Tree Series Transducers and Weighted Tree Automata

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December 2, 2003

1. Basic Definitions
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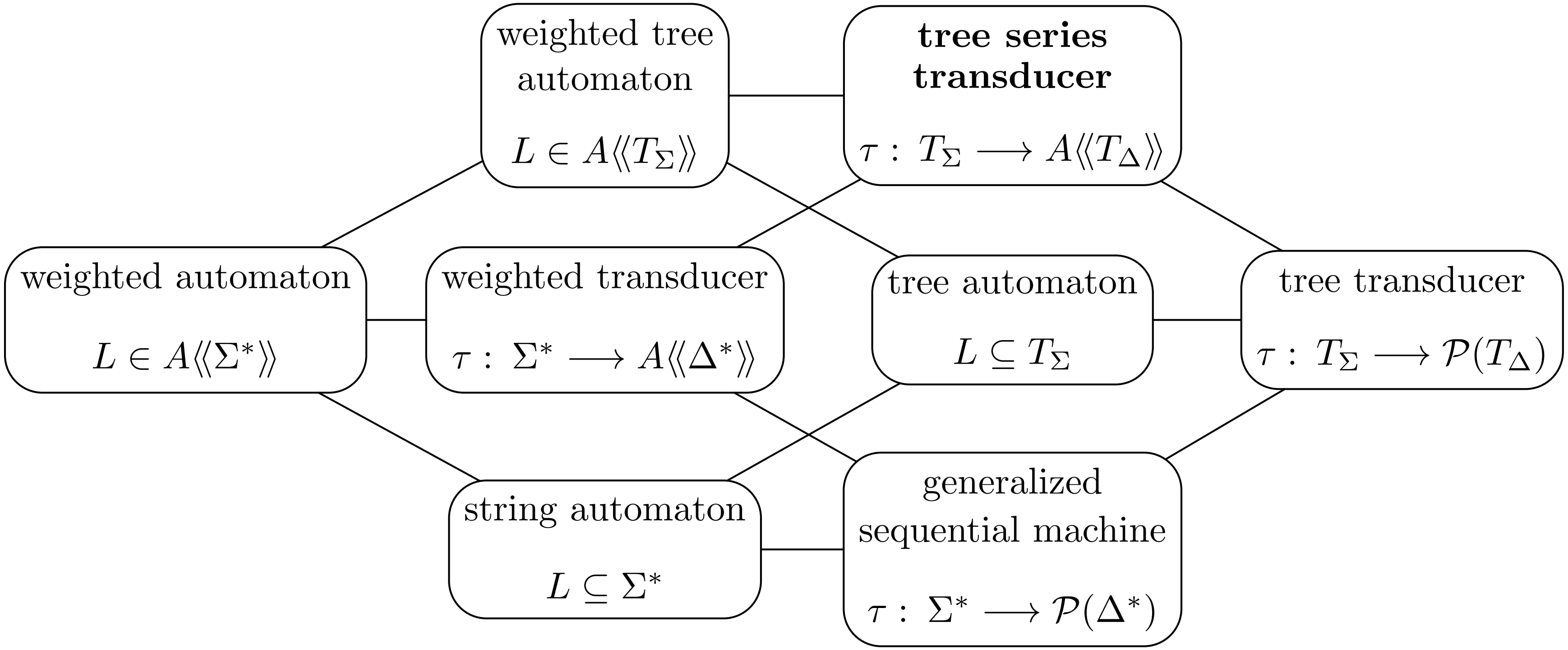
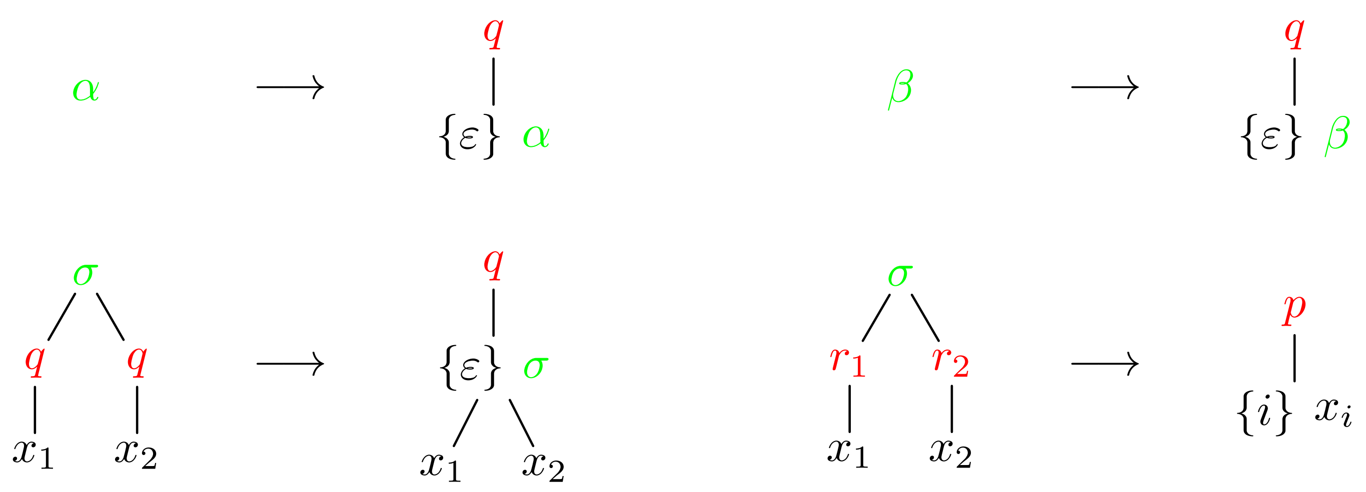
Generalization Hierarchy

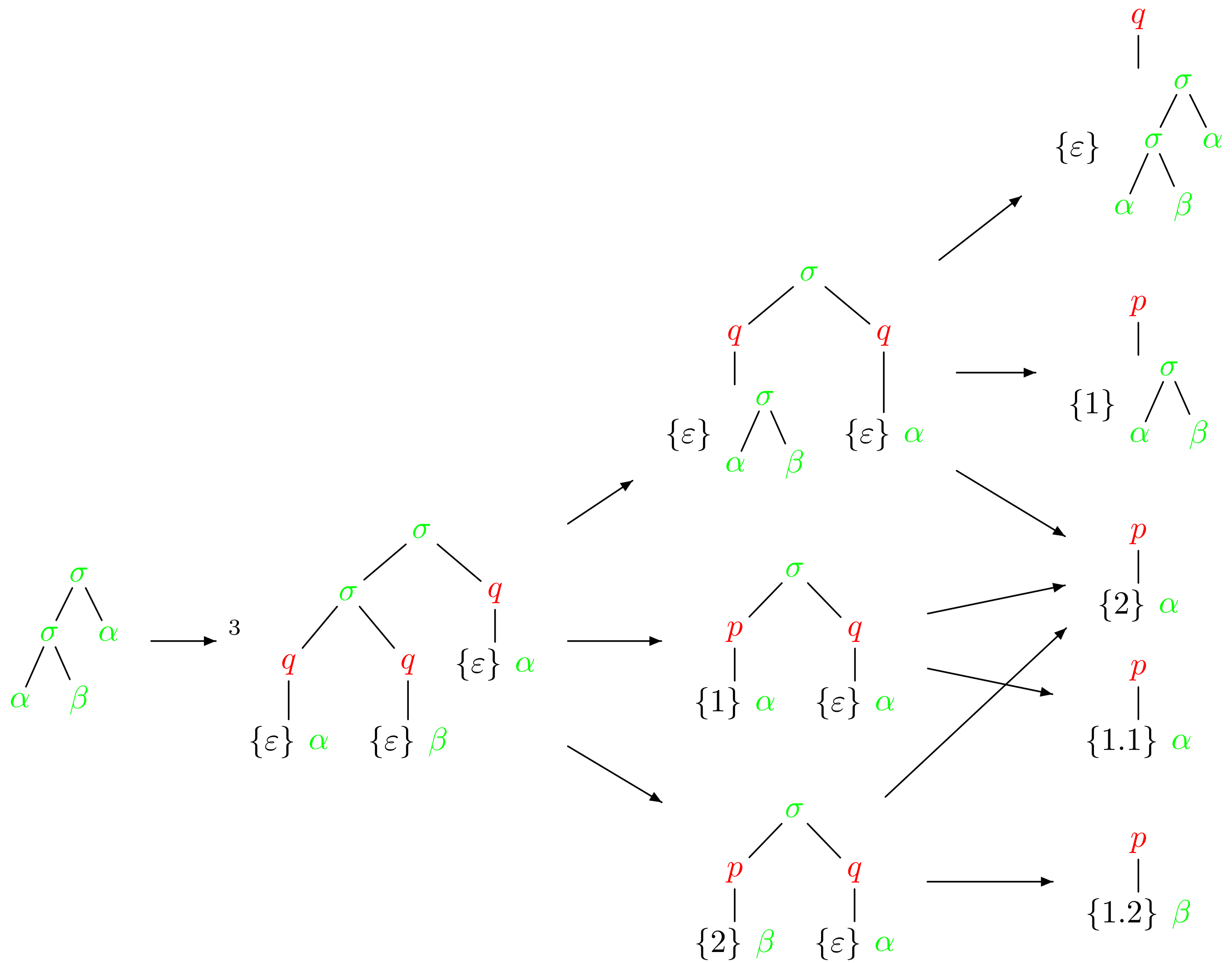


Bottom-Up Tree Series Transducers

$$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$$

- input and output ranked alphabet $\Sigma = \Delta = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$,
- states and final states $Q = F = \{p, q\}$,
- semiring $\mathcal{A} = \mathbb{P} = (\mathcal{P}(\mathbb{N}_+^*), \cup, \circ, \emptyset, \{\varepsilon\})$ with $P_1 \circ P_2 = \{\, ab \mid a \in P_1, b \in P_2 \,\}$, and
- tree representation μ

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Tree Series

- a *tree series* φ is a mapping of type $T_\Delta(V) \longrightarrow A$; (φ, t) is used to denote $\varphi(t)$
- the *class of all tree series* is denoted $A\langle\langle T_\Delta(V)\rangle\rangle$
- the *support* of a tree series φ is defined to be $\mathrm{supp}(\varphi) = \{\, t \in T_\Delta(V) \mid (\varphi, t) \neq \mathbf{0} \,\}$
- φ is *polynomial* iff its support is finite; the corresponding class is $A\langle T_\Delta(V)\rangle$
- Let $\varphi \in A\langle\langle T_\Delta(X_k)\rangle\rangle$, $(\psi_1, \ldots, \psi_k) \in A\langle\langle T_\Delta(V)\rangle\rangle^k$. *Substitution* of $(\psi_1, \ldots, \psi_k)$ into φ is

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_k) = \sum_{\substack{t \in \mathrm{supp}(\varphi) \\ (\forall i \in [k]):\, t_i \in \mathrm{supp}(\psi_i)}} \big((\varphi, t) \odot (\psi_1, t_1) \odot \cdots \odot (\psi_k, t_k)\big) t[t_1, \ldots, t_k].$$

Bottom-up Tree Series Transducers

$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, where

- Q and $F \subseteq Q$ are *finite* sets of states and final states, resp.,
- Σ and Δ are the input and output ranked alphabets, resp.,
- $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring
- μ is a family of mappings $(\mu_k)_{k \in \mathbb{N}}$ of type

$$\mu_k : \Sigma^{(k)} \longrightarrow A\langle\langle T_\Delta(X_k)\rangle\rangle^{Q \times Q^k}.$$

Semantics of Bottom-up Tree Series Transducers

$$\overline{\mu_k(\sigma)} : \left(A\langle\langle T_\Delta \rangle\rangle^{Q}\right)^k \longrightarrow A\langle\langle T_\Delta \rangle\rangle^{Q}$$

$$\overline{\mu_k(\sigma)}(R_1, \ldots, R_k)_q = \sum_{(q_1,\ldots,q_k)\in Q^k} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \longleftarrow \left((R_1)_{q_1}, \ldots, (R_k)_{q_k}\right).$$

Initial homomorphism: $h_\mu : T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle^{Q}$

$$h_\mu(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}(h_\mu(s_1), \ldots, h_\mu(s_k))$$

tree-to-tree-series transformation computed by M is $\tau_M : T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle$

$$\tau_M(s) = \sum_{q\in F} h_\mu(s)_q$$

Bottom-up Weighted Tree Automata

$M = (Q, \Sigma, \mathcal{A}, F, \mu)$, where

- Q and $F \subseteq Q$ are *finite* sets of states and final states, resp.,
- Σ is the input ranked alphabet, resp.,
- $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring
- μ is a family of mappings $(\mu_k)_{k \in \mathbb{N}}$ of type $\mu_k : \Sigma^{(k)} \longrightarrow A^{Q \times Q^k}$.

Semantics is similarly defined as it is for bottom-up tree series transducers.

A Semiring?

Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring. We define the following algebraic structure.

$$B = A\langle\langle T_\Delta \rangle\rangle^* \circ (\{\varepsilon\} \cup \{ (n, \varphi) \mid n \in \mathbb{N}_+, \varphi \in A\langle\langle T_\Delta(X_n) \rangle\rangle \}) \qquad S = \mathbb{N}^B$$

$\mathcal{S} = (S, \cup, \circ, \emptyset, \{\varepsilon\})$ with addition being defined for every element $b \in B$ and every two semiring elements $S_1, S_2 \in S$ by

$$(S_1 \cup S_2)(b) = S_1(b) + S_2(b). \tag{1}$$

This addition is trivially associative, commutative, and has unit element $\emptyset : B \longrightarrow \mathbb{N}$ which is defined for every $b \in B$ to be $\emptyset(b) = 0$.

The multiplication is defined for every element $b \in B$ and every two semiring elements $S_1, S_2 \in S$ by

$$(S_1 \circ S_2)(b) = \sum_{b_1, b_2 \in B,\, b = b_1 \leftarrow b_2} S_1(b_1) \cdot S_2(b_2). \tag{2}$$

Wrapping Substitution

On B we define the following operation $\longleftarrow\,:\ B^2 \longrightarrow B$:

$$\begin{aligned}
a \longleftarrow b &= a.b && \text{, if } a \in A\langle\langle T_\Delta\rangle\rangle^* \text{ or } b = \varepsilon,\\
a.(1,\varphi) \longleftarrow \psi.b &= a.(\varphi \longleftarrow_0 \psi).b,\\
a.(n,\varphi) \longleftarrow \psi.b &= a.(n-1, \varphi \longleftarrow_0 \psi) \longleftarrow b && \text{, if } n > 1,\\
a.(n,\varphi) \longleftarrow (m,\psi) &= a.(n-1+m, \varphi \longleftarrow_m \psi).
\end{aligned}$$

The substitutions $(\longleftarrow_k\,:\ A\langle\langle T_\Delta(X)\rangle\rangle \times A\langle\langle T_\Delta(X_k)\rangle\rangle \longrightarrow A\langle\langle T_\Delta(X)\rangle\rangle \mid k \in \mathbb{N})$ are defined as follows.

$$a \longleftarrow_k b \;=\; a[x_i/x_{i+k-1} \,|\, i > 1] \longleftarrow (b)$$

Lemma: $(a \longleftarrow b) \longleftarrow c \;=\; a \longleftarrow (b \longleftarrow c)$.

Lemma: $(\varphi \longleftarrow_m \psi) \longleftarrow_k \tau \;=\; \varphi \longleftarrow_{m-1+k} (\psi \longleftarrow_k \tau)$ with $m \neq 0$.

Remaining Questions and Literature

- Deterministic tree series transducers?
- Tree transducers, i.e., polynomial tree series transducers over $\mathbb{B}$
- Top-down tree series transducers?
- o-substitution?

Some References:

Seidl: *Finite Tree Automata with Cost Functions*, 1994

Kuich: *Formal Power Series over Trees*, 1997

Engelfriet, Fülöp, Vogler: *Bottom-up and Top-down Tree Series Transformations*, 2002

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