Geometric Algebra Approach to Fluid Dynamics

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Outline

- Introduction
- Vector Derivative
- Introduction to Fluid Dynamics
- Proposal of an Extension of the Stream Function
- Conclusion

Introduction

- Many real-world physical problems can be modeled as geometric problems
- Sometimes high-dimensional
- Sometimes requires “exotic” spaces
- Examples: Space-Time, String Theory, Phase Space, ...

Introduction

- Geometric algebra (GA) encodes information that otherwise has to be extracted laboriously
- GA generalizes well-known concepts
  - to higher dimensions and
  - to spaces of arbitrary metric
- Develop a fluid dynamics calculus based on geometric algebra
- Focus on the stream function

Vector Differentiation

- Vector derivative

$$\nabla = \sum \boldsymbol{e}_k (\boldsymbol{e}^k \cdot \nabla) = \sum \boldsymbol{e}_k \frac{\partial}{\partial x^k}$$

  - Vector valued
  - Properties of differential operator
- Directional derivative

$$(\boldsymbol{a} \cdot \nabla) = \frac{1}{2}(\boldsymbol{a} \nabla + \nabla \boldsymbol{a})$$

  - Scalar valued
  - Properties of differential operator
  - Grade preserving

Vector Differentiation

- Special derivatives of vector fields
  - Divergence of ***u(x)***

$$\dot{\nabla}_{x} \cdot \dot{\boldsymbol{u}}(\boldsymbol{x})$$

  - Curl of ***u(x)*** ("vorticity")

$$\dot{\nabla}_{x} \wedge \dot{\boldsymbol{u}}(\boldsymbol{x}) = \xi(\boldsymbol{x})$$

  - Material Derivative of ***u(x)***

$$\frac{D}{Dt} = (\boldsymbol{u}(\boldsymbol{x}, t) \cdot \dot{\nabla}_{x}) \dot{\boldsymbol{u}}(\boldsymbol{x}, t) + \frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t}$$

Introduction to Fluid Dynamics

- A fluid is a mass of moving particles in *n* dimensions
- Initial particle position

  $$\boldsymbol{x_0} = (x^1, \ldots, x^n)$$

- Particle position at time *t*

  $$\varphi(\boldsymbol{x_0}, t) = (\varphi^1(\boldsymbol{x_0}, t), \ldots, \varphi^n(\boldsymbol{x_0}, t))$$

- Velocity vector field describes flow

  $$\frac{\partial \varphi(\boldsymbol{x_0}, t)}{\partial t} = \boldsymbol{u}(\varphi, t)$$

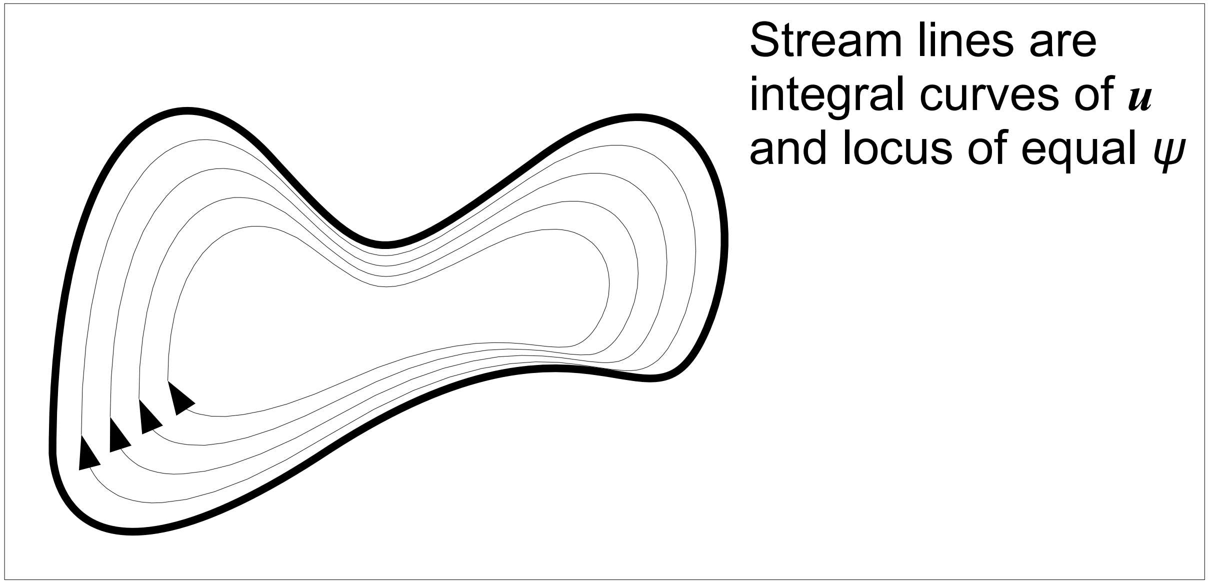
  - Depends on position and time, at which ***u*** is “measured”

Introduction to Fluid Dynamics

- Some properties of a flow are purely geometric
  - e.g. Divergence, Curl
- Others follow from physical reality
  - e.g. Existence of Mass Density, Conservation of Mass and Energy, Balance of Momentum, (In-)Compressibility, Viscosity
  - If flows are used as a model for other problems, only subsets of these physical properties may hold

The Stream Function

- Used to calculate flow potential and streamlines



- Usually defined for 2D flows by

$$\boldsymbol{u}=\nabla\times\psi\,,\quad \boldsymbol{u}=(u_{1,}u_{2,}0)\,,\quad \psi=(0,0,\psi)$$

The Stream Function

- In an *n*-dimensional, incompressible flow there exists a grade *n-2* function *ψ(**x**, t)* with

$$\nabla \wedge \psi = \boldsymbol{u} I_n^{-1}$$

- $$0 = \nabla \wedge (\nabla \wedge \psi)$$
$$= \nabla \wedge (\boldsymbol{u} I_n^{-1})$$
$$= (\nabla \cdot \boldsymbol{u}) I_n^{-1}$$

  which only holds exactly with incompressibility condition fulfilled

$$(\nabla \cdot \boldsymbol{u}) = 0$$

The Stream Function

- Ψ is not determined uniquely by that relation; let

$$\psi' = \psi + \nabla \wedge g, \quad grade(g) = n-3$$

$$\begin{aligned} \boldsymbol{u}' &= (\nabla \wedge \psi') I_n \\ &= \nabla \wedge (\psi + \nabla \wedge g) I_n \\ &= \boldsymbol{u} \end{aligned}$$

- To overcome this specify

$$\nabla \cdot \psi = m,$$

$$\begin{aligned} \nabla \cdot \psi' &= \nabla \cdot (\psi + \nabla \wedge g) \\ &= m + \nabla \cdot (\nabla \wedge g) \neq m \end{aligned}$$

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The Stream Function

- Relationship between Ψ and flow field's curl given by

$$\nabla \wedge \boldsymbol{u} = \nabla \wedge ((\nabla \wedge \psi) I_n)$$
$$= (\nabla \cdot ((\nabla \wedge \psi) I_n I_n)) I_n^{-1}$$
$$= (-1)^{\frac{1}{2} n(n-1)} (\nabla \cdot (\nabla \wedge \psi)) I_n^{-1}$$

- In 2D this simplifies to the known stream function with

$$\nabla \wedge \boldsymbol{u} = -\nabla^2 \psi I_n^{-1}$$
$$\nabla \cdot \psi = 0$$

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Conclusion

- Fluid dynamics is accessible to GA methods
- GA facilitates algebraic proofs and geometric understanding of fluid dynamics
- GA is suitable to generalize known concepts to arbitrary dimensions and “exotic” spaces

Thank you for your attention