2}$	$\not{p}\bar{4}_3\bar{2}\bar{2}$	$abT_c^{\frac{3}{4}}, bc$
R_{34}	$\not{p}4mm$	99	$P4mm$	$P4$	$\not{p}4$	a, b
R_{35}	$\not{p}4_2cm$	101	$P4_2cm$	P_c4	$\not{p}_c4$	$aT_c^{\frac{1}{2}}, b$
R_{36}	$\not{p}4cc$	103	$P4cc$	P_c4_c	$\not{p}_c4_c$	$aT_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}$
R_{37}	$\not{p}\bar{4}m2$	115	$P\bar{4}m2$	$P4\bar{2}$	$\not{p}4\bar{2}$	a, bc
R_{38}	$\not{p}\bar{4}c2$	116	$P\bar{4}c2$	$P_c4\bar{2}$	$\not{p}_c4\bar{2}$	$aT_c^{\frac{1}{2}}, bc$
R_{39}	$\not{p}4/mmm$	123	$P4/mmm$	$P42$	$\not{p}42$	a, b, c
R_{40}	$\not{p}4/mcc$	124	$P4/mcc$	P_c4_c2	$\not{p}_c4_c2$	$aT_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}, c$
R_{41}	$\not{p}4_2/mmc$	131	$P4_2/mmc$	$P4_c2$	$\not{p}4_c2$	$a, bT_c^{\frac{1}{2}}, c$

Table of rod groups

Trigonal rod groups. T_c is omitted.

Rod Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Rod Group Generators
Trigonal						
R_{42}	$\not{p}3$	143	$P3$	$P\bar{3}$	$\not{p}\bar{3}$	ab
R_{43}	$\not{p}3_1$	144	$P3_1$	$P\bar{3}_1$	$\not{p}\bar{3}_1$	$abT_c^{\frac{1}{3}}$
R_{44}	$\not{p}3_2$	145	$P3_2$	$P\bar{3}_2$	$\not{p}\bar{3}_2$	$abT_c^{\frac{2}{3}}$
R_{45}	$\not{p}\bar{3}$	147	$P\bar{3}$	$P\overline{62}$	$\not{p}\overline{62}$	abc
R_{46}	$\not{p}312$	149	$P312$	$P\bar{3}\bar{2}$	$\not{p}\bar{3}\bar{2}$	ab, bc
R_{47}	$\not{p}3_112$	151	$P3_112$	$P\bar{3}_1\bar{2}$	$\not{p}\bar{3}_1\bar{2}$	$abT_c^{\frac{1}{3}}, bc$
R_{48}	$\not{p}3_212$	153	$P3_212$	$P\bar{3}_2\bar{2}$	$\not{p}\bar{3}_2\bar{2}$	$abT_c^{\frac{2}{3}}, bc$
R_{49}	$\not{p}3m1$	156	$P3m1$	$P3$	$\not{p}3$	a, b
R_{50}	$\not{p}3c1$	158	$P3c1$	P_c3_c	$\not{p}_c3_c$	$aT_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}$
R_{51}	$\not{p}\bar{3}1m$	162	$P\bar{3}1m$	$P\bar{2}6$	$\not{p}6\bar{2}$	a, bc
R_{52}	$\not{p}\bar{3}1c$	163	$P\bar{3}1c$	$P\bar{2}_c6$	$\not{p}_c6\bar{2}$	$aT_c^{\frac{1}{2}}, bc$

Table of rod groups

Hexagonal rod groups. T_c is omitted.

Rod Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Rod Group Generators
R_{53}	$\not{p}6$	168	$P6$	$P\bar{6}$	$\not{p}\bar{6}$	ab
R_{54}	$\not{p}6_1$	169	$P6_1$	$P\bar{6}_1$	$\not{p}\bar{6}_1$	$abT_c^{\frac{1}{6}}$
R_{55}	$\not{p}6_2$	171	$P6_2$	$P\bar{6}_2$	$\not{p}\bar{6}_2$	$abT_c^{\frac{1}{3}}$
R_{56}	$\not{p}6_3$	173	$P6_3$	$P\bar{6}_3$	$\not{p}\bar{6}_3$	$abT_c^{\frac{1}{2}}$
R_{57}	$\not{p}6_4$	172	$P6_4$	$P\bar{6}_4$	$\not{p}\bar{6}_4$	$abT_c^{\frac{2}{3}}$
R_{58}	$\not{p}6_5$	170	$P6_5$	$P\bar{6}_5$	$\not{p}\bar{6}_5$	$abT_c^{\frac{5}{6}}$
R_{59}	$\not{p}\bar{6}$	174	$P\bar{6}$	$P\bar{3}2$	$\not{p}\bar{3}2$	ab, c
R_{60}	$\not{p}6/m$	175	$P6/m$	$P\bar{6}2$	$\not{p}\bar{6}2$	ab, c
R_{61}	$\not{p}6_3/m$	176	$P6_3/m$	$P\bar{6}_32$	$\not{p}\bar{6}_32$	$abT_c^{\frac{1}{2}}$
R_{62}	$\not{p}622$	177	$P622$	$P\bar{6}\bar{2}$	$\not{p}\bar{6}\bar{2}$	ab, bc
R_{63}	$\not{p}6_122$	178	$P6_122$	$P\bar{6}_1\bar{2}$	$\not{p}\bar{6}_1\bar{2}$	$abT_c^{\frac{1}{6}}, bc$
R_{64}	$\not{p}6_222$	180	$P6_222$	$P\bar{6}_2\bar{2}$	$\not{p}\bar{6}_2\bar{2}$	$abT_c^{\frac{1}{3}}, bc$
R_{65}	$\not{p}6_322$	182	$P6_322$	$P\bar{6}_3\bar{2}$	$\not{p}\bar{6}_3\bar{2}$	$abT_c^{\frac{1}{2}}, bc$
R_{66}	$\not{p}6_422$	181	$P6_422$	$P\bar{6}_4\bar{2}$	$\not{p}\bar{6}_4\bar{2}$	$abT_c^{\frac{2}{3}}, bc$
R_{67}	$\not{p}6_522$	179	$P6_522$	$P\bar{6}_5\bar{2}$	$\not{p}\bar{6}_5\bar{2}$	$abT_c^{\frac{5}{6}}, bc$
R_{68}	$\not{p}6mm$	183	$P6mm$	$P6$	$\not{p}6$	a, b
R_{69}	$\not{p}6cc$	184	$P6cc$	P_c6_c	$\not{p}_c6_c$	$aT_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}$
R_{70}	$\not{p}6_3cm$	185	$P6_3cm$	P_c6	$\not{p}_c6$	$aT_c^{\frac{1}{2}}, b$
R_{71}	$\not{p}\bar{6}m2$	187	$P\bar{6}m2$	$P32$	$\not{p}32$	a, b, c
R_{72}	$\not{p}\bar{6}c2$	188	$P\bar{6}c2$	P_c3_c2	$\not{p}_c3_c2$	$aT_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}, c$
R_{73}	$\not{p}6/mmm$	191	$P6/mmm$	$P62$	$\not{p}62$	a, b, c
R_{74}	$\not{p}6/mcc$	192	$P6/mcc$	P_c6_c2	$\not{p}_c6_c2$	$aT_c^{\frac{1}{2}}, bT_c^{\frac{1}{2}}, c$
R_{75}	$\not{p}6_3/mcm$	193	$P6_3/mcm$	P_c62	$\not{p}_c62$	$aT_c^{\frac{1}{2}}, b, c$

Layer Groups

Generating vectors a, b, c of triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, tetragonal, trigonal , hexagonal and hexagonally centered (Bravais symbol: H or h) cells for 3D layer (and rod) groups. Two translation directions a, b.

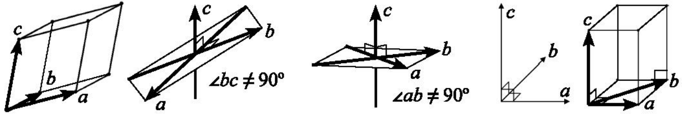


Fig. 3 From left to right: Triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, and tetragonal cell vectors a, b, c for rod and layer groups.

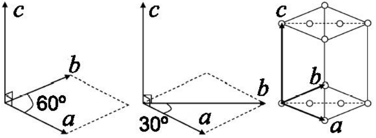


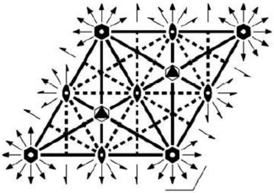
Fig. 4 Generating vectors a, b, c of trigonal (left), hexagonal (center) and hexagonally centered (right, Bravais symbol: H or h) cells for 3D rod and layer groups.

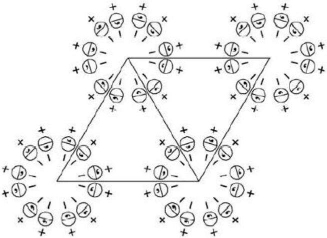
Tables of Crystallography (IUCr)

Depiction of layer group No. 80 in Int. Tables of Cryst. Vol. E. [7]

International Tables for Crystallography (2006). Vol. E, Layer group 80, pp. 388–389.

$p6/mmm$	$6/mmm$	Hexagonal/Hexagonal
No. 80	$p6/mmm$	Patterson symmetry $p6/mmm$





Triclinic/oblique and monoclinic/oblique and monoclinic/rectangular layer groups. T_a, T_b are omitted.

Layer Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Layer Group Generators
Triclinic/oblique						
L_1	$p1$	1	$P1$	$P\bar{1}$	$p\bar{1}$	
L_2	$p\bar{1}$	2	$P\bar{1}$	$P\overline{\bar{2}\bar{2}}$	$p\overline{\bar{2}\bar{2}}$	$a \wedge b \wedge c$
Monoclinic/oblique						
L_3	$p112$	3	$P2$	$P\bar{2}$	$p\bar{2}$	$a \wedge b$
L_4	$p11m$	6	Pm	$P1$	$p..1$	c
L_5	$p11a$	7	Pc	P_a1	$p_{..a}1$	$cT_a^{\frac{1}{2}}$
L_6	$p112/m$	10	$P2/m$	$P\bar{2}2$	$p\bar{2}2$	$a \wedge b, c$
L_7	$p112/a$	13	$P2/c$	$P_a2\bar{2}$	$p\bar{2}2_a$	$a \wedge b, cT_a^{\frac{1}{2}}$
Monoclinic/rectangular						
L_8	$p211$	3	$P2$	$P\bar{2}$	$p.\bar{2}$	$b \wedge c$
L_9	$p2_111$	4	$P2_1$	$P\bar{2}_1$	$p.\bar{2}_1$	$(b \wedge c)T_a^{\frac{1}{2}}$
L_{10}	$c211$	5	$C2$	$A\bar{2}$	$c.\bar{2}$	$b \wedge c, T_{a+b}^{1/2}$
L_{11}	$pm11$	6	Pm	$P1$	$p1$	a
L_{12}	$pb11$	7	Pc	P_a1	p_b1	$aT_b^{\frac{1}{2}}$
L_{13}	$cm11$	8	Cm	$A1$	$c1$	$a, T_{a+b}^{1/2}$
L_{14}	$p2/m11$	10	$P2/m$	$P2\bar{2}$	$p2\bar{2}$	$a, b \wedge c$
L_{15}	$p2_1/m11$	11	$P2_1/m$	$P2\bar{2}_1$	$p2\bar{2}_1$	$a, (b \wedge c)T_a^{\frac{1}{2}}$
L_{16}	$p2/b11$	13	$P2/c$	$P_a2\bar{2}$	$p_b2\bar{2}$	$aT_b^{\frac{1}{2}}, b \wedge c$
L_{17}	$p2_1/b11$	14	$P2_1/c$	$P_a2\bar{2}_2$	$p_b2\bar{2}_1$	$aT_b^{\frac{1}{2}}, (b \wedge c)T_a^{\frac{1}{2}}$
L_{18}	$c2/m11$	12	$C2/m$	$A2\bar{2}$	$c2\bar{2}$	$a, b \wedge c, T_{a+b}^{1/2}$

Table of layer groups

Orthorhombic/rectangular layer groups. T_a, T_b are omitted.

Layer Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Layer Group Generators
L_{19}	$p222$	16	$P222$	$P\bar{2}\bar{2}\bar{2}$	$p\bar{2}\bar{2}\bar{2}$	ab, bc
L_{20}	$p2_122$	17	$P222_1$	$P\bar{2}_1\bar{2}\bar{2}$	$p\bar{2}\bar{2}_1\bar{2}$	$ab, bcT_a{}^{\frac{1}{2}}$
L_{21}	$p2_12_12$	18	$P2_12_12$	$P\bar{2}_1\bar{2}_1\bar{2}$	$p\bar{2}\bar{2}_1\bar{2}_1$	$bcT_a{}^{\frac{1}{2}}, acT_b{}^{\frac{1}{2}}$
L_{22}	$c222$	21	$C222$	$C\bar{2}\bar{2}\bar{2}$	$c\bar{2}\bar{2}\bar{2}$	$ab, bc, T_{a+b}{}^{\frac{1}{2}}$
L_{23}	$pmm2$	25	$Pmm2$	$P2$	$p2$	a, b
L_{24}	$pma2$	28	$Pma2$	$P2_a$	$p2_a$	$a, bT_a{}^{\frac{1}{2}}$
L_{25}	$pba2$	32	$Pba2$	P_b2_a	p_b2_a	$aT_b{}^{\frac{1}{2}}, bT_a{}^{\frac{1}{2}}$
L_{26}	$cmm2$	35	$Cmm2$	$C2$	$c2$	$a, b, T_{a+b}^{1/2}$
L_{27}	$pm2m$	25	$Pmm2$	$P2$	$p..2$	a, c
L_{28}	$pm2_1b$	26	$Pmc2_1$	$P2_c$	$p.._b2$	$a, cT_b{}^{\frac{1}{2}}$
L_{29}	$pb2_1m$	26	$Pmc2_1$	$P2_c$	$p_b..2$	$aT_b{}^{\frac{1}{2}}, c$
L_{30}	$pb2b$	27	$Pcc2$	P_c2_c	$p_b.._b2$	$aT_b{}^{\frac{1}{2}}, cT_b{}^{\frac{1}{2}}$
L_{31}	$pm2a$	28	$Pma2$	$P2_a$	$p.._a2$	$a, cT_a{}^{\frac{1}{2}}$
L_{32}	$pm2_1n$	31	$Pmn2_1$	$P2_n$	$p.._n2$	$a, cT_{a+b}^{1/2}$
L_{33}	$pb2_1a$	29	$Pca2_1$	P_c2_a	$p_b.._a2$	$aT_b{}^{\frac{1}{2}}, cT_a{}^{\frac{1}{2}}$
L_{34}	$pb2n$	30	$Pnc2$	P_n2_c	$p_b.._n2$	$aT_b{}^{\frac{1}{2}}, cT_{a+b}^{1/2}$
L_{35}	$cm2m$	35	$Cmm2$	$C2$	$c..2$	$a, c, T_{a+b}^{1/2}$
L_{36}	$cm2e$	39	$Aem2$	A_b2	$c.._a2$	$a, cT_a{}^{\frac{1}{2}}, T_{a+b}^{1/2}$
L_{37}	$pmmm$	47	$Pmmm$	$P22$	$p22$	a, b, c
L_{38}	$pmaa$	49	$Pccm$	P_c2_c2	$p2_a2_a$	$a, bT_a{}^{\frac{1}{2}}, cT_a{}^{\frac{1}{2}}$
L_{39}	$pban$	50	$Pban$	$P_b2_a2_n$	$p_b2_a2_n$	$aT_b{}^{\frac{1}{2}}, bT_a^{1/2}, cT_{a+b}{}^{1/2}$
L_{40}	$pmam$	51	$Pmma$	$P22_a$	$p2_a2$	$a, bT_a{}^{\frac{1}{2}}, c$
L_{41}	$pmma$	51	$Pmma$	$P22_a$	$p22_a$	$a, b, cT_a{}^{\frac{1}{2}}$
L_{42}	$pman$	53	$Pmna$	$P2_n2_a$	$p2_a2_n$	$a, bT_a{}^{\frac{1}{2}}, cT_{a+b}^{1/2}$
L_{43}	$pbaa$	54	$Pcca$	$P_c2_c2_a$	$p_b2_a2_a$	$aT_b{}^{\frac{1}{2}}, bT_a{}^{\frac{1}{2}}, cT_a{}^{\frac{1}{2}}$
L_{44}	$pbam$	55	$Pbam$	P_b2_a2	p_b2_a2	$aT_b{}^{\frac{1}{2}}, bT_a{}^{\frac{1}{2}}, c$
L_{45}	$pbma$	57	$Pbcm$	P_b2_c2	p_b22_a	$aT_b{}^{\frac{1}{2}}, b, cT_a^{1/2}$
L_{46}	$pmmn$	59	$Pmmn$	$P22_n$	$p22_n$	$a, b, cT_{a+b}{}^{\frac{1}{2}}$
L_{47}	$cmmm$	65	$Cmmm$	$C22$	$c22$	$a, b, c, T_{a+b}^{1/2}$
L_{48}	$cmme$	67	$Cmme$	$C22_a$	$c22_a$	$a, b, cT_a{}^{\frac{1}{2}}, T_{a+b}^{1/2}$

Tetragonal/square layer groups. T_a, T_b are omitted.

Layer Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Layer Group Generators
Tetragonal/square						
L_{49}	$p4$	75	$P4$	$P\bar{4}$	$p\bar{4}$	ab
L_{50}	$p\bar{4}$	81	$P\bar{4}$	$P\overline{\bar{4}2}$	$p\overline{\bar{4}2}$	abc
L_{51}	$p4/m$	83	$P4/m$	$P\bar{4}2$	$p\bar{4}2$	ab, c
L_{52}	$p4/n$	85	$P4/n$	$P\bar{4}_n2$	$p\bar{4}_n2$	$ab, cT_b^{\frac{1}{2}}$
L_{53}	$p422$	89	$P422$	$P\bar{4}\bar{2}\bar{2}$	$p\bar{4}\bar{2}\bar{2}$	ab, bc
L_{54}	$p42_12$	90	$P42_12$	$P\bar{4}\bar{2}_1\bar{2}$	$p\bar{4}\bar{2}_1\bar{2}$	$ab, bcT_{2a-b}^{1/2}$
L_{55}	$p4mm$	99	$P4mm$	$P4$	$p4$	a, b
L_{56}	$p4bm$	100	$P4bm$	P_b4	p_b4	$aT_{a-b}^{1/2}, b$
L_{57}	$p\bar{4}2m$	111	$P\bar{4}2m$	$P\bar{2}4$	$p\bar{2}4$	ac, b
L_{58}	$p\bar{4}2_1m$	113	$P\bar{4}2_1m$	$P\bar{2}_14$	$p\bar{2}_14$	$acT_{a-b}^{1/2}, b$
L_{59}	$p\bar{4}m2$	115	$P\bar{4}m2$	$P4\bar{2}$	$p4\bar{2}$	a, bc
L_{60}	$p\bar{4}b2$	117	$P\bar{4}b2$	$P_b4\bar{2}$	$p_b4\bar{2}$	$aT_{a-b}^{1/2}, bc$
L_{61}	$p4/mmm$	123	$P4/mmm$	$P42$	$p42$	a, b, c
L_{62}	$p4/nbm$	125	$P4/nbm$	P_b42_n	p_b42_n	$aT_{a-b}^{1/2}, b, cT_b^{\frac{1}{2}}$
L_{63}	$p4/mbm$	127	$P4/mbm$	P_b42	p_b42	$aT_{a-b}^{1/2}, b, c$
L_{64}	$p4/nmm$	129	$P4/nmm$	$P42_n$	$p42_n$	$a, b, cT_b^{\frac{1}{2}}$

Table of layer groups

Trigonal/hexagonal and Hexagonal/hexagonal layer groups. T_a, T_b are omitted.

Layer Group #	Intern. Notat.	3D Space Group #	Intern. 3D SGN	Geom. 3D SGN	Geom. Notat.	Layer Group Generators
Trigonal/hexagonal						
L_{65}	$p3$	143	$P3$	$P\bar{3}$	$p\bar{3}$	ab
L_{66}	$p\bar{3}$	147	$P\bar{3}$	$P\overline{6\bar{2}}$	$p\overline{6\bar{2}}$	abc
L_{67}	$p312$	149	$P312$	$P\bar{3}\bar{2}$	$p\bar{3}\bar{2}$	ab, bc
L_{68}	$p321$	150	$P321$	$H\bar{3}\bar{2}$	$h\bar{3}\bar{2}$	ab, bc
L_{69}	$p3m1$	156	$P3m1$	$P3$	$p3$	a, b
L_{70}	$p31m$	157	$P31m$	$H3$	$h3$	a, b
L_{71}	$p\bar{3}1m$	162	$P\bar{3}1m$	$P\bar{2}6$	$p\bar{2}6$	ac, b
L_{72}	$p\bar{3}m1$	164	$P\bar{3}m1$	$P6\bar{2}$	$p6\bar{2}$	a, bc
Hexagonal/hexagonal						
L_{73}	$p6$	168	$P6$	$P\bar{6}$	$p\bar{6}$	ab
L_{74}	$p\bar{6}$	174	$P\bar{6}$	$P\bar{3}2$	$p\bar{3}2$	ab, c
L_{75}	$p6/m$	175	$P6/m$	$P\bar{6}2$	$p\bar{6}2$	ab, c
L_{76}	$p622$	177	$P622$	$P\bar{6}\bar{2}$	$p\bar{6}\bar{2}$	ab, bc
L_{77}	$p6mm$	183	$P6mm$	$P6$	$p6$	a, b
L_{78}	$p\bar{6}m2$	187	$P\bar{6}m2$	$P32$	$p32$	a, b, c
L_{79}	$p\bar{6}2m$	189	$P\bar{6}2m$	$H32$	$h32$	a, b, c
L_{80}	$p6/mmm$	191	$P6/mmm$	$P62$	$p62$	a, b, c

Conclusions

- We have devised a new Clifford geometric algebra representation for the 162 subperiodic space groups using versors (Clifford group, Lipschitz elements).
- In the future this may be extended to magnetic subperiodic space groups, the sign of the generators may achieve that.
- We expect that the present work forms a suitable foundation for interactive visualization software of subperiodic space groups [5].

Future work

Preview of how the rod groups 13: $\not{p}\bar{2}\bar{2}\bar{2}$ and 14: $\not{p}\bar{2}_1\bar{2}\bar{2}$, and the layer group 11: $p1$ and might be visualized in the future, based on the Space Group Visualizer.

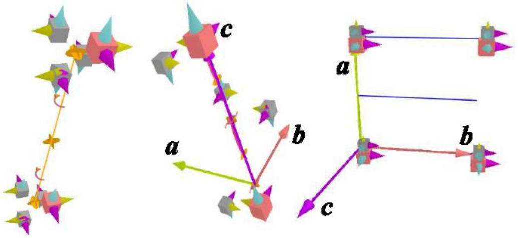


Fig. 5 How a future subperiodic space group viewer software might depict rod groups 13: $\not{p}\bar{2}\bar{2}\bar{2}$ and 14: $\not{p}\bar{2}_1\bar{2}\bar{2}$, and the layer group 11: $p1$, based on [5].

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News about Clifford Geometric Algebra

GA-Net

Electronic newsletter:

http://sinai.mech.fukui-u.ac.jp/GA-Net/index.html

GA-Net Updates (blog)

Immediate access to latest GA news:

http://gaupdate.wordpress.com/

Soli Deo Gloria

J. S. Bach (Leipzig)