Analyzing Real Vector Fields with Clifford Convolution and Clifford Fourier Transform

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Abstract Post-processing in computational fluid dynamics as well as processing of fluid flow measurements needs robust methods that can deal with scalar as well as vector fields. While image processing of scalar data is a well-established discipline, there is a lack of similar methods for vector data. This paper surveys a particular approach defining convolution operators on vector fields using geometric algebra. This includes a corresponding Clifford Fourier transform including a convolution theorem. Finally, a comparison is tried with related approaches for a Fourier transform of spatial vector or multivector data. In particular, we analyze the Fourier series based on quaternion holomorphic functions of Gürlebeck et al., the quaternion Fourier transform of Hitzer et al. and the biquaternion Fourier transform of Sangwine et al.

1 Fluid Flow Analysis

Fluid flow, especially of air and water, is usually modelled by the Navier-Stokes equations or simplifications like the Euler equations [1]. The physical fields in this model include pressure, density, velocity and internal energy [17]. These variables depend on space and often also on time. While there are mainly scalar fields, velocity is a vector field and of high importance for any analysis of numerical or physical experiments. Some numerical simulations use a discretization of the spatial domain and calculate the variables at a finite number of positions on a regular lattice (finite difference methods). Other methods split space into volume elements and assume a polynomial solution of a certain degree in each volume element (finite element methods or finite volume methods). These numerical methods create a large amount of data and its analysis, i.e. post-processing, usually uses computer graphics to cre-

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ate images. Research about this process is an important part of scientific visualization [22].
Besides numerical simulations, modern measurement techniques like particle image velocimetry (PIV) [8] create velocity vector field measurements on a regular lattice using laser sheets and image processing. Therefore, simulations and experiments in fluid mechanics create discretized vector fields as part of their output. The analysis of these fields is an important post-processing task. In the following, we will assume that there is a way to continuously integrate the data to simplify notations. Of course, one can also discretize the integrals conversely.
Flow visualization knows a lot of direct methods like hedgehogs, streamlines or streamsurfaces [22], but also techniques based on mathematical data analysis like topology or feature detection methods [10, 21, 18]. Since these methods are often not very robust, a transfer of image processing to vector data is an attractive approach. A look into any image processing book, e. g. by Jähne [15], reveals the importance of shift invariant linear filters based on convolution. Of course, there is vector data processing in image processing, but the usual techniques for optical flow, which is the velocity field which warps one image into another, do not really help as they concentrate mainly on non-continuities in the field which is not a typical event in fluid flow velocity fields.

2 Geometric Algebra

In classical linear algebra, there are several multiplications involving vectors. Some multiplications have led to approaches for a convolution on vector fields. Scalar multiplication can be easily applied and can be seen as a special case of componentwise multiplication of two vectors which has been used by Granlund and Knutsson [9]. The scalar product has been used by Heiberg et al. [11] as convolution operator. Obviously, one would like a unified convolution operator that incorporates these approaches and can be applied several times, in contrast to the scalar product version that creates a scalar field using two vector fields. Furthermore, one looks for all the nice theorems like convolution theorem with a suitable generalized Fourier transform, derivation theorem, shift theorem, and Parseval's theorem. Geometric algebra allows such a convolution [5, 6].
Let $\mathbb{R}^d,\ d=2,3$, be the Euclidean space with the orthonormal basis

$$\{e_1,e_2\} \text{ resp. } \{e_1,e_2,e_3\}. \tag{1}$$

We use the 2^d-dimensional real geometric algebras $G_d, d=2,3$, which have a associative, bilinear geometric product satisfying

$$1e_j = e_j, \qquad j=1,\dots,d \tag{2}$$

$$e_j^2 = 1, \qquad j=1,\dots,d \tag{3}$$

$$e_je_k = -e_ke_j, \quad j,k=1,\dots,d, j\neq k \tag{4}$$

Their basis is given by

$$\{1, e_1, e_2, i_2 := e_1e_2\} \quad \text{resp.} \quad \{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, i_3 := e_1e_2e_3\} \tag{5}$$

It can be verified quickly that the squares of i_2, i_3 and those of the bivectors e_1e_2, e_2e_3 and e_3e_1, are -1. General elements in G_d are called multivectors while elements of the form

$$v = \sum_{j=1}^{d} \alpha_j e_j \tag{6}$$

are called vectors, i. e. $v \in \mathbb{R}^d \subset G_d$. The geometric product of two vectors $a, b \in \mathbb{R}^d$ results in

$$ab = a \cdot b + a \wedge b \tag{7}$$

where $\cdot$ is the inner product and $\wedge$ is the outer product. We have

$$a \cdot b = |a||b|\cos(\alpha) \tag{8}$$

$$|a \wedge b| = |a||b|\sin(\alpha) \tag{9}$$

with the usual norm for vectors and the angle α from a to b.
Let F be a multivector-valued function (field) of a vector variable x defined on some region G of the Euclidean space $\mathbb{R}^d$, compare (19) for $d = 3$ and (23) for $d = 2$. We define the Riemann integral of F by

$$\int_G F(x)|dx| = \lim_{|\Delta x| \to 0, n \to \infty} \sum_{j=1}^{n} F(x_j)|\Delta x_j| \tag{10}$$

We define $\Delta x = dx_1 \wedge dx_2 i_2^{-1}$ $(d = 2)$ resp. $\Delta x = dx_1 \wedge dx_2 \wedge dx_3 i_3^{-1}$ $(d = 3)$ as the dual oriented scalar magnitude. The quantity $|\Delta x|$ is used here to make the integral grade preserving since dx is a vector within geometric algebra in general.

The directional derivative of F in direction r is

$$F_r(x) = \lim_{h \to 0} \frac{F(x + hr) - F(x)}{h} \tag{11}$$

with $r \in \mathbb{R}^3$, $h \in \mathbb{R}$. With the vector derivative

$$\nabla = \sum_{j=1}^{d} e_j \frac{\partial}{\partial e_j} \tag{12}$$

(vector valued), the complete (left) derivative of F is defined as

$$\nabla F(x) = \sum_{j=1}^{d} e_j F_{e_j}(x). \tag{13}$$

Similarly, we define the complete right derivative as

$$F(x)\nabla = \sum_{j=1}^{d} F_{e_j}(x)e_j. \tag{14}$$

Curl and divergence of a vector field f can be computed as scalar and bivector parts of (12)

$$\mathrm{curl} f = \nabla \wedge f = \frac{1}{2}(\nabla f - f\nabla), \quad \mathrm{div} f = \nabla \cdot f = \frac{1}{2}(\nabla f + f\nabla). \tag{15}$$

Readers interested in the basics and more applications of geometric algebra are also referred to [12] and [4]. That fluid flow dynamics is accessible by geometric algebra methods is also discussed in [3].

3 Clifford Convolution

Let $V, H : \mathbb{R}^d \to G_d$ be two multivector fields. As **Clifford convolution**, we define

$$(H * V)(r) := \int_{\mathbb{R}^d} H(\xi)V(r-\xi)|d\xi| \tag{16}$$

If both fields are scalar fields, this is the usual convolution in image processing. If H is a scalar field, e.g. a Gaussian kernel, and V a vector field, we get a scalar multiplication and can model smoothing of a vector field. If H is a vector field and V a multivector field, the simple relation

$$H(\xi)V(r-\xi) = H(\xi)\cdot V(r-\xi) + H(\xi)\wedge V(r-\xi) \tag{17}$$

shows that the scalar part of the result is Heiberg's convolution while the bivector part contains additional information. General multivector fields allow a closed operation in G_d, so that several convolutions can be combined . We have shown [5, 7] that this convolution can be used for the analysis of velocity vector fields from computational fluid dynamics (CFD) simulations and PIV measurements.

4 Clifford Fourier Transform

Our group has found a generalization of the Fourier transform of complex signals to multivector fields [6] that allows the generalization of the well-known theorems like the convolution theorem for the convolution defined in the previous section. There are different approaches of transforming multivector valued data, e.g. in [2], [14] and [19]. In section 5 we discuss the relation to ours.
Let $F : \mathbb{R}^d \to G_3$ be a multivector field. We define

$$\mathscr{F}\{F\}(u) := \int_{\mathbb{R}^d} F(x)\exp(-2\pi i_d x \cdot u)|dx| \tag{18}$$

as **Clifford Fourier transform (CFT)** with the inverse

$$\mathscr{F}^{-1}\{F\}(x) := \int_{\mathbb{R}^d} F(u)\exp(2\pi i_d x \cdot u)|du|. \tag{19}$$

In three dimensions, the CFT is a linear combination of four classical complex Fourier transforms as can be seen by looking at the real components. Since

$$\begin{aligned}
F(x) &= F_0(x) + F_1(x)e_1 + F_2(x)e_2 + F_3(x)e_3 + \\
&\quad F_{12}(x)e_1e_2 + F_{23}(x)e_2e_3 + F_{31}(x)e_3e_1 + F_{123}(x)e_1e_2e_3 && (20)\\
&= F_0(x) + F_1(x)e_1 + F_2(x)e_2 + F_3(x)e_3 + F_{12}(x)i_3e_3 + \\
&\quad F_{23}(x)i_3e_1 + F_{31}(x)i_3e_2 + F_{123}(x)i_3 && (21)\\
&= (F_0(x) + F_{123}(x)i_3)1 + (F_1(x) + F_{23}(x)i_3)e_1 + \\
&\quad (F_2(x) + F_{31}(x)i_3)e_2 + (F_3(x) + F_{12}(x)i_3)e_3 && (22)
\end{aligned}$$

holds, we get

$$\begin{aligned}
\mathscr{F}\{F\}(u) = {} & [\mathscr{F}\{F_0 + F_{123}i_3\}(u)]\,1 + && (23)\\
& [\mathscr{F}\{F_1 + F_{23}i_3\}(u)]\,e_1 + \\
& [\mathscr{F}\{F_2 + F_{31}i_3\}(u)]\,e_2 + \\
& [\mathscr{F}\{F_3 + F_{12}i_3\}(u)]\,e_3.
\end{aligned}$$

We have proven earlier [6] that the convolution, derivative, shift, and Parseval theorem hold. For vector fields, we can see that the CFT treats each component as a real signal that is transformed independently from the other components.

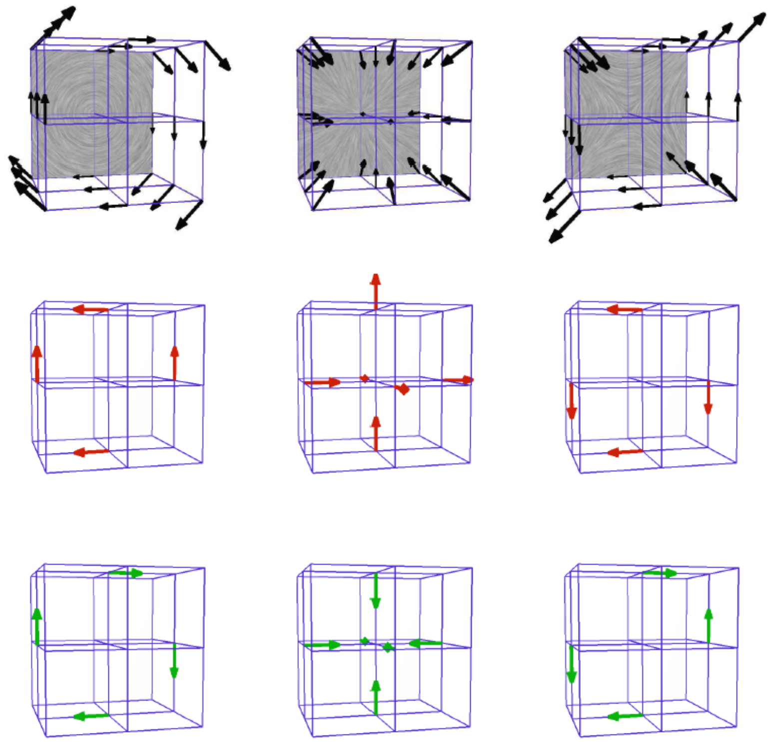

Fig. 1 **Top:** Various 3D patterns. **Middle:** The vector part of their DCFT. **Bottom:** The bivector part of their DCFT, displayed as normal vector of the plane. **Left:** $3 \times 3 \times 3$ rotation in one coordinate plane. **Middle:** $3 \times 3 \times 3$ convergence. **Right:** $3 \times 3 \times 3$ saddle line. The mean value of the discrete Clifford Fourier transform (DCFT) is situated in the center of the field. In 3D, the waves forming the patterns can be easily seen in the frequency domain. The magnitude of the bivectors of the DCFT is only half the magnitude of the corresponding vectors, though both are displayed with same length.

In two dimensions, the CFT is a linear combination of two classical complex Fourier transforms. We have

$$F(x) = F_0(x) + F_1(x)e_1 + F_2(x)e_2 + F_{12}(x)e_1e_2 \quad (24)$$

$$= F_0(x) + F_1(x)e_1 + F_2(x)e_1i_2 + F_{12}(x)i_2 \quad (25)$$

$$= 1[F_0(x) + F_{12}(x)i_2] + e_1[F_1(x) + F_2(x)i_2] \quad (26)$$

and obtain

$$\mathscr{F}\{F\}(u) = 1[\mathscr{F}\{F_0 + F_{12}i_2\}(u)] + e_1[\mathscr{F}\{F_1 + F_2i_2\}(u)]. \quad (27)$$

Regarding the convolution theorem, one has to separate the vector and spinor parts since i_2 does not commute with all algebra elements. With this restriction, the theorem hold again [6].

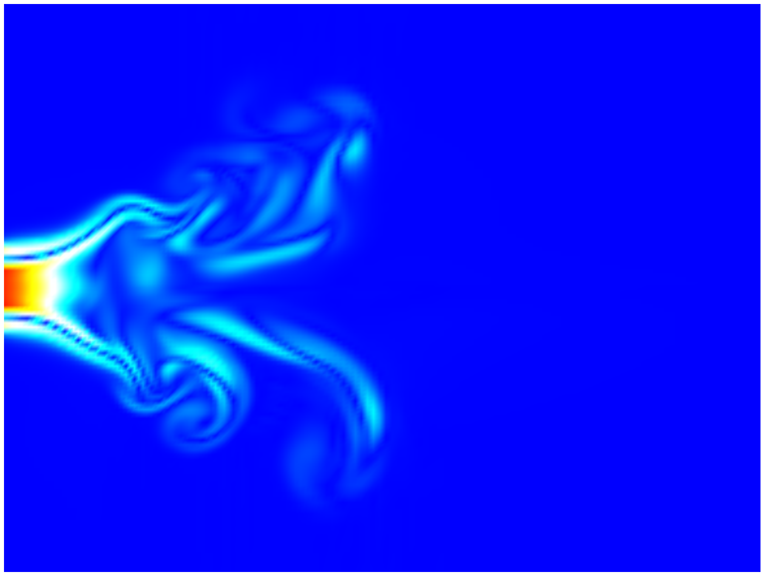

Fig. 2 A 2D slice of a turbulent swirling jet entering a fluid at rest. The image shows color coding of the absolute magnitude of the vectors. The colors are scaled from zero (blue) to the maximal magnitude (red).

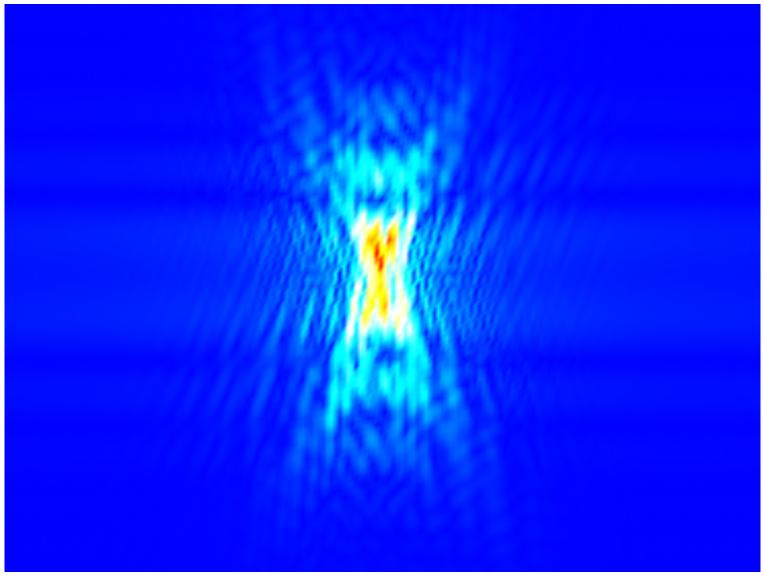

Fig. 3 This image shows a (fast) discrete Clifford Fourier transform of the data set. Zero frequency is located in the middle of the image. Vectors are treated as rotors when using Clifford algebra in the frequency domain, thus color coding is based on the magnitude of the transformed rotor. The scaling of the colors is the same as the last image. We zoomed in to get more information.

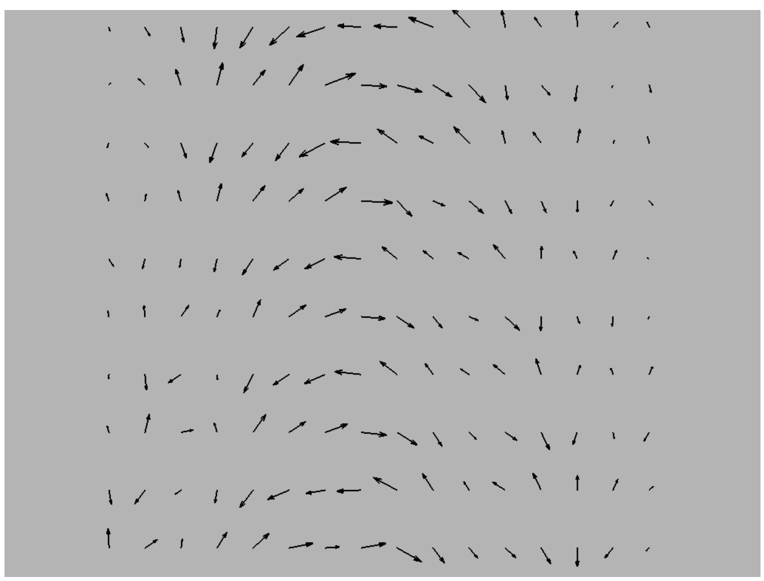

Fig. 4 In that image we zoom in further and take a look at the direction of the ”vectors” in a neighborhood of zero frequency.

5 Relation to other Fourier transforms

In recent years, other definitions of a Fourier transform have appeared in the literature which can be applied to vector fields. In this section, we compare our approach with the Fourier series based on quaternion analysis by Gürlebeck, Habetha and Sprößig [16], the biquaternion Fourier transform by Sangwine et al [20] and the quaternion Fourier transform by Hitzer in [13]. Therefore we use 3 important isomorphisms. First of all quaternions are isomorph to the even subalgebra G_3^+ of G_3 by

$$i \mapsto e_1e_2,\ j \mapsto e_2e_3,\ k \mapsto e_1e_3, \tag{28}$$

biquaternions are isomorph to G_3 by additionally

$$I \mapsto e_1e_2e_3 = i_3. \tag{29}$$

Further the quaternions are isomorph to the Clifford-Algebra $Cl_{0,2}$ of the Anti-Euklidean Space $\mathbb{R}^{0,2}$ by

$$i \mapsto e_1,\ i \mapsto e_2,\ k \mapsto e_1e_2, \tag{30}$$

which we will only use in for the definition of holomorphicity in section 5.1.

5.1 $\mathbb{H}$-holomorphic Functions and Fourier Series

We follow the definitions by Gürlebeck et al. [16] and identify the quaternions as in (29). Let $f : \mathbb{H} \to \mathbb{H}$ be a function with the real partial derivatives $\partial_k := \frac{\partial}{\partial q_k}$ we define the complete real differential as

$$df = \sum_{k=0}^{3} \partial_k f\, dq_k \tag{31}$$

and set

$$dq = dq_0 + \sum_{k=1}^{3} e_k dq_k \quad d\bar{q} = dq_0 - \sum_{k=1}^{3} e_k\, dq_k \tag{32}$$

This leads to

$$df = \frac{1}{2}\left(\sum_{k=0}^{3} \partial_k f\, e_k\right) d\bar{q} + \frac{1}{2}\left(\partial_0 f\, dq - \sum_{k=1}^{3} \partial_k f\, dq\, e_k\right). \tag{33}$$

A real C^1-function f is right $\mathbb{H}$**-holomorphic** in $G \subset \mathbb{H}$ if for every $q \in G$ and $h \to 0$ exist $a_k(q) \in \mathbb{H}$ with

$$f(q+h) = f(q) + \sum_{k=1}^{3} a_k(q)\,(h_k - h_0 e_k) + o(h) \tag{34}$$

and **left $\mathbb{H}$-holomorphic** if

$$f(q+h) = f(q) + \sum_{k=1}^{3} (h_k - h_0 e_k) a_k(q) + o(h) \tag{35}$$

with Landau Symbol $o(h)$. The h_0, h_k are the 4D coordinates of $h \in \mathbb{H}$. With the operator

$$\bar{\partial} := \frac{\partial}{\partial q_0} + \sum_{k=1}^{3} \frac{\partial}{\partial q_k} e_k, \tag{36}$$

we have

$$f \text{ is left } \mathbb{H}\text{-holomorphic} \Leftrightarrow \bar{\partial} f = 0 \tag{37}$$

$$f \text{ is right } \mathbb{H}\text{-holomorphic} \Leftrightarrow f \bar{\partial} = 0. \tag{38}$$

Let $\mathbb{B}_3 := \{q \in \mathbb{H} | |q| = 1\}$ be the unit sphere in $\mathbb{H}$. Let $L^2(\mathbb{B}_3)$ be the functions on $\mathbb{B}_3$ with existing integral of the squared function. Then one can write

$$L^2(\mathbb{B}_3) \cap \ker \bar{\partial} = \bigoplus_{k=0}^{\infty} H_k^+ \tag{39}$$

where H_k^+ are the homogenous $\mathbb{H}$-holomorphic polynomials. There is an orthogonal basis for this space that allows a Fourier series approximation [16].
If we look at a vector field

$$v : \mathbb{R}^3 \to \mathbb{R}^3 \subset G_3 \tag{40}$$

we have to find a related $\mathbb{H}$-holomorphic function $f : \mathbb{B}_3 \to \mathbb{H} \subset G_3$ to apply the construction above. We tried

$$v(x) = f(x) e_1 \overline{f(x)} \tag{41}$$

and

$$v(x) = f(x) e_1, \quad v(x) = e_1 f(x) \tag{42}$$

as well as

$$f(x) = v_1(x) e_1 e_2 + v_2(x) e_2 e_3 + v_3(x) e_1 e_3 \tag{43}$$

with $e_i \in G_3$.
In all examined cases, general linear vector fields v do not generate a $\mathbb{H}$-holomorphic function f, which makes applying the fourier series expansion to our purpose inconvenient. Options by using the concept of monogenicity alternatively may succeed, but are not worked out yet.

5.2 Biquaternion Fourier Transform

Let $\mathbb{H}_\mathbb{C}$ be the biquaternions, i. e.

$$\mathbb{H}_\mathbb{C} = \{q_0 + q_1 i + q_2 j + q_3 k | q_k = \Re(q_k) + I\Im(q_k) \in \mathbb{C}\} \tag{44}$$

with the algebra isomorphism $\mathbb{H}_\mathbb{C} \to G_3$ as in (28)+(29). Sangwine et al. [20] choose a $\mu \in G_3$ with $\mu^2 = -1$ and define the **right biquaternion Fourier transform (BiQFT)** for a signal $F : \mathbb{R}^3 \to G_3$ by

$$\mathscr{F}_r^\mu\{F\}(u) = \int_{\mathbb{R}^3} F(x)\exp(-2\pi\mu x \cdot u)|dx| \tag{45}$$

and the **left biquaternion Fourier transform** by

$$\mathscr{F}_l^\mu\{F\}(u) = \int_{\mathbb{R}^3} \exp(-2\pi\mu x \cdot u)F(x)|dx| \tag{46}$$

For $\mu = i_3$, this is the 3D-CFT. But for a pure bi-quarternion, i.e. a bivector, one can choose an orthogonal basis $\mu, \nu, \xi = \mu\nu$, with $\{\mu, \nu, \xi\}$ being quaternionic roots of -1, such that any $q \in G_3$ can be written as

$$\begin{aligned} q &= q_0 + q_1 e_1 e_2 + q_2 e_2 e_3 + q_3 e_1 e_3 \\ &= q_0 + \tilde{q}_1 \mu + \tilde{q}_2 \nu + \tilde{q}_3 \xi \\ &= (q_0 + \tilde{q}_1 \mu) + (\tilde{q}_2 + \tilde{q}_3 \mu)\nu \\ &= Simp(q) + Perp(q)\nu \end{aligned} \tag{47}$$

with $Simp(q), Perp(q)$ denoting the so-called simplex and perplex of q. For a pure bi-quarternion μ, the corresponding BiQFT fulfills

$$\mathscr{F}^{e_1 e_2} = T^{-1} \mathscr{F}^\mu T \tag{48}$$

with the linear operator $T(1) = 1, T(e_1 e_2) = \mu, T(e_2 e_3) = \nu, T(e_1 e_3) = \xi$, so any two BiQFT with pure bi-quarternion μ differ just by an orthogonal transformation. The BiQFT splits like the CFT in four independent classical complex Fourier transforms.

$$
\begin{aligned}
\mathscr{F}_r^{\mu}\{F\}(u) &= \int_{\mathbb{R}^3} F(x)\exp(-2\pi\mu x\cdot u)|dx| \qquad (49)\\
&= \int_{\mathbb{R}^3} (f_0(x)+\tilde{f}_1(x)\mu)\exp(-2\pi\mu x\cdot u)|dx| + \\
&\quad \int_{\mathbb{R}^3} (\tilde{f}_2(x)+\tilde{f}_3(x)\mu)\nu\exp(-2\pi\mu x\cdot u)|dx| \\
&= \int_{\mathbb{R}^3} (\Re(f_0(x))+\Re(\tilde{f}_1(x))\mu)\exp(-2\pi\mu x\cdot u)|dx| + \\
&\quad \int_{\mathbb{R}^3} (\Im(f_0(x))+\Im(\tilde{f}_1(x))\mu)\exp(-2\pi\mu x\cdot u)|dx| i_3 + \\
&\quad \int_{\mathbb{R}^3} (\Re(\tilde{f}_2(x))+\Re(\tilde{f}_3(x))\mu)\nu\exp(-2\pi\mu x\cdot u)|dx| \\
&\quad \int_{\mathbb{R}^3} (\Im(\tilde{f}_2(x))+\Im(\tilde{f}_3(x))\mu)\nu\exp(-2\pi\mu x\cdot u)|dx| i_3
\end{aligned}
$$

For a real vector field

$$
v : \mathbb{R}^3 \to \mathbb{R}^3 \subset G_3, \quad x \mapsto \sum_{k=1}^{3} v_k(x)e_k \qquad (50)
$$

we have

$$
\begin{aligned}
v(x) &= (-v_3(x)e_1e_2 - v_1(x)e_2e_3 + v_2(x)e_1e_3)i_3 \qquad (51)\\
&= ((-v_3(x)e_1e_2) + (-v_1(x) + v_2(x)e_1e_2)e_2e_3)i_3. \qquad (52)
\end{aligned}
$$

We get for $\mu = e_1e_2$:

$$
\begin{aligned}
\mathscr{F}_r^{i}\{v\}(u) &= \int_{\mathbb{R}^3} (-v_3(x)e_1e_2)\exp(-2\pi e_1e_2 x\cdot u)|dx| i_3 \qquad (53)\\
&\quad + \int_{\mathbb{R}^3} (-v_1(x)+v_2(x)e_1e_2)\exp(-2\pi e_1e_2 x\cdot u)|dx| i_3
\end{aligned}
$$

which means that the vector field is split in a purely complex signal $-v_3(x)e_1e_2$ and a complex signal $-v_1(x)+v_2(x)e_1e_2$ which are transformed independently by two classical Fourier transforms. In essence, the BiQFT means choosing a planar direction μ in $\mathbb{R}^3$, transforming the planar part of the vector field with a 2D-CFT in each plane parallel to μ, and transforming the scalar part orthogonal to μ as independent real signal. One can say that the BiQFT is the direct generalization of the 2D-CFT to three dimensions while the 3D-CFT looks at a vector field as three independent real signals.

5.3 Two-Sided Quaternion Fourier Transform

Another approach for transforming a function F including the main QFT Theorems can be found in [2] and its generalization in [13]. Hitzer also stated a Plancherel Theorem for the QFT and, together with Mawardi, extended the theory to higher dimensional Clifford Algebras in [14]. Let $F : \mathbb{R}^2 \to G_3^+$, then the QFT is defined as

$$\mathscr{F}\{F\}(u) = \int_{\mathbb{R}^2} e^{-2\pi e_1 e_2 x_1 u_1} F(x_1, x_2) e^{-2\pi e_2 e_3 x_2 u_2} |dx|. \tag{54}$$

The inverse QFT is given by

$$\mathscr{F}^{-1}\{F\}(x) = \int_{\mathbb{R}^2} e^{2\pi e_1 e_2 x_1 u_1} F(u_1, u_2) e^{2\pi e_2 e_3 x_2 u_2} |du|. \tag{55}$$

Though the usual decomposition of F into 4 real-valued resp. 2 complex-valued signals is possible via

$$F = F_0 + e_1 e_2 F_1 + (F_2 + e_1 e_2 F_3) e_2 e_3 \tag{56}$$

and there are several options to embed two real variables in a quaternion for applying the transform to a real vector field, the Two-Sided QFT is different from the 2D-CFT. Not only the multiplication from 2 sides and using 2 distinct axis of transformation at once leades to different numerical results, even if the fourier kernel is all right-sided, we cannot merge the 2 exponentials, because the functional equation does not hold for arbitrary quaternions.
Investigating the precise relationship of both transforms is left for future work.

6 Conclusion

It has been shown that a convolution of vector fields is a nice asset for the analysis of fluid flow simulations or physical velocity measurements. Geometric algebra allows a formulation of a suitable convolution as closed operation. Furthermore, one can define a Clifford Fourier transform in two and three dimensional Euclidean space that allows the well-known theorems like convolution theorem, derivative theorem and Parseval's theorem. Looking into the two CFT transforms reveals that they look at the vector field in a totally different manner, i.e. the 2D-CFT transforms the vector field as one complex signal while the 3D-CFT transforms the vector field as three independent real signals. This mismatch can be interpreted by the BiQFT of Sangwine et al. which needs an element $\mu \in G_3$ with $\mu^2 = -1$. For a pure bivector, this means choosing a planar direction in which the vector field is transformed as complex signal. The perpendicular part of the vector field is independently transformed as real signal. For $\mu = i_3$, one gets the 3D-CFT. We have also shown that several constructions do not allow a use of the Fourier series approach based on [16].

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