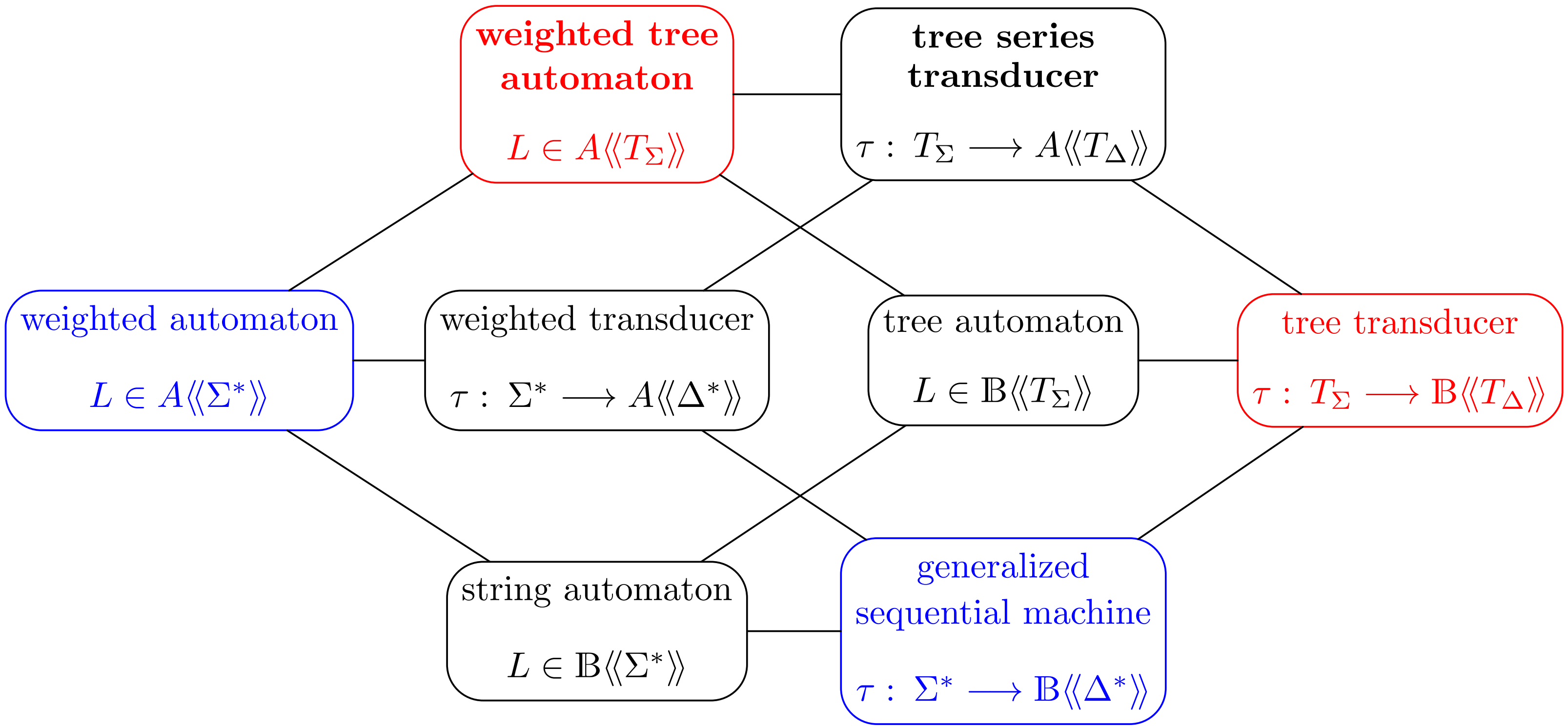
Relating Tree Transducers and Weighted Tree Automata

Andreas Maletti

June 2, 2004

1. Introduction
2. Bottom-Up Tree Transducers
3. Bottom-Up Weighted Tree Automata
4. Establishing a Relationship

Generalization Hierarchy



Known Relations and Problems

- *String-based:*

  Theorem: For every generalized sequential machine M, there exists a weighted automaton N such that $\|M\| = \|N\|$.

  Required semiring: Start with the monoid $(\Delta^*, \circ, \varepsilon)$ and extend it to the semiring $(\mathbb{B}\langle\langle \Delta^* \rangle\rangle, \vee, \circ, \widetilde{0}, 1\,\varepsilon) \cong (\mathcal{P}(\Delta^*), \cup, \circ, \emptyset, \{\varepsilon\})$.

  Theorem: For every weighted transducer M, there exists a weighted automaton N such that $\|M\| = \|N\|$.

- *Tree-based:*

  Problem: For every tree transducer M, does there exist a weighted tree automaton N such that $\|M\| = \|N\|$?

  Problem: For every tree series transducer M, does there exist a weighted tree automaton N such that $\|M\| = \|N\|$?

Tree Transducer — Syntax

A (bottom-up) tree transducer $M = (Q, \Sigma, \Delta, F, \mu)$ consists of

- non-empty finite sets Q and $F \subseteq Q$ of *states* and *final states*,
- ranked alphabets Σ and Δ, and
- a tree representation $\mu = (\mu_k)_{k \in \mathbb{N}}$ of mappings $\mu_k : \Sigma^{(k)} \longrightarrow \mathbb{B}\langle\langle T_\Delta(X_k)\rangle\rangle^{Q \times Q^k}$.

Example: Let $Q = F = \{q\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, $\Delta = \{1^{(1)}, 2^{(1)}, \varepsilon^{(0)}\}$, and

$$\mu_0(\alpha)_q = \{\varepsilon\} \qquad \mu_2(\sigma)_{q,(q,q)} = \{\varepsilon, 1(x_1), 2(x_2)\}.$$

Then $M_1 = (Q, \Sigma, \Delta, F, \mu)$.

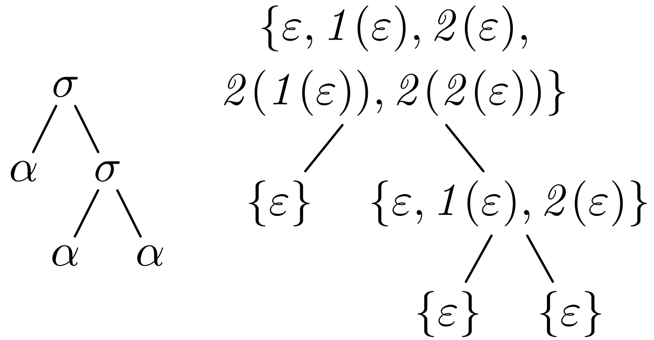
Tree Transducer — Semantics

Define $h_\mu : T_\Sigma \longrightarrow \mathbb{B}\langle\langle T_\Delta \rangle\rangle$ as

$$h_\mu(\sigma(s_1, \ldots, s_k))_q = \bigcup_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \longleftarrow_{\mathrm{IO}} (h_\mu(s_1)_{q_1}, \ldots, h_\mu(s_k)_{q_k})$$

and $\|M\| : T_\Sigma \longrightarrow \mathbb{B}\langle\langle T_\Delta \rangle\rangle$ as $(\|M\|, s) = \bigcup_{q \in F} h_\mu(s)_q$.

Example: $(\|M_1\|, s) = \mathrm{pos}(s)$.



Weighted Tree Automata — Syntax

A (bottom-up) weighted tree automaton $M = (Q, \Sigma, \mathcal{A}, F, \mu)$ consists of

- non-empty finite sets Q and $F \subseteq Q$ of *states* and *final states*,
- a ranked alphabet Σ,
- a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, and
- a tree representation $\mu = (\mu_k)_{k \in \mathbb{N}}$ of mappings $\mu_k : \Sigma^{(k)} \longrightarrow A^{Q \times Q^k}$.

Example: Let $Q = \{q, p\}$, $F = \{q\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$,
$\mathrm{Arct} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, and

$$\mu_0(\alpha) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{matrix} q \\ p \end{matrix} \qquad \mu_2(\sigma) = \begin{matrix} (q,q) & (q,p) & (p,q) & (p,p) \end{matrix} \atop \begin{pmatrix} -\infty & 1 & 1 & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \begin{matrix} q \\ p \end{matrix}$$

Then $M_2 = (Q, \Sigma, \mathcal{A}, F, \mu)$.

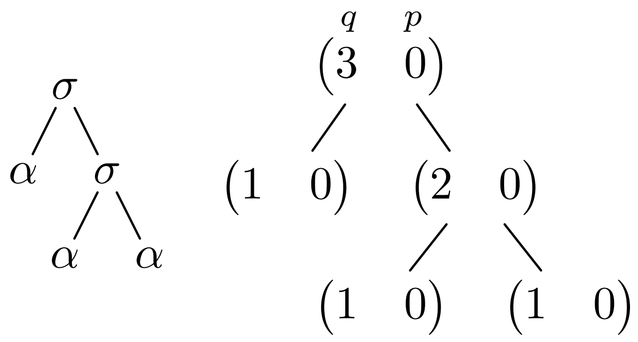
Weighted Tree Automata — Semantics

The mapping $h_\mu : T_\Sigma \longrightarrow A^Q$ is defined componentwise as

$$h_\mu(\sigma(s_1, \ldots, s_k))_q = \sum_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{q,(q_1, \ldots, q_k)} \odot \prod_{i \in [k]} h_\mu(s_i)_{q_i}.$$

The *tree series computed by* M is $(\|M\|, s) = \sum_{q \in F} h_\mu(s)_q$.

Example: Then M_2 computes $(\|M_2\|, s) = \mathrm{height}(s)$.



Establishing a Relationship I

Theorem: For every bottom-up tree transducer M, there exists a bottom-up weighted tree automaton N such that $\|M\| = \|N\|$.

Required semiring: Start with the monoid $(B, \longleftarrow, \{\varepsilon\})$ where

$$B = \mathbb{B}\langle\langle T_\Delta \rangle\rangle^* \circ \Big(\{\varepsilon\} \cup \{ (k, S) \mid k \in \mathbb{N}_+, S \in \mathbb{B}\langle\langle T_\Delta(X_k) \rangle\rangle \} \Big)$$

and $\longleftarrow : B^2 \longrightarrow B$ is defined for every $a \in \mathbb{B}\langle\langle T_\Delta \rangle\rangle^*$ as

$$\begin{aligned}
a \longleftarrow b &= a.b \\
b \longleftarrow \varepsilon &= b \\
a.(k, S) \longleftarrow T.b &= \begin{cases} a.(S \longleftarrow_{1,0} T).b & \text{, if } k = 1 \\ a.(k-1, S \longleftarrow_{k,0} T) \longleftarrow b & \text{, otherwise} \end{cases} \\
a.(k, S) \longleftarrow (n, T) &= a.(k+n-1, S \longleftarrow_{k,n} T)
\end{aligned}$$

with

$$S \longleftarrow_{k,n} T = S[x_2 \leftarrow x_{n+1}, \ldots, x_k \leftarrow x_{k+n-1}] \longleftarrow_{\mathrm{IO}} T$$

Establishing a Relationship II

Arrive at the semiring $(\mathcal{P}(B), \cup, \longleftarrow, \emptyset, \{\varepsilon\})$. Let $k \in \mathbb{N}$, $n \in \mathbb{N}_+$,

$$C_{k,n} = \{\, S_1 \dots S_k.(n, S) \mid S_1, \dots, S_k \in \mathbb{B}\langle\langle T_\Delta \rangle\rangle, S \in \mathbb{B}\langle\langle T_\Delta(X_n) \rangle\rangle \,\}$$

and $C_{k,0} = \mathbb{B}\langle\langle T_\Delta \rangle\rangle^k$. Clearly for every $C \in \mathcal{P}(B)$ we have the unique partition $C = \bigcup_{k,n \in \mathbb{N}} C'_{k,n}$ for $C'_{k,n} \subseteq C_{k,n}$. Define the homomorphism $\oplus$ on the partitions as follows.

$$S_1 \dots S_k.(n, S) \oplus T_1 \dots T_k.(n, T) = \{(S_1 \cup T_1) \dots (S_k \cup T_k).(n, S \cup T)\}$$

Then $C/_{\ker \oplus}$ is the desired semiring.

Theorem: For every bottom-up tree series transducer M over an idempotent semiring, there exists a bottom-up weighted tree automaton N such that $\|M\| = \|N\|$.

Conclusions

- the study of arbitrary weighted tree automata provides results for tree transducers (tree series transducers)
- e.g., a pumping lemma for tree series transducers can be derived from a pumping lemma for weighted tree automata
- unfortunately, few results for weighted tree automata over non-commutative semirings exist

Thank You for Your Attention.