New Tools for Computational Geometry

• the Rejuvenation of Screw Theory •

David Hestenes
Arizona State University

Motivation: Euclidean geometry supplies essential conceptual underpinnings for physics, engineering and design

Objective: Formulate Euclidean geometry to

- *facilitate* geometric modeling and analysis
- *optimize* computational efficiency
- specifically for *rigid body mechanics*

A fundamental problem in the Design of Mathematics

"The whole is simpler than its parts!" — J. Willard Gibbs

AGACSE. *Leipzig, August 2008*

Universal Geometric Algebra

Real Vector Space: $\mathbb{V}^{r,s} = \{a, b, c, \ldots\}$ dimension $r+s = n$

Geometric product: $a^2 = \pm |a|^2$ nondegenerate signature $\{r, s\}$

generates Real GA: $\mathbb{G}^{r,s} = \mathbb{G}(\mathbb{V}^{r,s}) = \{A, G, M \ldots\} = \{\text{Multivectors}\}$

Inner product: $a \cdot b \equiv \frac{1}{2}(ab + ba)$ Outer product: $a \wedge b \equiv \frac{1}{2}(ab - ba)$

$\Rightarrow$ $ab = a \cdot b + a \wedge b$ $a \wedge A_k \equiv \frac{1}{2}\left(aA_k + (-1)^k A_k a\right)$

k-blade: $a_1 \wedge a_2 \wedge \ldots \wedge a_k = \langle a_1 a_2 \ldots a_k \rangle_k \equiv A_k$ $\Rightarrow$ k-vector

$$a \cdot (a_1 \wedge a_2 \wedge \ldots \wedge a_k) = \sum_{j=1}^{k} (-1)^{j+1} a \cdot a_j (a_1 \wedge \ldots \wedge \breve{a}_j \wedge \ldots \wedge a_k)$$

Graded algebra: $\mathbb{G}^{r,s} = \sum_{k=0}^{n} \mathbb{G}_k^{r,s} = \left\{ A = \sum_{k=0}^{n} \langle A \rangle_k \right\}$

Reverse: $(a_1 \wedge a_2 \wedge \ldots \wedge a_k)^{\sim} = a_k \wedge \ldots \wedge a_2 \wedge a_1$ $\tilde{A} = \sum_{k=0}^{n} \langle \tilde{A} \rangle_k = \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle A \rangle_k$

Unit pseudoscalar: $\mathrm{I} = \langle \mathrm{I} \rangle_n$ $\mathrm{I}\tilde{\mathrm{I}} = (-1)^s$ $a \wedge \mathrm{I} = 0$

Dual: $A^* \equiv A\mathrm{I}$ Thm: $a \cdot A^* = a \cdot (A\mathrm{I}) = (a \wedge A)\mathrm{I}$

Group Theory with Geometric Algebra

versor (of order k): $\boxed{G = n_k \ldots n_2 n_1}$ $\quad G^{-1} = n_1^{-1} n_2^{-1} \ldots n_k^{-1}$ $\quad n_i^2 \neq 0$

Pin and Spin groups:

$$\mathrm{Pin}(r,\, s) = \left\{ G : GG^{-1} = 1 \right\} \supset \mathrm{Spin}(r,\, s) = \left\{ \text{even } G \right\}$$

Orthogonal group: $\mathrm{O}(r,\, s) = \left\{ \underline{G}(a) = \varepsilon G a G^{-1} = \sigma a' \right\}$

parity: $\varepsilon = \pm 1$

σ = scale factor

Advantage over matrix representations:

- Coordinate-free
- Simple composition laws: $G_2 G_1 = G_3$ $\quad \underline{G}_2 \underline{G}_1 = \underline{G}_3$
- Reducible to multiplication and reflection by vectors:
- Reflection in a hyperplane in $\mathbb{V}_{\mathrm{r,s}}$ with normal n_i : $\underline{G}_i(a) = -n_i a n_i^{-1}$

  $\Rightarrow$ *Cartan-Dieudonné Thm* (Lipschitz, 1880): $\underline{G} = \underline{G}_k \cdots \underline{G}_2 \underline{G}_1$

$\Rightarrow$ *Nearly all groups* [Doran et. al. (1993)"Lie Groups as Spin Groups"]

For example: All the classical groups!

In particular: Conformal group: $\mathrm{C}(r,\, s) \cong \mathrm{O}(r+1,\, s+1)$

Hence define: Conformal GA: $\mathbb{G}^{r+1,s+1}$

Euclidean GA

Homogeneous (conformal) model of Euclidean 3-space in $\mathbb{G}^{4,1} = \mathbb{G}(\mathbb{V}^{4,1})$

- Identify Euclidean points with vectors in the null cone:

  $\mathbb{N}^{4,1} \equiv \{x \in \mathbb{V}^{4,1}: \quad x^2 = 0\}$ (4 degrees of freedom)

- Choose a point at infinity $e = x_\infty$, and normalize all points to the hyperplane $\{x: \ e \cdot x = -1, \ e^2 = 0\}$, so

  $\mathbb{E}^3 \cong \mathbb{N}_e^{4,1} \equiv \{x \in \mathbb{V}^{4,1}: \quad x^2 = 0, \quad x \cdot e = -1\}$ (3 degrees of freedom)

Euclidean metric defined as follows:

Chord (displacement vector) between two points: $d_{21} = x_2 - x_1$ ($\neq$ a point)

Euclidean distance: $d_{21}{}^2 = (x_2 - x_1)^2 = -2x_2 \cdot x_1$

$$(x_2 - x_1)^2 = \cancel{x_2}{}^2 - 2x_2 \cdot x_1 + \cancel{x_1}{}^2$$

The invariance group of this metric is the **Euclidean group** $E(3) = \{\underline{G}\}$, a subgroup of the conformal group $C(3, 0) \cong O(4, 1)$ defined by the constraint: $\boxed{\underline{G}(e) = GeG^{-1} = e}$

Geometric Objects in 3D Euclidean Geometry:

Circle C determined by three points: $C = x_1 \wedge x_2 \wedge x_3$

Line L is a circle through infinity: $L = x_1 \wedge x_2 \wedge e$

Sphere S determined by four points: $S = x_1 \wedge x_2 \wedge x_3 \wedge x_4 = C \wedge x_4$

Plane P determined by three points: $P = x_1 \wedge x_2 \wedge x_3 \wedge e = x_4 \wedge L$

- Note the distinction between a geometric object (defined algebraically) and the set of points it determines (as in Euclid):

$$\mathbb{L}\text{ine} \equiv \{x \mid x \wedge L = 0\} \qquad \mathbb{P}\text{lane} \equiv \{x \mid x \wedge P = 0\}$$

- Intersection: Point x *lies on* object O if and only if $x \wedge O = 0$

or: $x \vee O \equiv x \cdot (\mathrm{I}^{-1} O) = \mathrm{I}^{-1}(\mathrm{x} \wedge O) = 0$

Dual forms for geometric objects and intersection with points:

$P = \mathrm{I}n$:	$x \wedge P = x \wedge (\mathrm{I}n) = (x \cdot n)\mathrm{I} = 0$	$\Rightarrow$	$x \cdot n = 0$
$L = \mathrm{I}B$:	$x \wedge L = x \wedge (\mathrm{I}B) = (x \cdot B)\mathrm{I} = 0$	$\Rightarrow$	$x \cdot B = 0$

Every line is the intersection of two planes:

$$P_1 \vee P_2 \equiv P_1 \cdot (\mathrm{I}^{-1} P_2) = P_1 \cdot n_2 = (\mathrm{I}n_1) \cdot n_2 = \mathrm{I}(n_1 \wedge n_2) = \mathrm{I}B$$

(Similar expressions for intersections of lines, planes, circles & spheres)

Invariant Euclidean Geometry

Algebraic axioms $\Leftrightarrow$ Synthetic descriptions $\Leftrightarrow$ Geometric figures

Basic geometric objects (**vectors**):

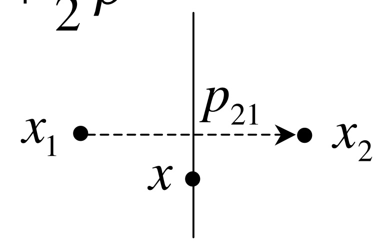
Points: $\{x \mid x^2 = 0,\ x\cdot e = -1\}$ $\quad e^2 = 0, \quad e = x_\infty$ $\qquad$ *nD / 3D / 2D*

Planes: $\{p \mid p^2 > 0,\ p\cdot e = 0\}$ $\qquad$ hyperplanes / planes /lines

Spheres: $\{s \mid s^2 = \rho^2 > 0,\ s\cdot e = -1\}$ $\qquad$ hyperspheres/spheres/circles

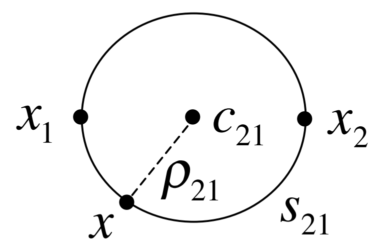
$$\textit{radius } \rho, \ \textit{center} \quad c = -\frac{1}{2}ses = -\frac{1}{2}(2e\cdot s - es)s = s + \frac{1}{2}\rho^2$$

Two points determine a plane: $\boxed{p_{21} = x_2 - x_1}$ ($\perp$ bisector)



Point *x on plane*: $\boxed{x\cdot p_{21} = 0} \Rightarrow x\cdot x_2 = x\cdot x_1 = \frac{1}{2}|x_2 - x_1|$

Two points determine a sphere: $\boxed{s_{21} = x_2 + x_1 = c_{21} + \frac{1}{2}\rho_{21}{}^2 e}$



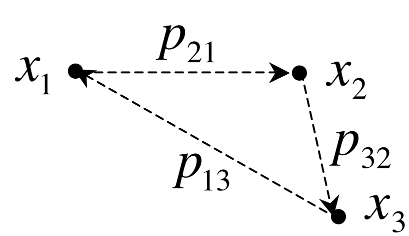
Point *x on sphere*: $x\cdot s_{21} = 0 \quad \Rightarrow \quad \boxed{p_{21}\cdot c_{21} = 0}$

$$\Rightarrow \quad |x - c_{21}|^2 = -2x\cdot c_{21} = \rho_{21}{}^2 \qquad |x - c_{21}| = \rho_{21}$$

Euclidean metric: $|x_i - x_j|^2 = p_{ij}{}^2 \geq 0$

Triangle: $p_{21} + p_{32} + p_{13} = 0$

Cosine law: $p_{21}{}^2 + p_{32}{}^2 + 2p_{21}\cdot p_{32} = p_{13}{}^2$



Projective Geometry

Projective transformations = nonsingular linear transformations:

$$\underline{\mathrm{f}}:\ x \mapsto x' = \underline{\mathrm{f}}(x) \qquad \text{with} \qquad \underline{\mathrm{f}}(e) = \sigma e$$

Problem: In general, this does not preserve the null property of points:

$$\underline{\mathrm{f}}:\ x^2 = 0 \mapsto \left[\underline{\mathrm{f}}(x)\right]^2 = \underline{\mathrm{f}}(x)\cdot\underline{\mathrm{f}}(x) = x\cdot\overline{\mathrm{f}}\underline{\mathrm{f}}\,(x) \overset{?}{\neq} 0$$

Solution (A. Lasenby): Extend the notion of points to include planes as points at ∞, thus composing a plane (of directions) at ∞:

Interior points: $\left\{x \mid x^2 = 0,\quad x\cdot e = 0\right\}$

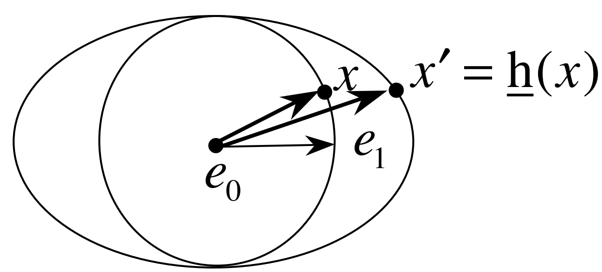
Boundary points: $\left\{n \mid n^2 = 1,\quad n\cdot e = 0\right\}$ = {unit chords}

Symmetric transformations: $\underline{\mathrm{h}} = \underline{\mathrm{G}}^{-1}\underline{\mathrm{h}}\,\underline{\mathrm{G}} \qquad \underline{\mathrm{G}} = \underline{\mathrm{R}}\,\underline{\mathrm{T}}_a \qquad \underline{\mathrm{f}} = \underline{\mathrm{G}}\underline{\mathrm{h}}$

Fixed point: $\underline{\mathrm{h}}(e_0) = e_0$

$$\underline{\mathrm{h}}(x) = x + \lambda(e_1\cdot x)e_1$$

$$\underline{\mathrm{h}}(e_1) = (1+\lambda)e_1$$



Affine transformations: $\underline{\mathrm{f}} = \underline{\mathrm{T}}_a\underline{\mathrm{h}}$

Inversive Geometry

- **Relation of point to sphere:** $s = e_0 - \frac{1}{2}\rho^2 e$

$$s^2 = \rho^2 \qquad s\cdot e = s\cdot e_0 = -1 \qquad -2x\cdot e_0 = (x-e_0)^2$$

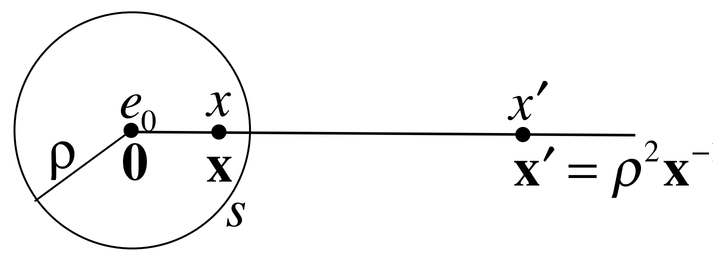
$$-2s\cdot x = -2(e_0 - \frac{1}{2}\rho^2 e)\cdot x = -2e_0\cdot x - \rho^2 = |x-e_0|^2 - \rho^2$$

$$\Rightarrow \quad \begin{array}{ll} s\cdot x > 0 & \text{iff } x \textit{ inside sphere} \\ s\cdot x = 0 & \text{iff } x \textit{ on sphere} \\ s\cdot x < 0 & \text{iff } x \textit{ outside sphere} \end{array}$$

- **Inversion** in a sphere: $\underline{s}(x) = -sxs^{-1} = \sigma x'$ $\quad\Rightarrow\quad$ $\sigma^2 x'^2 = x^2 = 0$

$$\sigma x' = -(-xs + 2s\cdot x)s^{-1} = x - \frac{2s\cdot x}{\rho^2}s$$

$$\sigma = 1 - \frac{2s\cdot x}{\rho^2} = -\frac{2e_0\cdot x}{\rho^2} = \frac{(x-e_0)^2}{\rho^2}$$



$$x' - e_0 = \frac{x-e_0}{\sigma} - \frac{(x-e_0)^2}{2}e \quad\Rightarrow\quad (x'-e_0)^2 = \frac{(x-e_0)^2}{\sigma^2} = \frac{\rho^4}{(x-e_0)^2}$$

- Line through sphere center: $\mathbf{x} \equiv x\wedge e_0 \wedge e = x\wedge s\wedge e$

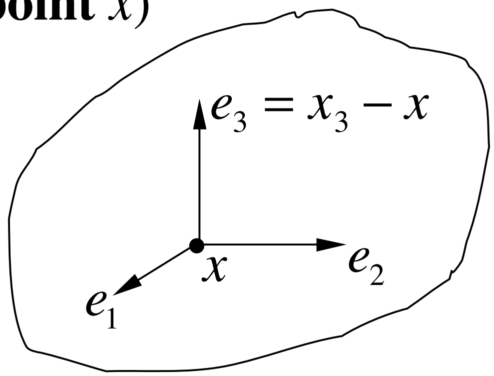
$$\Rightarrow \quad \mathbf{x}^2 = -2x\cdot e_0 = (x-e_0)^2 \quad\Rightarrow\quad \mathbf{x}' = \rho^2\mathbf{x}^{-1} = \frac{\rho^2}{\mathbf{x}^2}\mathbf{x}$$

- Inversion of infinity: $\rho^2\underline{s}(e) = -ses = -e_0 e e_0 = 2e_0$

Rigid Body Representation

Body frame: $e_k = x_k - x$ = (**chords** from a fixed **body point** x)

Orthonormalize: $e_j \cdot e_k = \delta_{jk}$



Embed the frame in a single algebraic object:

Flag (Selig) or Soma (Engels)

= point + line + plane (with common point)

$$F = x + L + P = x + \mathrm{I}Q$$

$$L = x \wedge x_1 \wedge e = x \wedge (x_1 - x) \wedge e = x \wedge e_1 \wedge e = \mathrm{I}n_2 n_3$$

$$P = x \wedge x_1 \wedge x_2 \wedge e = x \wedge e_1 \wedge e_2 \wedge e = e_2 \wedge L = \mathrm{I}n_3$$

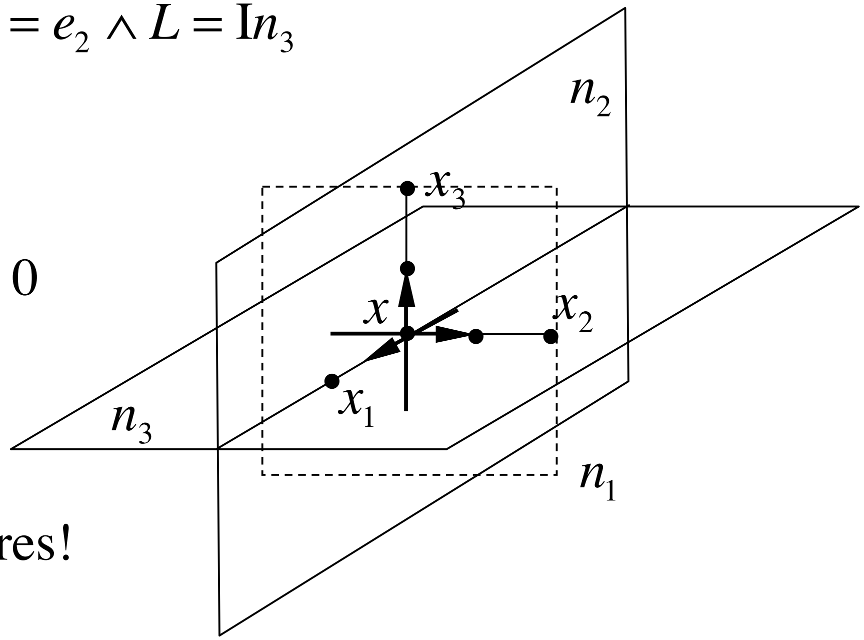
$$Q = n_3 + n_2 n_3 = (1 + n_2)n_3$$

$$n_j \cdot n_k = \delta_{jk}$$

$$x \cdot Q = 0 \qquad Q^2 = 0$$

$$\Leftrightarrow \quad x \wedge (L + P) = x \wedge (\mathrm{I}Q) = 0$$

Absolute conic!
(A. Lasenby)



Generalize by replacing planes with spheres!

Rigid displacement of a Body = Congruence

$$F = x + L + P \quad \to \quad F' = \underline{D}(F) = \underline{D}(x) + \underline{D}(L) + \underline{D}(P)$$

$$F = x + IQ \quad \to \quad F' = \underline{D}(F) = \underline{D}(x) + I\underline{D}(Q)$$

Decomposition into translation & rotation: $\underline{D} = \underline{T}_a \underline{R}_x$

Rotation defined by: $\underline{R}_x(x) = x$ Translation given by:

$$\underline{T}_a\colon\ x \ \mapsto\ x' = \underline{D}(x) = \underline{T}_a(x) = T_a x T_a^{-1}$$

$$T_a = e^{\frac{1}{2}ea} = 1 + \frac{1}{2}ea \quad \text{with} \quad 2a \cdot e = ae + ea = 0$$

$$T_a x T_a^{-1} = (1 + \frac{1}{2}ea)x(1 + \frac{1}{2}ae) = x + x \cdot (a \wedge e) - \frac{1}{2}aea$$

$$\boxed{x' - x = a + \frac{1}{2}(x+a)^2 e = n} = \text{bisecting plane!}$$

$$Q \ \to \ Q' = \underline{D}(Q) = \underline{R}_x(Q) = R_x Q R_x^{-1}$$ <u>One eqn. for body</u>

$$n_k' = \underline{R}_x(n_k) = R n_k R^{-1}$$ Three eqns.

$$e_k' = \underline{R}_x(e_k) = Re_k R^{-1}$$ Can find R from e_k' & e_k (NFCM)

$$F' = \underline{D}(F) = \underline{T}_a \underline{R}_x(F) = \underline{T}_a(x) + I\underline{R}_x(Q) = x' + IQ'$$

Relation of the conformal model to alternative models of Euclidean Geometry

Vector space model of Euclidean 3-space: $\mathbb{E}^3 \cong \mathbb{V}^3 = \{\mathbf{x}\}$

$\mathbb{G}^3 = \mathbb{G}(\mathbb{V}^3) = \{\alpha + \mathbf{a} + i\mathbf{b} + i\beta\}$ $\quad$ i = I = pseudoscalar

Advantages:

- Smoothly integrated with vector algebra:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + i\mathbf{a}\times\mathbf{b}$$

- Quaternion (spinor) forms for rotors and rotations:

Rotation: $\underline{R}(\mathbf{x}) = R\mathbf{x}R^{-1}$ (no matrices!)

Rotor: $R = \alpha + i\boldsymbol{\beta} = e^{-\frac{1}{2}i\mathbf{a}}$ (See NFCM)

Composition: $R_2 R_1 = R_3$

Matrix elements: $\alpha_{kj} = \mathbf{e}_k \cdot \boldsymbol{\sigma}_j = \langle R\boldsymbol{\sigma}_k R^{-1}\boldsymbol{\sigma}_j \rangle$

- $\cong$ {complex quaternions}

Drawback: The model is inhomogeneous! ($\mathbf{x} = \mathbf{0}$ distinguished)

$\Rightarrow$ Inhomogeneous rigid displacements: $\underline{G}(\mathbf{x}) = \underline{T}_{\mathbf{a}}\underline{R}(\mathbf{x}) = \underline{R}(\mathbf{x}) + \mathbf{a}$

Covariant Euclidean Geometry: relates conformal and vector space models

Conformal Split: $\mathbb{G}^{4,1} = \mathbb{G}^3 \otimes \mathbb{G}^{1,1}$

determined by choosing one point e_0 **as *origin*,**

so each point lies on the bundle of all lines through the origin

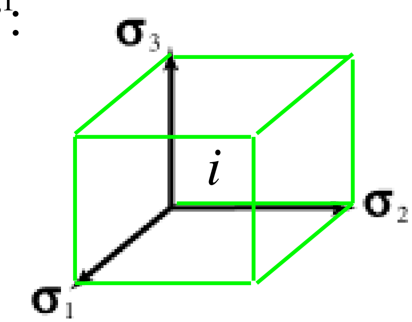
$\mathbb{G}^3 = \mathbb{G}_0{}^3 = \mathbb{G}(\mathbb{V}_0{}^3)$ is the geometric algebra of that bundle;

each point is then designated by a vector $\mathbf{x}$ in $\mathbb{V}_0{}^3 = \{\mathbf{x}\}$

The generating basis vectors for $\mathbb{G}^3$ are trivectors in $\mathbb{G}^{4,1}$:

$$\{\boldsymbol{\sigma}_k = e_k \wedge e \wedge e_0 = e_k(e \wedge e_0) = e_k E = E e_k\}$$

$$i = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (e_1E)(e_2E)(e_3E) = e_1e_2e_3E$$



Invariant pseudoscalar: $\mathrm{I} = i \qquad i^2 = -1$

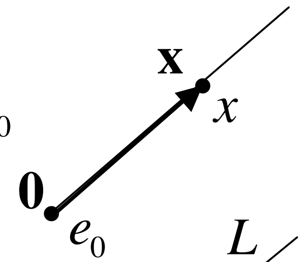
Warning: Outer products in $\mathbb{G}^3$ differ from outer products in $\mathbb{G}^{4,1}$!

Additive Split: $\mathbb{G}^{4,1} = \mathbb{G}(\mathbb{V}^3 \oplus \mathbb{V}^{1,1}) \equiv \mathbb{G}_+{}^3 \oplus \mathbb{G}^{1,1}$

Vector basis generating $\mathbb{G}_+{}^3$: $\{e_1, e_2, e_3\}$

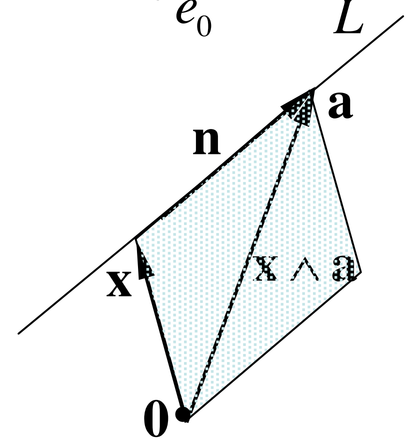
This is not algebraically associated with lines or a point origin,
and its pseudoscalar $\mathrm{I}_3 = e_1e_2e_3$ is not invariant.

Conformal splits for points and simplexes



- Point: $x = (\mathbf{x} + \frac{1}{2}\mathbf{x}^2 e + e_0)E = E(\mathbf{x} - \frac{1}{2}\mathbf{x}^2 e - e_0) = \mathbf{x}E + \frac{1}{2}\mathbf{x}^2 e - e_0$

  $\Leftrightarrow \quad \mathbf{x} = x \wedge e_0 \wedge e = x \wedge E \qquad E = e_0 \wedge e$



- Line: $L = x \wedge a \wedge e = \mathbf{x} \wedge \mathbf{a}e + (\mathbf{a} - \mathbf{x})E = (\mathbf{d}e + E)\mathbf{n}$

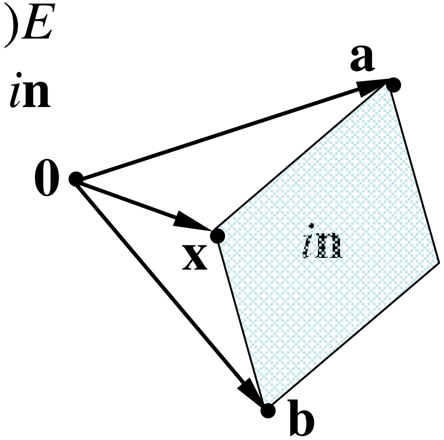
  tangent: $\mathbf{n} = \mathbf{a} - \mathbf{x}$

  moment: $\mathbf{x} \wedge \mathbf{a} = \mathbf{x} \wedge (\mathbf{a} - \mathbf{x}) = \mathbf{dn}$

  Plücker coordinates

  directance: $\mathbf{d} = (\mathbf{x} \wedge \mathbf{a})\mathbf{n}^{-1} = (\mathbf{x} \wedge \mathbf{n})\mathbf{n}^{-1} = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}^{-1})\mathbf{n}$

  (L = line vector = *spear*)



- Plane: $P = x \wedge a \wedge b \wedge e = \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b}e + (\mathbf{a} - \mathbf{x}) \wedge (\mathbf{b} - \mathbf{x})E$

  tangent: $(\mathbf{a} - \mathbf{x}) \wedge (\mathbf{b} - \mathbf{x}) = \mathbf{x} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{x} = i\mathbf{n}$

  moment: $\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b} = \mathbf{x} \wedge [(\mathbf{a} - \mathbf{x}) \wedge (\mathbf{b} - \mathbf{x})]$
  $= \mathbf{x} \wedge (i\mathbf{n}) = i(\mathbf{x} \cdot \mathbf{n})$

  dual form: $P = i(\mathbf{x} \cdot \mathbf{n}e + \mathbf{n}E) = in$

  $$n = x_2 - x_1 = (\mathbf{x}_2 - \mathbf{x}_1)E + \frac{1}{2}(\mathbf{x}_2{}^2 - \mathbf{x}_1{}^2)e$$

  $$= (\mathbf{x}_2 - \mathbf{x}_1)E + \frac{1}{2}(\mathbf{x}_2 + \mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)e = \mathbf{n}E + \mathbf{c} \cdot \mathbf{n}e$$

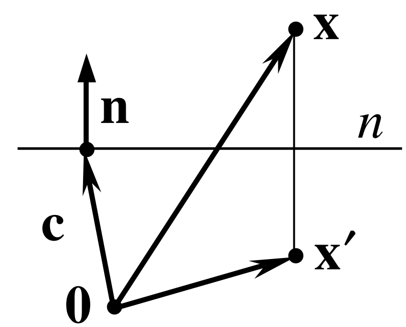
Translations and rotations from reflections

- **Reflection** in a (hyper)plane: $\underline{n}(x) = -nxn = x'$
  $= x - 2x \cdot nn$

$$e \cdot n = 0, \quad n^2 = 1, \quad E = e_0 \wedge e$$

$$c \cdot n = 0 \quad \Rightarrow \quad \text{\underline{Split}: } n = \mathbf{n}E + \mathbf{c}\cdot\mathbf{n}\,e$$

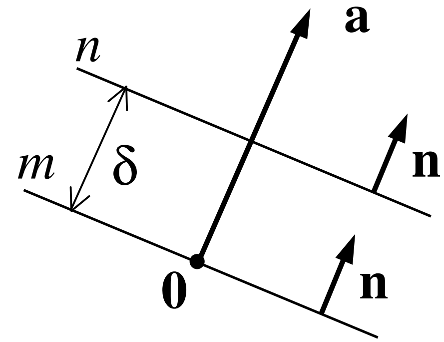
$$\Rightarrow \quad x \cdot n = \mathbf{n}\cdot(\mathbf{x}-\mathbf{c}) \qquad \mathbf{x}' = \mathbf{x} - 2(\mathbf{x}-\mathbf{c})\cdot\mathbf{n}\,\mathbf{n}$$



- **Translation** from parallel planes n and m:

$$G = mn = (\mathbf{n}E + 0)(\mathbf{n}E + \delta e)$$

$$= 1 + \frac{1}{2}\mathbf{a}e = T_{\mathbf{a}} \qquad \mathbf{a} = 2\mathbf{n}\delta$$

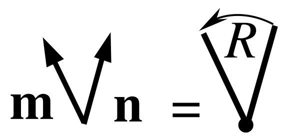


- **Rotation** by planes n and m intersecting at a point c:

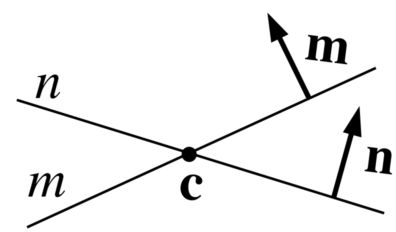
$$G = mn = (\mathbf{m}E + \mathbf{m}\cdot\mathbf{c}e)(\mathbf{n}E + \mathbf{n}\cdot\mathbf{c}e)$$

$$= \mathbf{mn} + e(\mathbf{m}\wedge\mathbf{n})\cdot\mathbf{c} = R + e(R\times\mathbf{c}) = T_{\mathbf{c}}^{-1}RT_{\mathbf{c}}$$

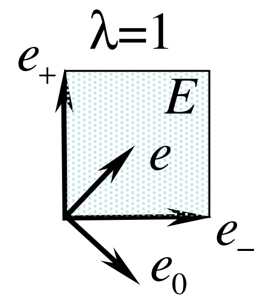
$$R = \mathbf{mn}$$



(Rotor as a directed arc)



2D Minkowski space $\mathbb{V}^{1,1}$ and its algebra $\mathbb{G}^{1,1} = \mathbb{G}(\mathbb{V}^{1,1})$



Null vector basis: $\{e,\ e_0 \mid e^2 = e_0{}^2 = 0,\ e\cdot e_0 = -1\}$

Orthonormal basis: $\left\{e_\pm = \frac{1}{\sqrt{2}}(\lambda e \mp \lambda^{-1} e_0),\ \lambda \neq 0,\ e_\pm{}^2 = \pm 1\right\}$

$\mathbb{G}^{1,1}$ basis: $\{1,\ e,\ e_0,\ E\}$ $\quad E = e_0 \wedge e = e_- e_+ \quad E^2 = 1$

$$e_0 e = E - 1 \qquad Ee = -eE = e, \quad e_0 E = -E e_0 = e_0$$

Matrix representation: $\mathbb{G}^{1,1} \simeq \mathbb{M}_2(\mathbb{R})$ = {real 2×2 matrices}

Basis: $e_+ \simeq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad e_- \simeq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad E \simeq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad 1 \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$M = \tfrac{1}{2}\left[A(1+E) + B(e_+ + e_-) + C(e_+ - e_-) + D(1-E)\right] \simeq [\mathrm{M}] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$\mathbb{G}^{4,1} = \mathbb{G}^3 \otimes \mathbb{G}^{1,1} \simeq \mathbb{M}_2(\mathbb{G}^3)$ ={c-quaternion-valued 2×2 matrices}

$\mathbb{M}_2(\mathbb{G}^3)$ Advantages: • Relation to literature on robotics and screws

Disadvantages: • Implicit assumption of *conformal split* (origin choice)
- *covariance* versus *invariance*
- Suppresses geometric meaning of matrix elements
- Redundancy in matrix elements

SE(3) = **Special Euclidean group**

$= \{\text{rigid displacements } \underline{D}\} \underset{2}{\cong} \{\text{twistors } D\}$

= subgroup of the conformal group C(3, 0) $\cong$ O(4, 1) defined by:

- $\underline{D}(e) = DeD^{-1} = e \quad \Leftrightarrow \quad De = eD$

Screw form: *Twistor* $D = e^{\frac{1}{2}S}$

- $D^{-1} = \tilde{D} = e^{-\frac{1}{2}S} \quad \Leftrightarrow \quad S = -\tilde{S} = \langle S \rangle_2$ (even parity)

$\Rightarrow S = i\mathbf{m} + e\mathbf{n} \qquad Se = eS \quad \Leftrightarrow \quad S \cdot e = 0$

S is called a *twist* (or *screw if* $\mathbf{n} \parallel \mathbf{m}$)

The *screw axis* (direction $\hat{\mathbf{m}}$) is called the *axode*

se(3) = Lie algebra of SE(3)

= an algebra of bivectors: $S_k = i\mathbf{m}_k + e\mathbf{n}_k$

closed under $S_1 \times S_2 = \frac{1}{2}(S_1 S_2 - S_2 S_1)$

$= i(\mathbf{m}_2 \times \mathbf{m}_1) + e(\mathbf{n}_2 \times \mathbf{m}_1 - \mathbf{n}_1 \times \mathbf{m}_2)$

Screw theory follows automatically!

Screws: $S_k = i\mathbf{m}_k + e\mathbf{n}_k$

Product: $S_1 S_2 = S_1 \cdot S_2 + S_1 \times S_2 + S_1 \wedge S_2$

Transform: $S_k' = \underline{U}(S_k) = U S_k U^{-1} = Ad_U\, S_k$ $\qquad \{\underline{U}\} = \mathrm{SE}(3)$

$S_1' S_2' = \underline{U}(S_1 S_2)$ Product preserving

$= U(S_1 \cdot S_2 + S_1 \times S_2 + S_1 \wedge S_2)U^{-1}$

Invariants: $\underline{U}(e) = e \qquad \underline{U}(i) = i$

$S_1' \wedge S_2' = \underline{U}(S_1 \wedge S_2) = S_1 \wedge S_2 = ie(\mathbf{m}_1 \cdot \mathbf{n}_2 + \mathbf{m}_2 \cdot \mathbf{n}_1)$

$S_1' \cdot S_2' = S_1 \cdot S_2 = -\mathbf{m}_1 \cdot \mathbf{m}_2$ (Killing Form)

Covariant: $S_1' \times S_2' = U(S_1 \times S_2)U^{-1}$

Coscrew (Ball's *reciprocal screw*): $S_k^* \equiv \langle S_k i e_0 \rangle_2 = \frac{1}{2}(S_k i e_0 + i e_0 S_k)$

$= i\mathbf{n}_k e \cdot e_0 - e_0 \mathbf{m}_k = -i\mathbf{n}_k - e_0 \mathbf{m}_k$

Invariant: $S_1^* \cdot S_2 = S_1 \cdot S_2^* = \langle S_1 \wedge S_2 i e_0 \rangle = \mathbf{m}_1 \cdot \mathbf{n}_2 + \mathbf{n}_1 \cdot \mathbf{m}_2$

Pitch: $h = \frac{1}{2}\frac{S^* \cdot S}{S \cdot S} = \mathbf{n} \cdot \mathbf{m}^{-1} \doteq \frac{|\mathbf{n}|}{|\mathbf{m}|} = \frac{\text{linear displacement}}{\text{angular displacement}} \quad (\mathbf{n} \wedge \mathbf{m} = 0)$

Rigid Displacement:

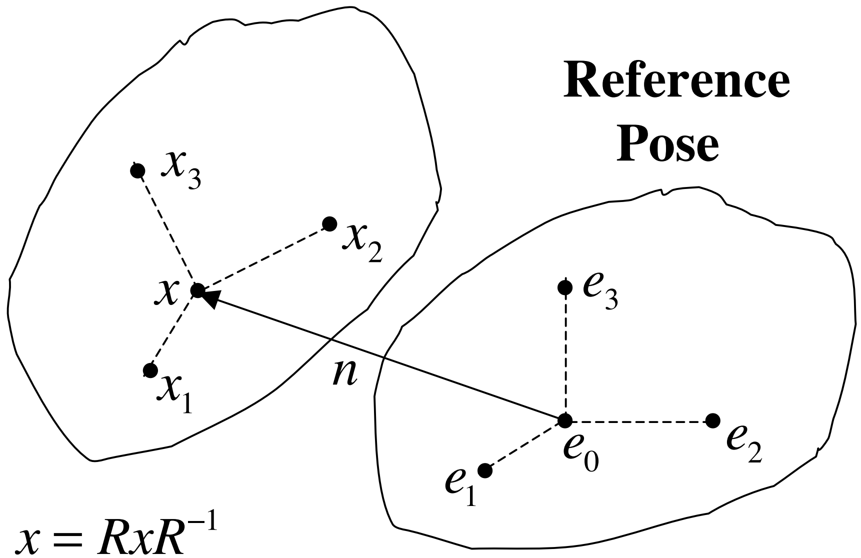
$$x_k = De_k D^{-1} \qquad D^{-1} = \tilde{D}$$

Translation defined by:

$$x = De_0 D^{-1} = Te_0 T^{-1} = e_0 + n$$

$$T = 1 + \frac{1}{2} ne$$

Defines conformal split: $D = RT \qquad x = RxR^{-1}$



Rigid Body Kinematics:

$$x_k(t) = De_k D^{-1} \qquad D = D(t)$$

$$\dot{D} = \frac{1}{2} VD \qquad \tilde{V} = -V = \langle V \rangle_2 \qquad \dot{D}^{-1} = -\frac{1}{2} D^{-1} V$$

$$\dot{x}_k = V \cdot x_k$$

$$e = Ee$$

$$\dot{R} = -\frac{1}{2} i\boldsymbol{\omega} R \qquad \dot{T} = \frac{1}{2}\dot{n}e = \frac{1}{2}\dot{x}e = \frac{1}{2}\dot{x}eT = \frac{1}{2}\dot{\mathbf{x}}eT \qquad \dot{\mathbf{x}} = \dot{x} \wedge E$$

$$\dot{D} = \dot{R}T + R\dot{T} = (-i\boldsymbol{\omega} + R\,e\dot{\mathbf{x}}\,R^{-1})RT = \frac{1}{2} VD$$

$$\Rightarrow \boxed{V = -i\boldsymbol{\omega} + e\mathbf{v}} \qquad \mathbf{v} = R\,\dot{\mathbf{x}}\,R^{-1}$$

(Note: $\mathbf{v} = \dot{\mathbf{x}}$ for $R(t) = 1$)

Rigid Body Dynamics

Comomentum: $P = \underline{M}V = i\underline{I}\boldsymbol{\omega} + m\mathbf{v}e_0 = i\mathbf{l} - \mathbf{p}e_0$

(a coscrew)

Coforce (**wrench**): $W = i\boldsymbol{\Gamma} - \mathbf{f}e_0$ $\qquad \dot{\mathbf{p}} = \mathbf{f}$

Equation of motion: $\dot{P} = W$ $\qquad \Rightarrow \qquad \dot{\mathbf{l}} = \underline{I}(\dot{\boldsymbol{\omega}}) + \boldsymbol{\omega} \times \underline{I}(\boldsymbol{\omega}) = \boldsymbol{\Gamma}$

Kinetic energy: $K \equiv \frac{1}{2} V \cdot P = \frac{1}{2} V \cdot \underline{M}V = \frac{1}{2}(\boldsymbol{\omega} \cdot \mathbf{l} + \mathbf{v} \cdot \mathbf{p})$

Power: $\dot{K} = V \cdot W = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma} + \mathbf{v} \cdot \mathbf{f}$

Change of base point: $e_0 \mapsto e_0' = e_0 - r_0 = T_0^{-1} e_0 T_0$

Chasles' Thm: $V \mapsto V' = V + e\boldsymbol{\omega} \times \mathbf{r}$

Poinsot's Thm: $W \mapsto W' = W - i\,\mathbf{r} \times \mathbf{f} = i(\boldsymbol{\Gamma} + \mathbf{r} \times \mathbf{f}) - e_0\mathbf{f}$

$P \mapsto P' = P - i\mathbf{r} \times \mathbf{p} == i(\mathbf{l} + \mathbf{r} \times \mathbf{p}) - \mathbf{p}e_0$

These theorems are related by the kinetic energy invariant:

$$2K = V' \cdot P' = (V + e\boldsymbol{\omega} \times \mathbf{r}) \cdot (P - i\mathbf{r} \times \mathbf{p})$$
$$= \boldsymbol{\omega} \cdot (\mathbf{l} - \mathbf{r} \times \mathbf{p}) + (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{p} = \boldsymbol{\omega} \cdot \mathbf{l} + \mathbf{v} \cdot \mathbf{p} = V \cdot P$$

Proof of Chasles: $D' = DT_0 = T'D$ $\qquad T' = RT_0R^{-1} = 1 + \frac{1}{2}\mathbf{r}e$

$$\dot{D}' = \dot{T}'D + T'\dot{D} = \frac{1}{2}e\dot{\mathbf{r}}T'D + \frac{1}{2}VDT_0 = \frac{1}{2}(e\dot{\mathbf{r}} + V)D' \qquad \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

Matrix form for Screw Mechanics

Screw transform:

Base point shift: $\mathbf{r} = \mathbf{x}_P - \mathbf{x}_Q$

$$\begin{bmatrix} \mathbf{v}_Q \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{r}\times \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_P - \mathbf{r}\times\boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} V_Q &= V_P - e\,\mathbf{r}\times\boldsymbol{\omega} \\ &= e\,(\mathbf{v}_P + \boldsymbol{\omega}\times\mathbf{r}) - i\boldsymbol{\omega} \end{aligned}$$

$$\hat{V}_Q = \hat{X}_S \hat{V}_P$$

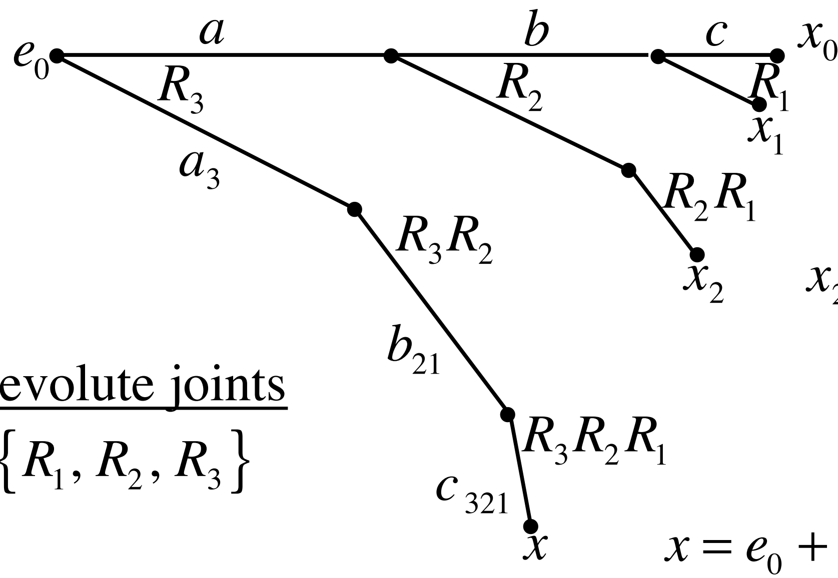
Coscrew transform:

$$\begin{bmatrix} \mathbf{f} \\ \boldsymbol{\Gamma}_Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\mathbf{r}\times & 1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\Gamma}_P \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\Gamma}_P + \mathbf{r}\times\mathbf{f} \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} W_Q &= W_P - i\,\mathbf{r}\times\mathbf{f} \\ &= i\,(\boldsymbol{\Gamma}_P + \mathbf{r}\times\mathbf{f}) - e_0\mathbf{f} \end{aligned}$$

$$\hat{W}_Q = \hat{X}_{CS} \hat{W}_P$$

Recall the drawbacks of the matrix representation.

Linked Rigid Bodies



Reference Pose

$$x_0 = e_0 + a + b + c$$

$$x_1 = e_0 + a + b + R_1 c R_1^{-1} = e_0 + a + b + c_1$$

$$x_2 = e_0 + a + R_2(b + R_1 c R_1^{-1})R_2^{-1} = e_0 + a + b_2 + c_{21}$$

Revolute joints

$$\{R_1, R_2, R_3\}$$

General Pose

$$x = e_0 + R_3[a + R_2(b + R_1 c R_1^{-1})R_2^{-1}]R_3^{-1} = e_0 + a_3 + b_{32} + c_{321}$$

$$\mathbf{x} = x \wedge E = R_3[\mathbf{a} + R_2(\mathbf{b} + R_1 \mathbf{c} R_1^{-1})R_2^{-1}]R_3^{-1} = \mathbf{a}_3 + \mathbf{b}_{32} + \mathbf{c}_{321}$$

Kinematics

$$\dot{R}_k = -\tfrac{1}{2} i \boldsymbol{\omega}_k R_k$$

$$\dot{R}_{32} = -\tfrac{1}{2} i \boldsymbol{\omega}_{32} R_{32}$$

$$\boldsymbol{\omega}_{32} = \boldsymbol{\omega}_3 + R_3 \boldsymbol{\omega}_2 R_3^{-1}$$

$$\boldsymbol{\omega}_{321} = \boldsymbol{\omega}_3 + R_3 \boldsymbol{\omega}_2 R_3^{-1} + R_3 R_2 \boldsymbol{\omega}_1 R_2^{-1} R_3^{-1}$$

$$\dot{\mathbf{x}} = \boldsymbol{\omega}_3 \times \mathbf{a}_3 + \boldsymbol{\omega}_{32} \times \mathbf{b}_{32} + \boldsymbol{\omega}_{321} \times \mathbf{c}_{321}$$