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Factorization, Join (and Meet) of Blades

Efficient algorithms for factorization of blades and and computing the join of blades.

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Motivation

Blade factorization: $\mathbf{B}_k = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_k$.

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- Conversion to other (LA-compatible) representations.
- As a building block of other algorithms.

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Applications of the join:

- True union of subspaces.
- Computing the meet.

In my implementation the join is interwoven with factorization, so factorization must be discussed first.

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Blade Representation

The algorithms in this talk are based on the *additive presentation*. Blades are represented as a sum of basis blades.

Example of basis for 3-D space:

$$\{\ \underbrace{1}_{\textit{grade 0}}\ ,\ \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\textit{grade 1}}\ ,\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_1 \wedge \mathbf{e}_3}_{\textit{grade 2}}\ ,\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\textit{grade 3}}\}.$$

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Example of FastFactorization

Suppose our input blade is:

$$\mathbf{B} = 1.0\,\mathbf{e}_1{\wedge}\mathbf{e}_2{\wedge}\mathbf{e}_3 - 0.5\,\mathbf{e}_1{\wedge}\mathbf{e}_3{\wedge}\mathbf{e}_4 + 0.25\,\mathbf{e}_2{\wedge}\mathbf{e}_3{\wedge}\mathbf{e}_4 - 0.75\,\mathbf{e}_1{\wedge}\mathbf{e}_2{\wedge}\mathbf{e}_4.$$

FastFactorization factorizes this to:

$$\begin{aligned}
\mathbf{b}_1 &= 1.0\,\mathbf{e}_1 && + 0.25\,\mathbf{e}_4,\\
\mathbf{b}_2 &= 1.0\,\mathbf{e}_2 && + 0.5\,\mathbf{e}_4,\\
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\end{aligned}$$

such that $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3$.

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such that $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3$.

The coordinates of the factors are $\pm$ the coordinates of the input blade! How does this work?

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Basic Factorization Algorithm

Algorithm $Factorization(\mathrm{B})$:

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1. Find the largest basis blade $\mathbf{F}$ in the representation of $\mathbf{B}$. I.e., $\mathbf{F} = \mathbf{e}_i \wedge \mathbf{e}_j \wedge \ldots \wedge \mathbf{e}_k$.

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2. Project the basis vectors of $\mathbf{F}$ onto $\mathbf{B}$.
   Use orthogonal projection: $\mathbf{b}_i = (\mathbf{e}_i \rfloor \mathbf{B}) \rfloor \mathbf{B}^{-1}$.
   The $\mathbf{b}_i$ will be independent.

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3. Compute the scale β such that $\mathbf{B} = \beta\, \mathbf{b}_i \wedge \mathbf{b}_j \wedge \ldots \wedge \mathbf{b}_k$.

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This works but is a bit slow (in our implementation, $50\times$ to $100\times$ slower than a simple bilinear outer product).
$\rightarrow$The projection is expensive!

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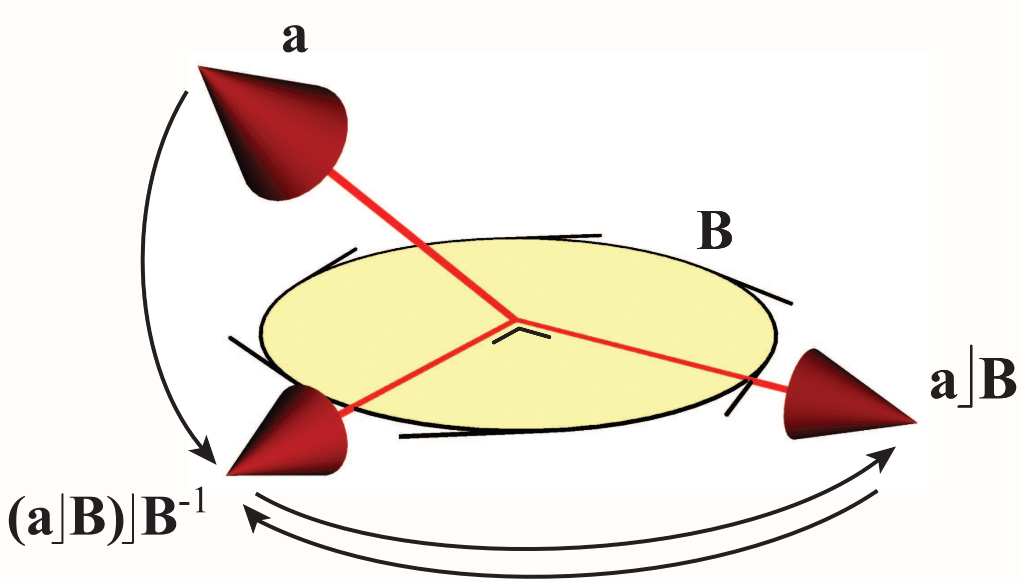
Orthogonal Projection Shortcut



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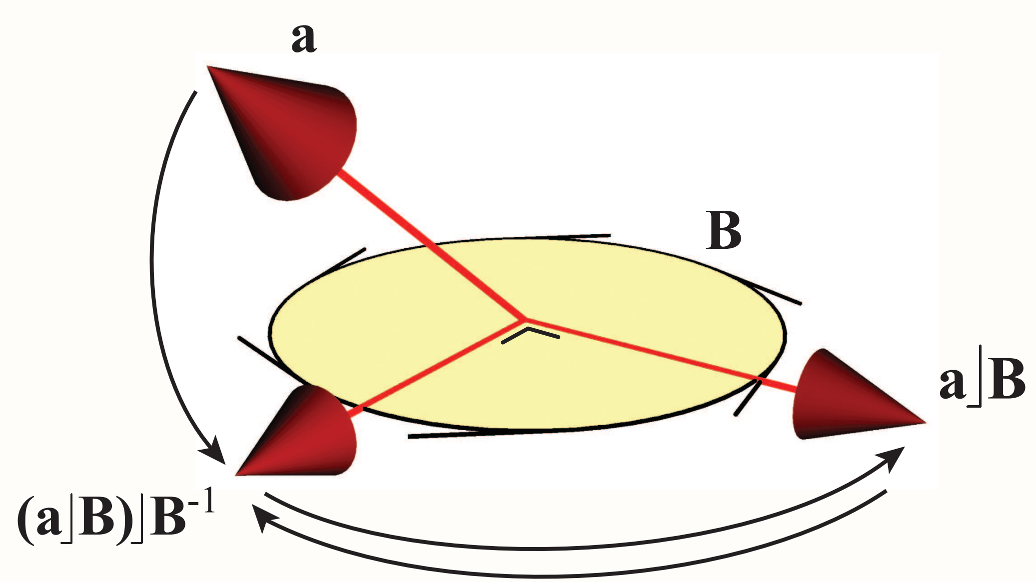

Instead of doing a true projection $\mathbf{b}_i = (\mathbf{e}_i \rfloor \mathbf{B}) \rfloor \mathbf{B}^{-1}$,
we do a 'pseudo projection' $\mathbf{b}_i = (\mathbf{e}_i \rfloor \mathbf{F}) \rfloor \mathbf{B}^{-1}$.

The pseudo projection is computationally cheap because it amounts to simply selecting coordinates from $\mathbf{B}$.

FastFactorization Algorithm

Algorithm $\mathrm{FastFactorization}(\mathbf{B})$:

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FastFactorization Algorithm

Algorithm $\mathrm{FastFactorization}(\mathbf{B})$:

Let $\mathbf{B}$ be a k-blade, with $1 < k < n$.
The algorithm computes a factorization
$\mathbf{B} = \beta\, \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_k$, where β is a scalar:

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1. Find the basis blade $\mathbf{F} = \mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \ldots \wedge \mathbf{f}_k$ to which the absolute largest coordinate of $\mathbf{B}$ refers. The $\mathbf{f}_i$ are basis vectors. Let β be the coordinate that refers to $\mathbf{F}$.

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2. Compute $\mathbf{B}_s = \mathbf{B}/\beta$.

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2. Compute $\mathbf{B}_s = \mathbf{B}/\beta$.
3. For each $\mathbf{f}_i$ compute: $\mathbf{b}_i = (\mathbf{f}_i \rfloor \mathbf{F}^{-1}) \rfloor \mathbf{B}_s$.

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2. Compute $\mathbf{B}_s = \mathbf{B}/\beta$.
3. For each $\mathbf{f}_i$ compute: $\mathbf{b}_i = (\mathbf{f}_i \rfloor \mathbf{F}^{-1}) \rfloor \mathbf{B}_s$.

Because the k vectors $\mathbf{b}_i$ are linearly independent and all contained in $\mathbf{B}$, they must form a factorization of $\mathbf{B}_s$.

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FastFactorization ‘Proof’

(The full proof in the paper).
Again, suppose our input blade is:

$$\mathbf{B} = 1.0\,\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3 - 0.5\,\mathbf{e}_1\wedge\mathbf{e}_3\wedge\mathbf{e}_4 + 0.25\,\mathbf{e}_2\wedge\mathbf{e}_3\wedge\mathbf{e}_4 - 0.75\,\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_4.$$

Then $\mathbf{F} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$, and the factors are:

$$\begin{array}{rcllll} \mathbf{b}_1 & = & 1.0\,\mathbf{e}_1 & & & +\,0.25\,\mathbf{e}_4, \\ \mathbf{b}_2 & = & & 1.0\,\mathbf{e}_2 & & +\,0.5\,\mathbf{e}_4, \\ \mathbf{b}_3 & = & & & 1.0\,\mathbf{e}_3 & -\,0.75\,\mathbf{e}_4. \end{array}$$

The diagonal typesetting of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ should make it obvious that the $\mathbf{b}_i$ are linearly independent.

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FastFactorization Code Generation

We used code generation to implement FastFactorization.

One function was generated for each valid combination of basis blade and grade.

Example of a generated function:

```
void factorE234grade3(const float *B, float **b) {
  b[2][0] = B[0];
  b[1][0] = -B[1];
  b[0][0] = B[2];
  b[0][1] = b[1][2] = b[2][3] = B[3];
  b[2][4] = B[6];
  b[1][4] = -B[8];
  b[0][4] = B[9];
  b[0][2] = b[0][3] = b[1][1] = b[1][3] = b[2][1] = b[2][2] = 0.0f;
}
```

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FastFactorization Implementation

The full FastFactorization implementation amounts to:

- Filter out trivial special cases (hand written).
- Find largest coordinate / basis blade (hand written).
- Call the appropriate factorization function (generated) via a lookup table .

FastFactorization Benchmarks

Benchmark: Factorize millions of random blades.

Used one CPU on a Core2Duo 1.83Ghz.
Compiled using VS2005.

n	3	4	5	6
factorizations per second	15M	9.2M	5.2M	2.8M
relative to O.P.	5.1×	5.1×	3.4×	3.8×

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The Join (and the Meet)

The join $\mathbf{A} \cup \mathbf{B}$ is the union of $\mathbf{A}$ and $\mathbf{B}$.

The join is a non-linear product, for example in general
$\mathbf{A} \cup (\mathbf{B} + \mathbf{C}) \neq \mathbf{A} \cup \mathbf{B} + \mathbf{A} \cup \mathbf{C}$.

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The meet $\mathbf{A} \cap \mathbf{B}$ can be (most?) efficiently computed from the join using $\mathbf{A} \cap \mathbf{B} = (\mathbf{B} \rfloor (\mathbf{A} \cup \mathbf{B})^{-1}) \rfloor \mathbf{A}$.

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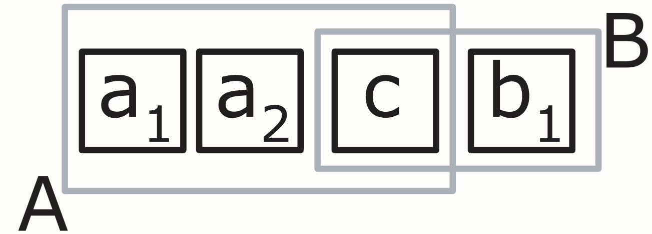
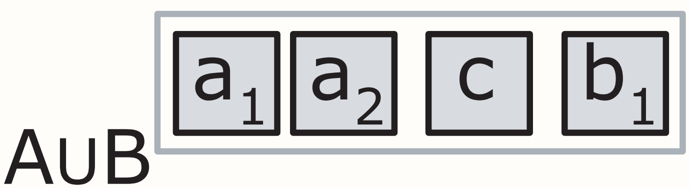
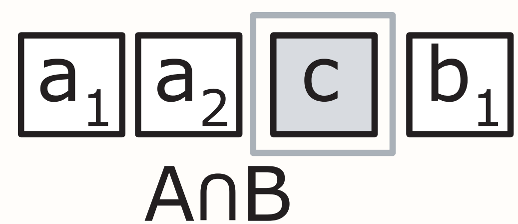
The Join, Meet and Delta Product Illustrated

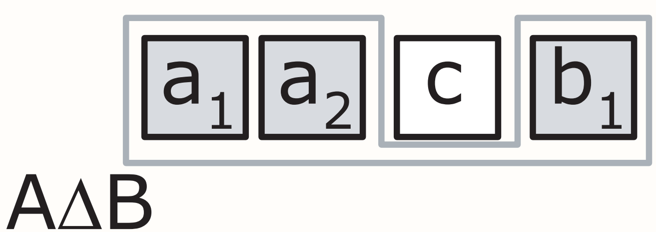
$A = a_1 \wedge a_2 \wedge c$

$B = c \wedge b_1$









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   (c) Compute $\mathbf{H} = \mathbf{J} \wedge \mathrm{unit}(\mathbf{b}_i)$.
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6. Return $\mathbf{J}$.

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The delta product Δ (geometric symmetric difference) is used:

$$\mathrm{grade}(\mathbf{A} \cup \mathbf{B}) = \frac{\mathrm{grade}(\mathbf{A}) + \mathrm{grade}(\mathbf{B}) + \mathrm{grade}(\mathbf{A}\Delta\mathbf{B})}{2}.$$

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The StableFastJoin Algorithm

Algorithm $\mathrm{StableFastJoin}(\mathbf{A}, \mathbf{B}, \epsilon, \delta)$:

Start with steps 1-5 of $\mathrm{FastJoin}(\mathbf{A}, \mathbf{B}, \epsilon)$.

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6. If $(\mathrm{grade}(\mathbf{J}) = n)$
or $(\mathrm{grade}(\mathbf{J}) = \mathrm{grade}(\mathbf{A}) + \mathrm{grade}(\mathbf{B}))$,
return $\mathbf{J}$. Otherwise:

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7. Compute $\mathrm{grade}(\mathbf{A} \cup \mathbf{B})$ using the delta product.

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7. Compute $\mathrm{grade}(\mathbf{A} \cup \mathbf{B})$ using the delta product.
8. While $(\mathrm{grade}(\mathbf{J}) < \mathrm{grade}(\mathbf{A} \cup \mathbf{B}))$
   (a) For all valid i, compute $\mathbf{b}_i = (\mathbf{f}_i \rfloor \mathbf{F}^{-1}) \rfloor \mathbf{B}$.
   Set $\mathbf{b}_m$ to that $\mathbf{b}_i$ which leads to the largest $\|\mathbf{J} \wedge \mathbf{b}_i\|$.
   (b) Update $\mathbf{J} \leftarrow \mathbf{J} \wedge \mathbf{b}_m$.

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   (b) Update $\mathbf{J} \leftarrow \mathbf{J} \wedge \mathbf{b}_m$.
9. Return $\mathbf{J}$.

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FastJoin Implementation

Code generation is used to generate the core of FastJoin.
Example of a generated function (step 5b/5c of algorithm):

```
void factorAndOuterProductE35G3(const float *J, const float *B, float *H) {
  H[0] = J[3] * B[5] - J[2] * B[6] + J[0] * B[9];
  H[1] = J[6] * B[5] - J[5] * B[6];
  H[2] = J[8] * B[5] - J[7] * B[6] - J[4] * B[9];
  H[3] = J[9] * B[5] - J[5] * B[9];
  H[4] = J[9] * B[6] - J[6] * B[9];
  return B[5] * B[5] + B[6] * B[6] + B[9] * B[9];
}
```

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FastJoin Implementation

Code generation is used to generate the core of FastJoin.
Example of a generated function (step 5b/5c of algorithm):

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  H[3] = J[9] * B[5] - J[5] * B[9];
  H[4] = J[9] * B[6] - J[6] * B[9];
  return B[5] * B[5] + B[6] * B[6] + B[9] * B[9];
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Implementation of the delta product is optimized using code generation and lazy evaluation.

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Implementation of the delta product is optimized using code generation and lazy evaluation.

The approach is limited to 6-D due to code size!
Above 6-D a conventional (hand-written) approach can be used (about $2\times$ slower).

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FastJoin Benchmarks

Benchmark: Compute the join of millions of random blades. Pairs of blades were generated such that they shared a common factor of a random grade.

Used one CPU on a Core2Duo 1.83Ghz.
Compiled using VS2005.

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n	3	4	5	6
FastJoin (absolute)	7.4M	5.4M	3.1M	1.8M
FastJoin (relative)	9.8×	8.7×	5.8×	6.4×
StableFastJoin (absolute)	7.4M	5.2M	2.6M	1.6M
StableFastJoin (relative)	9.8×	9.1×	7.0×	6.8×
Gram-Schmidt (relative)	12×	12×	7.9×	8.0×

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The Meet

The meet can be directedly computed by factorizing the dual of the delta product.

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The Meet

The meet can be directedly computed by factorizing the dual of the delta product.

But:
-Expensive full evaluation of delta product is always required.
-Generated code is larger.

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Discussion / Summary

- Fastest possible factorization algorithm?

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