A matricial Hilbert transform on closed surfaces in Hermitean Clifford analysis

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Outline

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definition (Hilbert)

consider **finite energy signals** f, i.e. $f \in L_2(\mathbb{R})$, so

$$\|f\|^2_{L_2(\mathbb{R})} = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt < +\infty$$

then, the **Hilbert transform $\mathcal{H}[f]$** of f

$$\begin{aligned}\mathcal{H}[f](t) &= \frac{1}{\pi} \operatorname{Pv} \int_{-\infty}^{+\infty} \frac{f(x)}{t-x} \, dx \\ &= \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{|t-x|>\epsilon} \frac{f(x)}{t-x} \, dx \\ &= (H * f)(t)\end{aligned}$$

with the **Hilbert kernel H**

$$H(x) = \operatorname{Pv} \frac{1}{\pi x}$$

1. the Hilbert operator $\mathcal{H}$ is **translation invariant**, i.e.

$$\tau_a[\mathcal{H}[f]] \;=\; \mathcal{H}[\tau_a[f]] \;, \quad \text{with } \tau_a[f](t) = f(t-a) \;, \; a \in \mathbb{R}$$

2. the Hilbert operator $\mathcal{H}$ is **dilation invariant**, i.e.

$$d_a[\mathcal{H}[f]] \;=\; \mathcal{H}[d_a[f]] \;, \quad \text{with } d_a[f](t) = \frac{1}{\sqrt{a}} f(\frac{t}{a}) \;, \; a > 0$$

3. the Hilbert operator $\mathcal{H}$ is a **bounded linear operator on $L_2(\mathbb{R})$**, and is a fortiori **norm preserving**, i.e.

$$\|\mathcal{H}[f]\|_{L_2(\mathbb{R})} \;=\; \|f\|_{L_2(\mathbb{R})}$$

properties (Hardy & Titchmarsh)

4. $\mathcal{H}^2 = -\mathbf{1}$ on $L_2(\mathbb{R})$, and hence $\mathcal{H}^{-1} = -\mathcal{H}$

5. the Hilbert operator $\mathcal{H}$ is **unitary** on $L_2(\mathbb{R})$, i.e.

$$\mathcal{H}\,\mathcal{H}^* = \mathcal{H}^*\,\mathcal{H} = \mathbf{1}$$

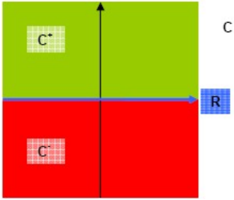
with its adjoint $\mathcal{H}^*$ given by $-\mathcal{H}$, i.e.

$$(\mathcal{H}[f], g)_{L_2(\mathbb{R})} = (f, -\mathcal{H}[g])_{L_2(\mathbb{R})}$$

6. the Hilbert operator $\mathcal{H}$ **commutes with differentiation**, i.e. if f and f' are in $L_2(\mathbb{R})$, then

$$\mathcal{H}\left[\frac{d}{dt}f(t)\right](x) = \frac{d}{dx}\left[\mathcal{H}[f](x)\right]$$

let $\mathbb{C}^\pm = \{x + iy \in \mathbb{C} : y \gtrless 0\}$



the **Cauchy integral $\mathcal{C}$[f]**

$$\begin{aligned}\mathcal{C}[f](z) &= \mathcal{C}[f](x,y) \\ &= \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{(x-t)+iy}\,dt\ , \quad y \neq 0\end{aligned}$$

is **holomorphic** in $\mathbb{C}^\pm$, i.e.

$$\partial_{z^c}\mathcal{C}[f](z) = \tfrac{1}{2}(\partial_x + i\partial_y)\,\mathcal{C}[f](x,y) = 0\ , \quad y \neq 0$$

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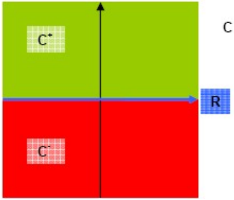

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7. the Hilbert transform **arises** in a natural way **by considering boundary limits** (in L_2 sense) **of the Cauchy integral**, i.e.

$$\lim_{\substack{y \to 0\pm \\ NT}} \mathcal{C}[f](x,y) = \pm\frac{1}{2}\, f(x) + \frac{1}{2}\, i\, \mathcal{H}[f](x)\ , \quad \text{for a.e. } x \in \mathbb{R}$$

(Plemelj–Sokhotzki formulae)

- **signal processing**
- metallurgy (theory of elasticity)
- Dirichlet boundary value problems (potential theory)
- dispersion relation in high energy physics, spectroscopy and wave equations
- wing theory
- harmonic analysis

notice: applications were not discussed in my research

some applications: the analytic signal

$$f(t)\text{: "nice" real valued function} \quad \rightarrow \quad \left\{ \begin{array}{l} \text{amplitude} \\ \text{phase} \\ \text{frequency} \end{array} \right. ?$$

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possible approach: associated **analytic signal**

$$f_{\mathcal{A}}(t) \quad = \quad f(t) + i\,\mathcal{H}[f](t)$$

$f(t)$: "nice" real valued function ➜ $\left\{ \begin{array}{l} \text{amplitude} \\ \text{phase} \\ \text{frequency} \end{array} \right.$?

possible approach: associated **analytic signal**

$$f_{\mathcal{A}}(t) \;=\; f(t) + i\,\mathcal{H}[f](t) \;=\; \rho(t)\,\cos(\theta(t)) + i\,\rho(t)\,\sin(\theta(t))$$

with

$$\begin{aligned} \rho(t) &= \textbf{instantaneous amplitude} \\ \theta(t) &= \textbf{instantaneous phase} \\ \frac{d\theta}{dt}(t) &= \textbf{instantaneous frequency} \end{aligned}$$

Outline

- 1D signal processing:

  1D Hilbert transform is used to construct analytic signal in order to extract interesting features (amplitude, phase, frequency) of real 1D signal

- mD signal processing (e.g. images (2D), video signals (3D)):

  mD Hilbert transform as a tool to obtain interesting mD signal information

- $\mathbb{R}^m = \langle e_1, e_2, \ldots, e_m \rangle$ with

$$\begin{cases} e_j^2 = e_j e_j = -1 \\ e_j e_k = -e_k e_j \ , \quad j \neq k \end{cases}$$

non–commutative multiplication!

- $\mathbb{R}_{0,m} = \langle 1, \{ e_{j_1} e_{j_2} \ldots e_{j_h} \ : \ 1 \leq j_1 < j_2 < \cdots < j_h \leq m \} \rangle$

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- $(X_1, \ldots, X_m) \quad \longleftrightarrow \quad$ **vector** $\underline{X} = \sum_{j=1}^{m} e_j X_j$

- **norm** of a vector: $|\underline{\mathbf{X}}|^2 = -\underline{X}^2 = \sum_{j=1}^{m} X_j^2$

- **conjugation** (anti–involution)

$$\overline{e_j} = -e_j \qquad \text{and} \qquad \overline{a\,b} = \overline{b}\,\overline{a}$$

- **Dirac operator**: $\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \, \partial_{X_j}$

  fundamental solution: $E(\underline{X}) = \frac{1}{a_m} \frac{\overline{\underline{X}}}{|\underline{X}|^m}$, i.e. $\partial_{\underline{X}} E(\underline{X}) = \delta(\underline{X})$

- $f : \Omega \overset{\text{open}}{\subset} \mathbb{R}^m \rightarrow \mathbb{R}_{0,m}$ **monogenic** on $\Omega \iff \partial_{\underline{X}} f = 0$ on Ω

- **Laplace operator**: $\Delta = \sum_{j=1}^{m} \partial_{X_j}^2 = -\partial_{\underline{X}} \, \partial_{\underline{X}}$

- **Dirac operator**: $\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \, \partial_{X_j}$

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- **Laplace operator**: $\boldsymbol{\Delta} = \sum_{j=1}^{m} \partial_{X_j}^2 = -\partial_{\underline{X}} \, \partial_{\underline{X}}$

harmonic analysis in $\mathbb{R}^m$

$\boldsymbol{\Delta} \mathbf{f} = \mathbf{0}$, with $\Delta = \sum_{j=1}^{m} \partial_{X_j}^2$

1D complex analysis (**holomorphic** functions)

$\boldsymbol{\partial_{z^c} f = 0}$, with $\partial_{z^c} = \frac{1}{2} (\partial_x + i \, \partial_y)$

↓ **refinement** ↓ **generalization**

Euclidean Clifford analysis in $\mathbb{R}^m$ (**monogenic** functions)

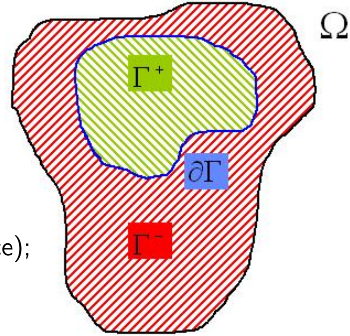
$\boldsymbol{\partial_{\underline{X}} f = 0}$, with $\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \, \partial_{X_j}$ and $\Delta = -\partial_{\underline{X}}^2$

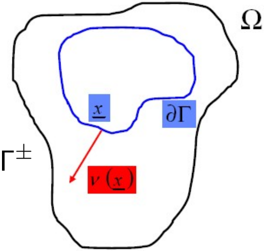
$\Omega \overset{\text{open}}{\subset} \mathbb{R}^m$

$$\Gamma \subset \Omega: \left\{ \begin{array}{l} m\text{–dimensional} \\ \text{compact differentiable} \\ \text{oriented} \end{array} \right\} \text{manifold}$$

with C^∞ smooth boundary $\partial\Gamma$ (= closed surface);

$\Gamma^+ = \overset{\circ}{\Gamma}$; $\Gamma^- = \Omega \setminus \Gamma$





Euclidean Cauchy integral for functions $f \in L_2(\partial\Gamma)$

$$\mathcal{C}[f](\underline{Y}) = \int_{\partial\Gamma} E(\underline{X} - \underline{Y})\, \widetilde{d\sigma}_{\underline{X}}\, f(\underline{X})\,, \qquad \underline{Y} \in \Gamma^\pm$$

is monogenic on $\Gamma^\pm$

- $f \in L_2(\partial\Gamma) \iff \|f\|^2_{L_2(\partial\Gamma)} = \int_{\partial\Gamma} |f(\underline{X})|^2\, dS(\underline{X}) < \infty$
- **$\widetilde{d\sigma}_{\underline{X}}$ = vector valued oriented surface element on $\partial\Gamma$**

$$= \sum e_j (-1)^{j-1} (dX_1 \wedge \cdots \wedge dX_{j-1} \wedge dX_{j+1} \wedge \cdots \wedge dX_m)$$

$$= \nu(\underline{X})\, dS(\underline{X})$$

- **$\nu(\underline{X})$ = outward pointing unit normal vector in $\underline{X}$ on $\partial\Gamma$**

definition

the Euclidean Hilbert transform arises in a natural way by considering boundary limits (in L_2 sense) of the Euclidean Cauchy integral, i.e.

$$\lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^\pm}} \mathcal{C}[f](\underline{Y}) \;=\; \pm\frac{1}{2}\,f(\underline{U}) + \frac{1}{2}\,\mathcal{H}[f](\underline{U})\;, \quad \underline{U}\in\partial\Gamma$$

with the Euclidean Hilbert transform

$$\mathcal{H}[f](\underline{U}) \;=\; 2\,\mathrm{Pv}\int_{\partial\Gamma} E(\underline{X}-\underline{U})\,\widetilde{d\sigma_{\underline{X}}}\,f(\underline{X})\;, \quad \underline{U}\in\partial\Gamma$$

properties

1. $\mathcal{H}$ is a bounded linear operator on $L_2(\partial\Gamma)$
2. $\mathcal{H}^2 = \mathbf{1}$ on $L_2(\partial\Gamma)$
3. $\mathcal{H}^* = \nu\,\mathcal{H}\,\nu$ on $L_2(\partial\Gamma)$, i.e. $\langle\mathcal{H}[f], g\rangle = \langle f, \nu\mathcal{H}[\nu g]\rangle$

Outline

$\mathbb{C}_{2n}$ = complex Clifford algebra over $\mathbb{R}^{2n}$

$$\mathbb{C}_{2n} = \mathbb{C} \otimes \mathbb{R}_{0,2n} = \mathbb{R}_{0,2n} \oplus i\, \mathbb{R}_{0,2n}$$

- why $m = 2n$?

  $\mathbb{R}^m$: **complex structure J**, i.e. $J \in SO(m)$, with $J^2 = -\mathbf{1}$

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- why $m = 2n$?

  $\mathbb{R}^m$: **complex structure J**, i.e. $J \in SO(m)$, with $J^2 = -\mathbf{1}$

- $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}) \xrightarrow{J} (-e_{n+1}, \ldots, -e_{2n}, e_1, \ldots, e_n)$

- Euclidean objects $\overset{\frac{1}{2}(1 \pm i\,J)}{\longmapsto}$ Hermitean objects

- $\lambda \in \mathbb{C}_{2n} \longleftrightarrow \lambda = a + ib\,, \quad a, b \in \mathbb{R}_{0,2n}$

  Hermitean conjugation: $\lambda^\dagger = \overline{a} - i\overline{b}$

Witt basis

$$\begin{cases} \mathfrak{f}_j = \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\, e_{n+j}) \ , & j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\, e_{n+j}) \ , & j = 1, \ldots, n \end{cases}$$

properties

- **isotropy**

$$(\mathfrak{f}_j)^2 = 0 \ , \quad (\mathfrak{f}_j^\dagger)^2 = 0 \ , \qquad j = 1, \ldots, n$$

- **Grassmann identities**

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = 0 \ , \qquad j, k = 1, \ldots, n$$

$$\mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0 \ , \qquad j, k = 1, \ldots, n$$

- **duality identities**

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk} \ , \qquad j, k = 1, \ldots, n$$

vector variables

- Euclidean vector

$$\underline{X} = \sum_{j=1}^{2n} e_j X_j = \sum_{j=1}^{n} (e_j x_j + e_{n+j} y_j)$$

- isotropic **Hermitean vectors**

$$\underline{Z} = \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] = \sum_{j=1}^{n} \mathfrak{f}_j z_j , \qquad z_j = x_j + i y_j$$

$$\underline{Z}^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger z_j^c , \qquad z_j^c = x_j - i y_j$$

- notice that

$$\underline{X} = \underline{Z} - \underline{Z}^\dagger \qquad \text{and} \qquad |\underline{X}|^2 = |\underline{Z}|^2 = |\underline{Z}^\dagger|^2 = \underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger\,\underline{Z}$$

Dirac operators

- Euclidean Dirac operator

$$\partial_{\underline{X}} = \sum_{j=1}^{2n} e_j \, \partial_{X_j} = \sum_{j=1}^{n} (e_j \, \partial_{x_j} + e_{n+j} \, \partial_{y_j})$$

- isotropic **Hermitean Dirac operators**

$$\partial_{\underline{Z}^\dagger} = \frac{1}{4} (\mathbf{1} + iJ)[\partial_{\underline{X}}] = \sum_{j=1}^{n} \mathfrak{f}_j \, \partial_{z_j^c} \, , \qquad \partial_{z_j^c} = \frac{1}{2} \left(\partial_{x_j} + i \, \partial_{y_j}\right)$$

$$\partial_{\underline{Z}} = -\frac{1}{4} (\mathbf{1} - iJ)[\partial_{\underline{X}}] = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \partial_{z_j} \, , \qquad \partial_{z_j} = \frac{1}{2} \left(\partial_{x_j} - i \, \partial_{y_j}\right)$$

- notice that

$$\partial_{\underline{X}} = 2 \left(\partial_{\underline{Z}^\dagger} - \partial_{\underline{Z}}\right) \quad \text{and} \quad \Delta = 4 \left(\partial_{\underline{Z}} \partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger} \partial_{\underline{Z}}\right)$$

- $g : \Omega \to \mathbb{C}_{2n}$ is (Hermitean or) **h–monogenic** if and only if

$$\partial_{\underline{Z}} \, g = 0 \quad \text{and} \quad \partial_{\underline{Z}^\dagger} \, g = 0$$

harmonic analysis in $\mathbb{R}^m$

$\Delta f = 0$, with $\Delta = \sum_{j=1}^m \partial_{X_j}^2$

1D complex analysis (**holomorphic** functions)

$\partial_{z^c} f = 0$, with $\partial_{z^c} = \frac{1}{2}(\partial_x + i\,\partial_y)$

↓ **refinement** ↓ **generalization**

Euclidean Clifford analysis in $\mathbb{R}^m$ (**monogenic** functions)

$\partial_{\underline{X}} f = 0$, with $\partial_{\underline{X}} = \sum_{j=1}^m e_j\,\partial_{X_j}$ and $\Delta = -\partial_{\underline{X}}^2$

↓ **refinement**

Hermitean Clifford analysis in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (**h–monogenic** functions)

$$\begin{cases} \partial_{\underline{Z}}\ f = 0 \\ \partial_{\underline{Z}^\dagger} f = 0 \end{cases} \quad \text{with} \quad \begin{cases} \partial_{\underline{Z}} = \sum_{j=1}^n \mathfrak{f}_j^\dagger\,\partial_{z_j} \\ \partial_{\underline{Z}^\dagger} = \sum_{j=1}^n \mathfrak{f}_j\,\partial_{z_j^c} \end{cases}$$

$$\text{and} \quad \partial_{\underline{X}} = 2\left(\partial_{\underline{Z}^\dagger} - \partial_{\underline{Z}}\right)$$

- further development: need for a Cauchy integral formula for h–monogenic functions

- put:

$$\mathcal{E}(\underline{Z},\underline{Z}^\dagger) = -(\mathbf{1}+i\,J)[E(\underline{X})]$$
$$\mathcal{E}^\dagger(\underline{Z},\underline{Z}^\dagger) = (\mathbf{1}-i\,J)[E(\underline{X})]$$

introducing the particular circulant (2×2) matrices

$$\mathcal{D}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix}, \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

one obtains that

$$\mathcal{D}_{(\underline{\mathbf{Z}},\underline{\mathbf{Z}}^\dagger)}\,\boldsymbol{\mathcal{E}}(\underline{Z},\underline{Z}^\dagger) = \boldsymbol{\delta}(\underline{Z})$$

- $g_1, g_2 \in C^1(\Omega; \mathbb{C}_{2n})$ ➜ $\mathbf{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$

  $\mathbf{G}_2^1$ **H–monogenic** if and only if

$$\mathcal{D}_{(\underline{Z},\underline{Z}^\dagger)}\,\mathbf{G}_2^1 = \mathbf{O} \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} \partial_{\underline{Z}}\,[g_1] + \partial_{\underline{Z}^\dagger}[g_2] & = & 0 \\ \partial_{\underline{Z}^\dagger}[g_1] + \partial_{\underline{Z}}\,[g_2] & = & 0 \end{array} \right.$$

  where $\mathbf{O}$ denotes the matrix with zero entries

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  where $\mathbf{O}$ denotes the matrix with zero entries

- important observation

$$\mathbf{G}_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \text{ is } \textbf{H–monogenic} \quad \Longleftrightarrow \quad \mathbf{g} \text{ is } \textbf{h–monogenic}$$

- **volume element on Γ**:

$$dW(\underline{Z}, \underline{Z}^\dagger) = (dz_1 \wedge dz_1^c) \wedge (dz_2 \wedge dz_2^c) \wedge \cdots \wedge (dz_n \wedge dz_n^c)$$

- **matricial Hermitean oriented surface element on $\partial\Gamma$**:

$$d\Sigma_{(\underline{Z}, \underline{Z}^\dagger)} = \begin{pmatrix} d\sigma_{\underline{Z}} & -d\sigma_{\underline{Z}^\dagger} \\ -d\sigma_{\underline{Z}^\dagger} & d\sigma_{\underline{Z}} \end{pmatrix}$$

with

$$d\sigma_{\underline{Z}} = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \widehat{dz_j} \, , \qquad d\sigma_{\underline{Z}^\dagger} = \sum_{j=1}^{n} \mathfrak{f}_j \, \widehat{dz_j^c}$$

and

$$\widehat{dz_j} = (dz_1 \wedge dz_1^c) \wedge \cdots \wedge ([dz_j] \wedge dz_j^c) \wedge \cdots \wedge (dz_n \wedge dz_n^c)$$
$$\widehat{dz_j^c} = (dz_1 \wedge dz_1^c) \wedge \cdots \wedge (dz_j \wedge [dz_j^c]) \wedge \cdots \wedge (dz_n \wedge dz_n^c)$$

Outline

1. **Clifford–Stokes theorems**

Theorem (Euclidean Clifford–Stokes theorem)

Let f and g be functions in $C^1(\Omega; \mathbb{R}_{0,2n})$, then

$$\int_{\partial\Gamma} f(\underline{X})\, \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}) = \int_{\Gamma} \left[(f\partial_{\underline{X}})\, g + f\,(\partial_{\underline{X}} g) \right] \widetilde{dV}(\underline{X})$$

Theorem (Hermitean Clifford–Stokes theorem)

Let f_1, f_2, g_1 and g_2 be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\mathbf{F}_2^1$ and $\mathbf{G}_2^1$ be their respective associated matrix functions, then

$$\int_{\partial\Gamma} \mathbf{F}_2^1(\underline{X})\, \mathbf{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \mathbf{G}_2^1(\underline{X})$$
$$= \int_{\Gamma} \left[(\mathbf{F}_2^1\, \mathcal{D}_{(\underline{Z},\underline{Z}^\dagger)})\, \mathbf{G}_2^1 + \mathbf{F}_2^1\, (\mathcal{D}_{(\underline{Z},\underline{Z}^\dagger)}\, \mathbf{G}_2^1) \right] dW(\underline{Z},\underline{Z}^\dagger)$$

2 Cauchy–Pompeiu formulae

Theorem (Euclidean Cauchy–Pompeiu formula)

Let $g \in C^1(\Omega; \mathbb{R}_{0,2n})$, then

$$\int_{\partial\Gamma} E(\underline{X}-\underline{Y})\widetilde{d\sigma}_{\underline{X}} g - \int_\Gamma E(\underline{X}-\underline{Y})\left(\partial_{\underline{X}} g\right) \widetilde{dV}(\underline{X}) = \begin{cases} 0\ , & \underline{Y} \in \Gamma^- \\ g(\underline{Y})\ , & \underline{Y} \in \Gamma^+ \end{cases}$$

Theorem (Hermitean Cauchy–Pompeiu formula)

Let g_1 and g_2 be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\mathbf{G}_2^1$ be the associated matrix function, then

$$\int_{\partial\Gamma} \mathcal{E}(\underline{Z}-\underline{V})\, \mathbf{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \mathbf{G}_2^1 - \int_\Gamma \mathcal{E}(\underline{Z}-\underline{V}) \left(\mathcal{D}_{(\underline{Z},\underline{Z}^\dagger)} \mathbf{G}_2^1\right) dW(\underline{Z}, \underline{Z}^\dagger)$$

$$= \begin{cases} \mathbf{O}\ , & \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \mathbf{G}_2^1(\underline{Y})\ , & \underline{Y} \in \Gamma^+ \end{cases}$$

3 Cauchy integral formulae

Theorem (Euclidean Cauchy integral formula)

If the function $g : \Omega \to \mathbb{R}_{0,2n}$ is **monogenic**, then

$$\int_{\partial\Gamma} E(\underline{X} - \underline{Y})\, \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}) \;=\; \begin{cases} 0\,, & \underline{Y} \in \Gamma^- \\ g(\underline{Y})\,, & \underline{Y} \in \Gamma^+ \end{cases}$$

Theorem (Hermitean Cauchy integral formula I)

Let g_1 and g_2 be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\mathbf{G}_2^1$ be the associated matrix function. If $\mathbf{G}_2^1$ is **H–monogenic**, then

$$\int_{\partial\Gamma} \mathcal{E}(\underline{Z} - \underline{V})\, \mathbf{d\Sigma}_{(\underline{Z}, \underline{Z}^\dagger)}\, \mathbf{G}_2^1(\underline{X}) \;=\; \begin{cases} \mathbf{O}\,, & \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \mathbf{G}_2^1(\underline{Y})\,, & \underline{Y} \in \Gamma^+ \end{cases}$$

3 Cauchy integral formulae

Theorem (Euclidean Cauchy integral formula)

If the function $g : \Omega \to \mathbb{R}_{0,2n}$ is **monogenic**, then

$$\int_{\partial\Gamma} E(\underline{X} - \underline{Y})\, \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}) = \begin{cases} 0\,, & \underline{Y} \in \Gamma^- \\ g(\underline{Y})\,, & \underline{Y} \in \Gamma^+ \end{cases}$$

Theorem (Hermitean Cauchy integral formula II)

If the function $g : \Omega \to \mathbb{C}_{2n}$ is **h–monogenic**, then

$$\int_{\partial\Gamma} \mathcal{E}(\underline{Z}-\underline{V})\, \mathbf{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \mathbf{G}_0(\underline{X}) = \begin{cases} \mathbf{O}\,, & \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \mathbf{G}_0(\underline{Y})\,, & \underline{Y} \in \Gamma^+ \end{cases}$$

Hermitean Cauchy integral $\mathcal{C}$

$$\mathcal{C}[\mathbf{G}_2^1](\underline{Y}) = \int_{\partial\Gamma} \mathcal{E}(\underline{Z} - \underline{V})\, \mathbf{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \mathbf{G}_2^1(\underline{X}) \,, \qquad \underline{Y} \in \Gamma^\pm$$

for functions $g_1, g_2 \in C^0(\partial\Gamma)$

Hermitean Hilbert transform $\mathcal{H}$

- **definition**

$$\lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^\pm}} \mathcal{C}[\mathbf{G}_2^1](\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\pm\frac{1}{2}\, \mathbf{G}_2^1(\underline{U}) + \frac{1}{2}\, \mathcal{H}[\mathbf{G}_2^1](\underline{U}) \right), \quad \underline{U} \in \partial\Gamma$$

for functions $g_1, g_2 \in L_2(\partial\Gamma)$

- **properties**
  1. $\mathcal{H}$ is a bounded linear operator on $\mathbf{L_2}(\partial\Gamma)$
  2. $\mathcal{H}^2 = \mathbb{E}_2$ on $\mathbf{L_2}(\partial\Gamma)$
  3. $\mathcal{H}^* = \mathcal{V}\,\mathcal{H}\,\mathcal{V}$ on $\mathbf{L_2}(\partial\Gamma)$

To read: Hermitean Clifford analysis

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