Weighted Tree Automata over Multioperator Monoids

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May 17, 2006

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4 Rational Operations and Expressions

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Motivation

- Wta important in a number of applications (NLP)
- No general determinization for weighted wta
- Usually nondeterminism needed for projection
- Simple essential functions difficult to implement
- Predetermined order of multiplication
- Aim: Systematic study of general-weighted wta

Multioperator-Monoid

Definition (Kuich '98)

$(A, +, 0, \Omega)$ Multioperator-Monoid (short: M-monoid), if

- $(A, +, 0)$ commutative monoid
- (A, Ω) algebra

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Example (Kuich '98)

Let (A, Σ) Σ-algebra. Then $(\mathcal{P}(A), \cup, \emptyset, \Omega)$ M-monoid with $\Omega = \{ \overline{\sigma} \mid \sigma \in \Sigma \}$ where

$$\overline{\sigma} \colon \mathcal{P}(A)^{\mathrm{rk}(\sigma)} \to \mathcal{P}(A)$$
$$(S_1, \ldots, S_k) \mapsto \{ \sigma(s_1, \ldots, s_k) \mid s_1 \in S_1, \ldots, s_k \in S_k \}$$

Distributive M-monoid

Definition (Kuich '98)

$(A, +, 0, \Omega)$ distributive M-monoid (short DM-monoid), if

- $(A, +, 0, \Omega)$ M-monoid
- for every $\omega \in \Omega$
$$\omega(\dots, 0, \dots) = 0$$
- for every $\omega \in \Omega$
$$\omega(\dots, a + b, \dots) = \omega(\dots, a, \dots) + \omega(\dots, b, \dots)$$

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Example (Kuich '98)

$(\mathcal{P}(A), \cup, \emptyset, \Omega)$ as before is DM-monoid

Semiring as DM-monoid

Example (Kuich '98)

Let $(A, +, \cdot, 0, 1)$ semiring. Then $(A, +, 0, \Omega)$ DM-monoid with $\Omega = \{ \overline{a}^k \mid a \in A, k \in \mathbb{N} \}$ where

$$\overline{a}^k \colon A^k \to A$$
$$(a_1, \ldots, a_k) \mapsto a_1 \cdot \ldots \cdot a_k \cdot a$$

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This DM-monoid is used to "simulate" weighted tree automata.

A DM-monoid of Tree Series

Definition

Let $(A, +, \cdot, 0, 1)$ semiring, $\psi, \psi_1, \dots, \psi_n \in A\langle T_\Sigma(Z)\rangle$

$$\psi \leftarrow (\psi_1, \dots, \psi_n) = \sum_{t, t_1, \dots, t_n \in T_\Sigma(Z)} (\psi, t) \cdot (\psi_1, t_1) \cdot \ldots \cdot (\psi_n, t_n).t[t_1, \dots, t_n]$$

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Example (M. '04)

Let $(A, +, \cdot, 0, 1)$ semiring. Then $(A\langle T_\Sigma(Z)\rangle, +, \widetilde{0}, \Omega)$ DM-monoid with $\Omega = \{\, \overline{\psi}^k \mid \psi \in A\langle T_\Sigma(Z)\rangle, k \in \mathbb{N} \,\}$ where

$$\overline{\psi}^k \colon A\langle T_\Sigma(Z)\rangle^k \to A\langle T_\Sigma(Z)\rangle$$
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This DM-monoid is used to “simulate” (polynomial) tree series transducers.

Weighted Tree Automata

Definition (M. ’04)

$(Q, \Sigma, Z, \underline{A}, F, \mu, \nu)$ weighted tree automaton (short: wta), if

- Q finite set
- Σ ranked alphabet
- Z finite set (of *variables*)
- $\underline{A} = (A, +, 0, \Omega)$ M-monoid

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$$\mu_k \colon \Sigma^{(k)} \to \left(\Omega^{(k)}\right)^{Q^k \times Q}$$

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- $\nu\colon Z \to \left(\Omega^{(1)}\right)^Q$

Example Wta

Example

DM-monoid $\underline{T} = (\mathbb{N} \cup \{\infty\}, \min, \infty, \Omega)$ with $\Omega = \{\, \overline{\mathrm{m}}^k \mid k \in \mathbb{N}\,\} \cup \{\mathrm{id}\}$ where

$$\overline{\mathrm{m}}^k \colon \mathbb{N}^k \longrightarrow \mathbb{N}$$
$$(n_1, \ldots, n_k) \mapsto 1 + \max(n_1, \ldots, n_k)$$

Wta $(\{\star\}, \Sigma, Z, \underline{T}, F, \mu, \nu)$ with

- $F_\star = \mathrm{id}$
- $\mu_k(\sigma)_{(\star,\ldots,\star),\star} = \overline{\mathrm{m}}^k$
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- This wta computes the height of the input tree deterministically.
- How do we handle variables?

Towards Uniform Mappings

Definition

$(\mathrm{Ops}(A), +_{\mathrm{ops}}, 0_{\mathrm{ops}}, \Omega_{\mathrm{ops}})$ with $\mathrm{Ops}^k(A) = A^{A^k}$ and $\mathrm{Ops}(A) = \bigcup_{i \in \mathbb{N}} \mathrm{Ops}^i(A)$ and

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- $\Omega_{\mathrm{ops}} = (\omega^{l_1,\dots,l_k}_{\mathrm{ops}})_{k\in\mathbb{N},\omega\in\Omega^{(k)},l_1,\dots,l_k\in\mathbb{N}}$ where

$$\omega^{l_1,\dots,l_k}_{\mathrm{ops}} : \mathrm{Ops}^{l_1}(A) \times \cdots \times \mathrm{Ops}^{l_k}(A) \to \mathrm{Ops}^{l_1+\cdots+l_k}(A)$$
$$(\omega_1, \dots, \omega_k) \mapsto f \quad \text{with} \quad f(\vec{a_1}, \dots, \vec{a_k}) = \omega(\omega_1(\vec{a_1}), \dots, \omega_k(\vec{a_k}))$$

Observations

Definition

$\underline{A}$ sum closed, if $(\omega_1 +_{\mathrm{ops}}^{k} \omega_2) \in \Omega^{(k)}$ for every $\omega_1, \omega_2 \in \Omega^{(k)}$

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Theorem

Let $(A, +, 0, \Omega)$ *M-monoid,* $l = l_1 + \cdots + l_k$

$$(\omega +^k_{\text{ops}} \omega')^{l_1,\ldots,l_k}_{\text{ops}}(\omega_1, \ldots, \omega_k)$$
$$= \omega^{l_1,\ldots,l_k}_{\text{ops}}(\omega_1, \ldots, \omega_k) +^l_{\text{ops}} (\omega')^{l_1,\ldots,l_k}_{\text{ops}}(\omega_1, \ldots, \omega_k)$$

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Let $(A, +, 0, \Omega)$ *DM-monoid,* $l = l_1 + \cdots + l_k$

$$\omega_{\mathrm{ops}}^{l_1,\ldots,l_k}(\ldots, \omega_1 +_{\mathrm{ops}}^{l_j} \omega_2, \ldots) = \omega_{\mathrm{ops}}^{l_1,\ldots,l_k}(\ldots, \omega_1, \ldots) +_{\mathrm{ops}}^{l} \omega_{\mathrm{ops}}^{l_1,\ldots,l_k}(\ldots, \omega_2, \ldots)$$

Uniform Mappings

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Definition

Let $(A, +, 0)$ monoid. Monoid $(\mathrm{Umaps}(\Sigma, Z, A), +^{\mathrm{u}}, \widetilde{0}^{\mathrm{u}})$ with

- $\mathrm{Umaps}(\Sigma, Z, A) = \{\, \psi\colon T_\Sigma(Z) \to \mathrm{Ops}(A) \mid \psi$ uniform $\}$
- $(\psi +^{\mathrm{u}} \psi', t) = (\psi, t) +^{|t|_Z}_{\mathrm{ops}} (\psi', t)$
- $(\widetilde{0}^{\mathrm{u}}, t) = 0^{|t|_Z}_{\mathrm{ops}}$

Semantics of Wta

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Let $M = (Q, \Sigma, Z, \underline{A}, F, \mu, \nu)$ wta.

- $R_M = T_{\langle \Sigma, Q \rangle}(\langle Z, Q \rangle)$ runs of M

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- weight of run r

$$c_M(\langle \sigma, q \rangle(r_1, \ldots, r_k)) = (\mu_k(\sigma)_{(q_1, \ldots, q_k), q})_{\text{ops}}^{l_1, \ldots, l_k}(c_M(r_1), \ldots, c_M(r_k))$$
$$c_M(\langle z, q \rangle) = \nu(z)_q$$

where $q_i = \pi_2(r_i(\varepsilon))$

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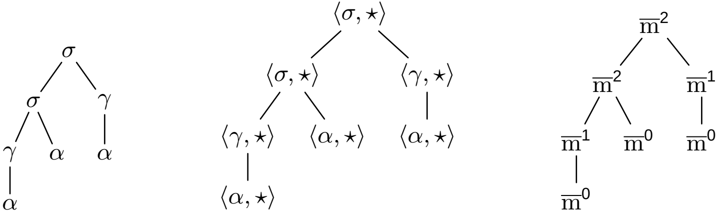
- $(S(M)_q, t) = \sum^{\mathrm{u}}_{r \in R_M(t,q)} c_M(r)$
- $(S(M), t) = \sum^{\mathrm{u}}_{q \in Q} (F_q)^{l}_{\text{ops}}((S(M)_q, t))$

Illustration of Runs

- $t = \sigma(\sigma(\gamma(\alpha), \alpha), \gamma(\alpha))$
- 

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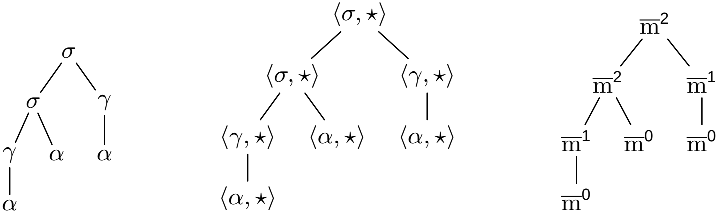
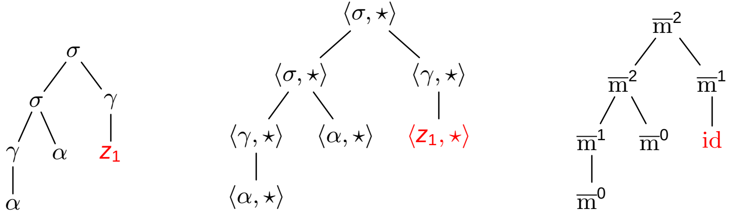
- 
- $(S(M), t) = \text{height}(t) = 4$

An Input Tree With Variables

- $t' = \sigma(\sigma(\gamma(\alpha), \alpha), \gamma(z_1))$
- 

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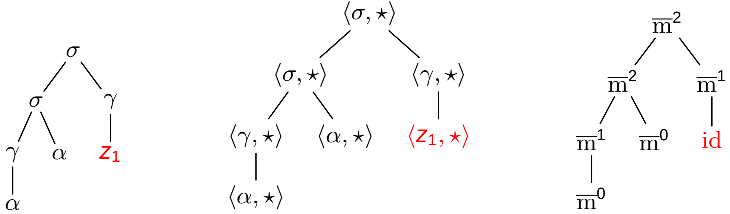
- 

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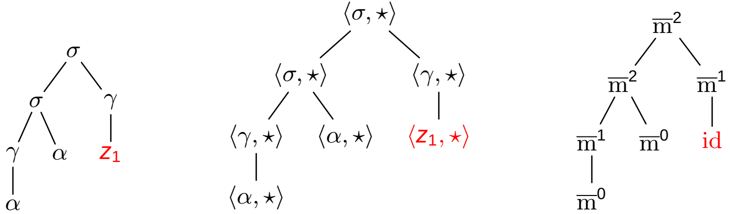
- 

- $(S(M), t') = \lambda x.2 + \max(2, x)$
- or equivalently: $(S(M), t') = \omega$ such that for every $t \in T_\Sigma$

$$\omega(\text{height}(t)) = \text{height}(t'[t])$$

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$$\mathrm{top}_{\sigma,\omega}(\psi_1, \ldots, \psi_k) = \sum^{\mathrm{u}}_{t_1,\ldots,t_k \in T_\Sigma(Z)} \omega^{l}_{\mathrm{ops}}((\psi_1, t_1), \ldots, (\psi_k, t_k)).\sigma(t_1, \ldots, t_k)$$

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- z-catenation $\cdot_z$ ($z \in Z$)

$$\psi \cdot_z \psi' = \sum^{\mathrm{u}}_{s,t_1,\dots,t_k \in T_\Sigma(Z)} \Big((\psi, s) \circ_{W,V} ((\psi', t_1), \dots, (\psi', t_k))\Big).s[z \leftarrow (t_1, \dots, t_k)]$$

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- z-iteration $[\cdot]^*_z$ ($z \in Z$) by $(\psi^*_z, t) = (\psi_z^{\mathrm{height}(t)+1}, t)$

$$\psi^0_z = \widetilde{0}^{\mathrm{u}} \quad \text{and} \quad \psi^{n+1}_z = (\psi \cdot_z \psi^n_z) +^{\mathrm{u}} \mathrm{id}.z$$

Rational Expressions

Definition

$\mathrm{Rat}(\Sigma, Z, \underline{A})$ smallest R s.t.

- $\omega.z \in R$ and $[\![\omega.z]\!] = \omega.z$ ($z \in Z$ and $\omega \in \Omega^{(1)}$)

Rational Expressions

Definition

$\mathrm{Rat}(\Sigma, Z, \underline{A})$ smallest R s.t.

- $\omega.z \in R$ and $[\![\omega.z]\!] = \omega.z$ ($z \in Z$ and $\omega \in \Omega^{(1)}$)
- $\mathrm{top}_{\sigma,\omega}(r_1, \ldots, r_k) \in R$ provided that $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $\omega \in \Omega^{(k)}$, and $r_1, \ldots, r_k \in R$

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- Provided that $r \in R$ and $([\![r]\!], z) = 0^1_{\mathrm{ops}}$ then $r^*_z \in R$ and $[\![r^*_z]\!] = [\![r]\!]^*_z$

Z-normalized Wta

Definition

Wta $(Q, \Sigma, Z, \underline{A}, F, \mu, \nu)$ Z-normalized, if for every $z \in Z$

- $\nu(z)_q \in \{0^1_{\text{ops}}, \text{id}\}$
- $\nu(z)_q \neq 0^1_{\text{ops}}$ for at most one $q \in Q$

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Example

Let $M = (Q, \Sigma, \emptyset, \Sigma, \underline{A}, F, \mu, \nu)$ wta. Then

$$M' = (Q, \Sigma, Q, \Sigma, \underline{A}, F, \mu, \nu')$$

where for every $q \in Q$

- $\nu'(q)_q = \mathrm{id}$ and $\nu'(q)_p = 0^1_{\mathrm{ops}}$ for every $p \in Q \setminus \{q\}$

is Q-normalized. Note $S(M')|_{T_\Sigma} = S(M)$.

The Central Recursion

Definition

Let $M = (Q, \Sigma, Z, \underline{A}, F, \mu, \nu)$ wta

$$(S(M)_q^P, t) = \begin{cases} \sum^{\mathrm{u}}_{r \in R_M^P(t,q)} c_M(r) & \text{if } t \in T_\Sigma(Z) \setminus Z, \\ 0^1_{\mathrm{ops}} & \text{if } t \in Z \end{cases}$$

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Theorem (cf. Droste, Pech, Vogler '05)

Let $\underline{A}$ *DM-monoid,* $M = (Q, \Sigma, Q, \underline{A}, F, \mu, \nu)$ Q*-normalized wta,* $\nu(q)_q = \mathrm{id}$ *for every* $q \in Q$*,* $P \subseteq Q$*,* $p \in Q \setminus P$

$$S(M)_q^{P \cup \{p\}} = S(M)_q^P \cdot_p \left(S(M)_p^P\right)_p^*$$

The Main Theorem

Theorem

Let $\underline{A}$ *DM-monoid,* $M = (Q, \Sigma, \emptyset, \underline{A}, F, \mu, \nu)$ *wta*

$$S(M) \in \underline{A}^{\mathrm{rat}}\langle\langle T_\Sigma(Q)\rangle\rangle|_{T_\Sigma}$$

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Proof.

Recursion as seen and

$$S(M)^{\emptyset}_q = \sum^{\mathrm{u}}_{k\in\mathbb{N},\sigma\in\Sigma^{(k)},q_1,\dots,q_k\in Q} \mathrm{top}_{\sigma,\mu_k(\sigma)_{(q_1,\dots,q_k),q}}(\mathrm{id}.q_1,\dots,\mathrm{id}.q_k)$$

□

Main Corollary

Corollary

Let $\underline{A}$ DM-monoid

$$\underline{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle \subseteq \underline{A}^{\mathrm{rat}}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle|_{T_\Sigma}$$

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Definition

Let $\psi \in A\langle\langle T_\Sigma\rangle\rangle$, $t \in T_\Sigma(Z)$

$$(\text{lift}_Z(\psi), t) = \begin{cases} (\psi, t) & \text{if } t \in T_\Sigma \\ 0_{\text{ops}}^{|t|_Z} & \text{otherwise} \end{cases}$$

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Theorem

Let $\underline{A}$ DM-monoid

$$\mathrm{lift}_{Q_\infty}(\underline{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle) \subseteq \underline{A}^{\mathrm{rat}}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle$$

Open Problem, but about to be solved

Conjecture

$\underline{A}$ DM-monoid

$$\underline{A}^{\mathrm{rat}}\langle\langle T_\Sigma(Z)\rangle\rangle \subseteq \underline{A}^{\mathrm{rec}}\langle\langle T_\Sigma(Z)\rangle\rangle$$

References

- W. Kuich: *Formal Power Series over Trees.* DLT'98, p. 61–101. University of Thessaloniki, 1998.
- A. Maletti: *Relating Tree Series Transducers and Weighted Tree Automata.* DLT'04, LNCS 3340, p. 321–333. Springer, 2004.
- M. Droste, Ch. Pech, H. Vogler: *Kleene Theorem for Recognizable Tree Series.* Theory of Computing Systems 38: 1–38, 2005.
- Z. Fülöp, A. Maletti, H. Vogler: *Weighted Tree Automata over Multioperator Monoids.* Manuscript, 2006.

Thank you for your attention!