A Logical Characterization for Weighted Event-Recording Automata

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Abstract. We aim to generalize Büchi’s fundamental theorem on the coincidence of recognizable and MSO-definable languages to a weighted timed setting. For this, we investigate subclasses of weighted timed automata and show how we can extend existing timed MSO logics with weights. Here, we focus on the class of weighted event-recording automata and define a weighted extension of the full logic \text{MSO}_e(\Sigma) introduced by D’Souza. We show that every weighted event-recording automaton can effectively be transformed into a corresponding sentence of our logic and vice versa. The methods presented in the paper can be adopted to weighted versions of timed automata and Wilke’s logic of relative distance. The results indicate the robustness of weighted timed automata models and may be used for specification purposes.

Introduction

Recently, the model of \textit{weighted timed automata} has received much attention in the real-time community as it can be used to model continuous consumption of resources [2, 3, 6, 5, 12]. The goal of this paper is to generalize Büchi’s and Elgot’s fundamental theorems about the coincidence of languages recognizable by finite automata and languages definable by sentences in a monadic second-order (MSO) logic [7, 16] to weighted timed automata. For this, we introduce a weighted timed MSO logic, which may be used for specifying quantitative aspects of timed automata, e.g. \textit{how often} a certain property is satisfied by the system.

In this paper, we focus on a weighted version of event-recording automata, a subclass of timed automata introduced by Alur et al. [1]. Recent results on event-recording automata include works on alternative characterizations using regular expressions [8] and MSO logic [15], real-time logics [25, 18], and inference/learning [17]. The main advantage of event-recording automata is that they - as opposed to timed automata - always can be determinized. This simplifies some of our constructions compared to the ones necessary for the class of weighted timed automata.

Our work is motivated by recent works on \textit{weighted logics} by Droste and Gastin [9, 11]. The authors introduce a weighted MSO logic for characterizing the behaviour of weighted automata defined over a semiring. They extend classical
MSO logic with formulas of the form $k$ (for $k$ an element of the semiring), which may be used to define the weight of a transition of a weighted automaton. They show that the behaviour of weighted automata coincides with the semantics of sentences of a fragment of the logic. Recently, this result has been generalized to weighted settings of infinite words [13], trees [14], pictures [21], traces [22], texts [19] and nested words [20].

Here, we aim to generalize the result to a weighted timed setting. The basis of our work is the MSO logic $\text{MSO}_{\text{er}}(\Sigma)$ introduced by D’Souza and used for the logical characterization of event-recording automata [15]. We extend it with two kinds of weighted formulas whose semantics correspond to the weights of edges and locations, respectively, in weighted event-recording automata. For proving a Büchi-type theorem we show that for every sentence $\varphi$ in our logic there is a weighted event-recording automaton whose behaviour corresponds to the semantics of $\varphi$ and vice versa.

For this, we use parts of the proofs presented by Droste and Gastin [11]. However, in the weighted timed setting we are faced with two new problems. First, due to the weights assigned to locations, the Hadamard product, which is used for defining the semantics of conjunction in our logic, does not preserve recognizability. Second, there are formulas $\varphi$ such that there are no weighted event-recording automata whose behaviours correspond to the semantics of $\forall x.\varphi$ and $\forall X.\varphi$, respectively. To overcome these problems, we define a suitable fragment of our logic, for which, with the support of some new notions and techniques, we are able to show the result.

1 (Weighted) Event-Recording Automata

Let $\Sigma$, $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote an alphabet, the natural numbers and the positive reals, respectively. A timed word is a non-empty finite sequence $(a_1, t_1) ... (a_k, t_k) \in (\Sigma \times \mathbb{R}_{\geq 0})^+$ such that the sequence $\ell = t_1 ... t_k$ of timestamps is non-decreasing. Sometimes we denote a timed word as above by $(\bar{a}, \bar{t})$, where $\bar{a} \in \Sigma^+$. We write $T\Sigma^+$ for the set of timed words over $\Sigma$. A set $L \subseteq T\Sigma^+$ is called a timed language. With $\Sigma$ we associate a set $C_\Sigma = \{x_a | a \in \Sigma\}$ of event-recording clock variables ranging over $\mathbb{R}_{\geq 0}$. The variable $x_a$ measures the time distance between the current event in a timed word and the last occurring $a$. Formally, given a timed word $w = (a_1, t_1) ... (a_k, t_k)$, we let $\text{dom}(w)$ be the set $\{1, ..., k\}$ and define for every $i \in \text{dom}(w)$ a clock valuation function $\gamma^w_i : C_\Sigma \to \mathbb{R}_{\geq 0} \cup \{\bot\}$ by

$$\gamma^w_i(x_a) = \begin{cases} t_i - t_j & \text{if there exists a } j \text{ such that } 1 \leq j < i \text{ and } a_j = a, \\ \bot & \text{and for all } m \text{ with } j < m < i, \text{ we have } a_m \neq a \\ \text{otherwise.} \end{cases}$$

We define clock constraints $\phi$ over $C_\Sigma$ to be conjunctions of formulas of the form $x = \bot$ or $x \sim c$, where $x \in C_\Sigma$, $c \in \mathbb{N}$, and $\sim \in \{<, \leq, =, \geq, >\}$. We use $\Phi(C_\Sigma)$ to
denote the set of all clock constraints over $C$. A clock valuation $\gamma^w$ satisfies $\phi$, written $\gamma^w \models \phi$, if $\phi$ evaluates to true according to the values given by $\gamma^w$. We further use $|w|$ to denote the length of $w$. An event-recording automaton (ERA) over $\Sigma$ is a tuple $A = (S, S_0, S_f, E)$, where

- $S$ is a finite set of locations (states),
- $S_0 \subseteq S$ is a set of initial locations,
- $S_f \subseteq S$ is a set of final locations,
- $E \subseteq S \times \Sigma \times \Phi(C_\Sigma) \times S$ is a finite set of edges.

For $w$ as above, we let a run of $A$ on $w$ be a finite sequence $s_0 \xrightarrow{a_1,\phi_1} s_1 \xrightarrow{a_2,\phi_2} \ldots \xrightarrow{a_k,\phi_k} s_k$ of edges $e_i = (s_{i-1}, a_i, \phi_i, s_i) \in E$ such that $\gamma^w \models \phi_i$ for all $1 \leq i \leq k$. We say that $r$ is successful if $s_0 \in S_0$ and $s_k \in S_f$. We define the timed language $L(A) = \{w \in \Sigma^+ \mid$ there is a successful run of $A$ on $w\}$. We say that a timed language $L \subseteq \Sigma^+$ is ERA-recognizable over $\Sigma$ if there is an ERA $A$ over $\Sigma$ such that $L(A) = L$.

Remark 1. The methods presented in this paper can easily be extended to event-clock automata additionally equipped with event-predicting clock variables [1] of the form $y_a$ measuring the time distance to the next occurring $a$.

Next, we recall the notion of quasi-event-recording automata introduced by D’Souza [15]. First we give the background on why we need this notion. In this paper, we want to give a Büchi-type theorem for the class of weighted event-recording automata. So let us recall how Büchi’s theorem is proved in the classical setting (see e.g. Thomas [26]). One part is to show that for every sentence $\varphi$ in the MSO logic there is a finite automaton recognizing $L(\varphi)$. This is done by induction on the structure of the logic. For the induction base one shows that for every atomic formula of the logic there is a corresponding finite automaton. Recall that such a finite automaton is defined over a so-called extended alphabet which encodes the current assignments of the free variables (see Sect. 2). In the induction step one shows that recognizable timed languages are closed under the constructs of the logic, i.e., under negation, disjunction and existential quantification. For showing these closure properties, one makes use of projections. First and foremost, existential quantification corresponds to the operation of projection. So closure of recognizable languages under existential quantification follows from the fact that recognizable languages are closed under projection. Unfortunately, this is not the case for the class of ERA-recognizable timed languages [1]. However, D’Souza showed that existential quantification in his logic MSOer$(\Sigma)$ corresponds to closure under projection of a subclass of ERA-recognizable timed languages, namely the class of quasi-event-recording automata-recognizable timed languages.

Let $U$ be a (possibly infinite) set of letters, called the universe. Consider a finite partition of this universe, given by a function $g$ from $U$ to a finite indexing set $C$. In the following, we fix $(U, g, C)$ and let $\Gamma \subseteq U$. A quasi-event-recording automaton (qERA) over $\Gamma$ is defined in the same way as ERA except that for every $a \in \Gamma$, the event-recording clock variable $x_a$ records the time to the last
event \( b \in \Gamma \) such that \( g(b) = g(a) \). The definition of the clock valuation function \( \gamma \) is changed accordingly. A timed language \( L \subseteq TU^+ \) is \( \text{qERA-recognizable} \) w.r.t. \((U, g, C)\) if there is a \( \text{qERA} \ A \) over \( \Gamma \) for some \( \Gamma \subseteq U \) such that \( L(A) = L \).

A \( \text{qERA} \ A \) is \text{deterministic} if \( |S_0| = 1 \) and whenever \((s, a, \phi_1, \phi_2)\) and \((s, a, \phi_2, s_2)\) are two different edges in \( A \), then for all clock valuations \( \gamma \) we have \( \gamma \neq \phi_1 \land \phi_2 \). \( A \) is \text{unambiguous} if for every accepted timed word \( w \in L(A) \) there is exactly one successful run of \( A \) on \( w \). A timed language is called \text{deterministically \( \text{qERA-recognizable} \) \( (\text{unambiguously \( \text{qERA-recognizable}, \) respectively) w.r.t.} \((U, g, C)\) if there is a deterministic (unambiguous, respectively) \( \text{qERA} \) over \( \Gamma \) for some \( \Gamma \subseteq U \) recognizing it.

Let \( \Gamma, \Delta \subseteq U \) be finite and \( \pi : \Gamma \rightarrow \Delta \) be a mapping. The \text{renaming} \( \pi(w) \) of a timed word \( w \in TT^+ \) is the timed word \( w' \in T\Delta^+ \) such that \( \text{dom}(w') = \text{dom}(w) \), \( a'_i = \pi(a_i) \) and \( t'_i = t_i \) for all \( i \in \text{dom}(w) \). We say that \( \pi \) is \text{valid} w.r.t. \((U, g, C)\) if for each \( a \in \Gamma \) we have \( g(a) = g(\pi(a)) \).

**Proposition 1.** \[1, 15\] The class of \( \text{qERA-recognizable} \) timed languages is closed under boolean operations and equal to the class of deterministically \( \text{qERA-recognizable} \) timed languages. The class of \( \text{qERA-recognizable} \) timed languages is closed under valid renamings.

We extend \( \text{(q)ERA} \) to be equipped with weights taken from a commutative semiring. For this, we let \( \mathcal{K} \) be a \text{commutative semiring}, i.e., an algebraic structure \( \mathcal{K} = (K, +, \cdot, 0, 1) \) such that \((K, +, 0)\) and \((K, \cdot, 1)\) are commutative monoids, multiplication distributes over addition and 0 is absorbing. As examples consider the semiring of natural numbers \((\mathbb{N}, +, \cdot, 0, 1)\), the Boolean semiring \(((0, 1), \lor, \land, 0, 1)\) and the tropical semiring \((R_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)\). Furthermore, we let \( F \) be a family of functions from \( R_{\geq 0} \) to \( K \). For instance, if \( K \) is the tropical semiring, \( F \) may be the family of linear functions of the form \( f(\delta) = k \cdot \delta \) mapping every \( \delta \in R_{\geq 0} \) to the value \( k \cdot \delta \) in \( K \) (for some \( k \in R_{\geq 0} \)).

Given \( f_1, f_2 \in F \), we define the pointwise product \( f_1 \circ f_2 \) of \( f_1 \) and \( f_2 \) by \( (f_1 \circ f_2)(\delta) = f_1(\delta) \cdot f_2(\delta) \).

A \text{weighted event-recording automaton} (\text{WERA}) over \( \Sigma, \mathcal{K} \) and \( F \) is a tuple \( A = (S, S_0, S_f, E, C) \) such that \((S, S_0, S_f, E)\) is an \( \text{ERA} \) over \( \Sigma \) and \( C = \{C_\Sigma\} \cup \{C_s|s \in S\} \) is a cost function, where \( C_\Sigma : E \rightarrow K \) assigns a weight to each edge, and \( C_s \in F \) gives us the weight for staying in location \( s \) per time unit for each \( s \in S \). Similarly, if \( \Sigma \subseteq U \), we define \text{weighted quasi-event-recording automata} (\text{qWERA}) over \( \mathcal{K}, \Sigma \) and \( F \). A (\text{q})\text{WERA} \( A \) maps to each timed word \( w \in T\Sigma^+ \) a weight in \( K \) as follows: first, we define the \text{running weight} \( \text{rwt}(r) \) of a run \( r \) as above to be \( \prod_{i \in \text{dom}(w)} C_{s_{i-1}}(t_i - t_{i-1}) \cdot C_\Sigma(e_i) \), where \( t_0 = 0 \). Then, the \text{behaviour} \( \| A \| : T\Sigma^+ \rightarrow K \) of \( A \) is given by \( \{\| A \|, w\} = \sum\{\text{rwt}(r): r \text{ is a successful run of } A \text{ on } w\} \). A function \( T : T\Sigma^+ \rightarrow K \) is called a \text{timed series}. A timed series \( T \) is said to be \( \text{WERA-recognizable} \) over \( \mathcal{K}, \Sigma \) and \( F \) if there is a \( \text{WERA} \ A \) over \( \mathcal{K}, \Sigma \) and \( F \) such that \( \| A \| = T \). Equivalently, a timed series \( T : TU^+ \rightarrow K \) is said to be \( \text{qWERA-recognizable} \) over \( \mathcal{K}, \Sigma \) and \( F \) w.r.t. \((U, g, C)\) if there is a \( \text{qWERA} \ A \) over \( \mathcal{K}, \Sigma \) and \( F \) for some \( \Sigma \subseteq U \) such that \( \| A \| = T \). We define the function
1 : \mathbb{R}_{\geq 0} \rightarrow K \text{ by } \delta \rightarrow 1 \text{ for every } \delta \in \mathbb{R}_{\geq 0}. \text{ In the following, we fix a commutative semiring } K \text{ and a family } \mathcal{F} \text{ of cost functions from } \mathbb{R}_{\geq 0} \text{ to } K \text{ containing } 1.

For \( L \subseteq TU^+ \), the characteristic series \( 1_L \) is defined by \((1_L, w) = 1 \) if \( w \in L \), \( 0 \) otherwise. Notice that a qERA \( \mathcal{A} \) over \( \Sigma \) can be seen as a qWERA over the Boolean semiring, \( \Sigma \) and the family of constant functions. The timed series recognized by such a qWERA is the characteristic series \( 1_{L(\mathcal{A})} \). However, due to the determinizability of qERA, \( 1_{L(\mathcal{A})} \) can also be recognized over arbitrary semirings:

**Lemma 1.** If \( L \subseteq TU^+ \) is qERA-recognizable w.r.t. \((U, g, C)\), then \( 1_L \) is qWERA-recognizable over \( K \) and \( \mathcal{F} \) w.r.t. \((U, g, C)\).

**Proof.** Let \( L \subseteq TU^+ \) be qERA-recognizable w.r.t. \((U, g, C)\). By Prop.1, there is a deterministic qERA \( \mathcal{A} = (S, S_0, S_f, E) \) over \( \Sigma \) such that \( L(\mathcal{A}) = L \) for some \( \Sigma \subseteq U \). We define a cost function \( C \) by \( C_e(e) = 1 \) for every \( e \in E \) and \( C_s = 1 \) for every \( s \in S \). Then for the qWERA \( \mathcal{A}' = (S, S_0, S_f, E, C) \) over \( K \), \( \Sigma \) and \( \mathcal{F} \) we have \( \|A\| = 1_L \). Thus, \( 1_L \) is qWERA-recognizable over \( K \) and \( \mathcal{F} \) w.r.t. \((U, g, C)\).

Given timed series \( T, T_1, T_2 \) and \( k \in K \), we define the sum \( T_1 + T_2 \), the Hadamard product \( T_1 \odot T_2 \) and the scalar products \( k \cdot T \) and \( T \cdot k \) pointwise, i.e., by \((T_1 + T_2, w) = (T_1, w) + (T_2, w), (T_1 \odot T_2, w) = (T_1, w) \cdot (T_2, w), (k \cdot T, w) = k \cdot (T, w) \) and \((T \cdot k, w) = (T, w) \cdot k \) respectively. If \( K \) is the Boolean semiring, then + and \( \odot \) correspond to the union and intersection of timed languages, respectively.

Let \( \pi : \Delta \rightarrow \Delta \) be a renaming and \( \mathcal{T} : \mathcal{T} \Delta^+ \rightarrow K \) a timed series. We define the renaming \( \overline{\pi}(T) : T \Delta^+ \rightarrow K \) of \( T \) by \((\overline{\pi}(T), u) = \sum_{\pi(w) = u} (T, w) \) for all \( u \in T \Delta^+ \). Notice that the sum in the equation is finite. A renaming \( \overline{\pi} \) is valid w.r.t. \((U, g, C)\) if the underlying renaming \( \pi \) is valid w.r.t. \((U, g, C)\). For the timed series \( T : \mathcal{T} \Delta^+ \rightarrow K \) we define the inverse renaming \( \overline{\pi}^{-1}(T) : T \mathcal{T}^+ \rightarrow K \) by \((\overline{\pi}^{-1}(T), w) = (T, \pi(w)) \) for each \( w \in T \mathcal{T}^+ \). An inverse renaming \( \overline{\pi}^{-1} \) is valid w.r.t. \((U, g, C)\) if the underlying renaming \( \pi \) is valid w.r.t. \((U, g, C)\).

Later in the paper, we need closure properties of qERA-recognizable timed series under these operations. We will show that sum, scalar products and valid (inverse) renamings preserve recognizability of timed series. In contrast to this, in general both qWERA- and WERA-recognizable timed series are not closed under the Hadamard product. We illustrate this in the next example for WERA-recognizable timed series, but notice that the same holds for qWERA-recognizable timed series.

**Example 1.** Let \( K = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0) \), \( \Sigma = \{a\} \) and \( \mathcal{F} \) be the family of linear functions of the form \( C : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \). We define two WERA \( \mathcal{A}' \) over \( K \), \( \Sigma \) and \( \mathcal{F} \) for each \( i = 1, 2 \) by \( \mathcal{A}' = (\{p^i, q^i\}, \{p^i\}, \{\{p^i, a, true, q^i\}\}, C^i) \) with \( C_{k}^i((p^i, a, true, q^i)) = 0 \), \( C_{a}^i \) arbitrary, \( C_{p}^i(\delta) = 2 \cdot \delta \) and \( C_{q}^i(\delta) = 3 \cdot \delta \) for each \( \delta \in \mathbb{R}_{\geq 0} \). Let \( w \in T \Sigma^+ \). If \( w \neq (a, t) \) for some \( t \in \mathbb{R}_{\geq 0} \), then \( \|A\|, w) = 0 \) for each \( i = 1, 2 \) and thus \((\|A^1\| \odot \|A^2\|, w) = 0 \). So let \( w = (a, t) \) for some \( t \in \mathbb{R}_{\geq 0} \). Then we have \((\|A^1\| \odot \|A^2\|, w) = 2 \cdot t + 3 \cdot t = 5 \cdot t \). Clearly, this timed series is WERA-recognizable over the family of linear functions. If \( K \) and \( \mathcal{F} \) are as above,
for building a WERA recognizing the Hadamard product of the behaviours of two given WERA, we can use the usual product automaton construction together with defining a cost function such that the cost of each edge and location equals the pointwise product of the costs of the two corresponding edges and locations in the original WERA. This can be done since the pointwise product of each pair of linear functions is a linear function and thus in \( \mathcal{F} \). However, this is not always the case. For instance, assume that \( \mathcal{A}^i \) are WERA over \( \mathcal{K}, \Sigma \) and \( \mathcal{F} \) for some \( \Sigma \subseteq U \). We say that \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \) are non-interfering if for all pairs \((s_1, s_2) \in S^1 \times S^2\), whenever there is a run from \((s_1, s_2)\) into \(S^1_f \times S^2_f\), then \(c^1_{s_1} = 1\) or \(c^2_{s_2} = 1\). Observe that this implies \(C^1_{s_1} \subseteq C^2_{s_2} \subseteq \mathcal{F}\). This enables us to use a product automaton construction for building a qWERA over \( \mathcal{K}, \Sigma \) and \( \mathcal{F} \) recognizing \( \|A^1\| \circ \|A^2\|\). Also notice that the premise of the condition is decidable for the whole class of weighted timed automata including qWERA \([2]\). Two timed series \(T_1, T_2 : TU^+ \rightarrow \mathcal{K}\) are non-interfering over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, g, C)\) if there are qWERA \(\mathcal{A}^1\) and \(\mathcal{A}^2\) over \(\mathcal{K}, \Sigma \) and \(\mathcal{F}\) for some \(\Sigma \subseteq U\) such that \(\|A^i\| = T_i\) for \(i = 1, 2\) and \(\mathcal{A}^1\) and \(\mathcal{A}^2\) are non-interfering.

**Lemma 2.** 1. If for all \(f_1, f_2 \in \mathcal{F}\) we have \(f_1 \circ f_2 \in \mathcal{F}\), then qWERA-recognizable timed series over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, g, C)\) are closed under \(\circ\).

2. If \(T_1\) and \(T_2\) are non-interfering over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, g, C)\), then \(T_1 \circ T_2\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, g, C)\).

**Proof.** We show the proof for 2. Let \(T_1, T_2 : TU^+ \rightarrow \mathcal{K}\) be two non-interfering series over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, g, C)\). By definition, there exist two qWERA \(\mathcal{A}^i = (S^i, S^i_0, S^i_f, E^i, C^i)\) \((i = 1, 2)\) over \(\mathcal{K}, \Sigma \) and \(\mathcal{F}\) such that \(\mathcal{A}^i = T_i\) for \(i = 1, 2\) and \(\mathcal{A}^1\) and \(\mathcal{A}^2\) are non-interfering. Define \(S' = \{(s_1, s_2) \in S^1 \times S^2 \mid C^1_{s_1} \neq 1 \text{ and } C^2_{s_2} \neq 1\}\) and put \(A = (S, S_0, S_f, E, C)\), where

- \(S = (S^1 \times S^2) \setminus S'\)
- \(S_0 = (S^1_0 \times S^2_0) \setminus S'\)
- \(S_f = (S^1_f \times S^2_f) \setminus S'\)
- \((s_1, s_2, a, \phi_1 \land \phi_2, (s_1', s_2')) \in E \text{ iff } (s_1, a, \phi_1, s_1') \in E^1, (s_2, a, \phi_2, s_2') \in E^2\)
- \(C^1_E((s_1, s_2), a, \phi_1 \land \phi_2, (s_1', s_2')) = C^1_E((s_1, a, \phi_1, s_1')) \cdot C^1_E((s_2, a, \phi_2, s_2'))\)
- \(C_{(s_1, s_2)} = C^1_{s_1} \circ C^2_{s_2}\) for every \((s_1, s_2) \in S\).

Intuitively, \(\mathcal{A}\) is the classical product automaton, but we remove all “bad” pairs of locations both of whose cost functions do not equal 1. As a consequence, we obtain \(C_{(p, q)} \in \mathcal{F}\) for every \((p, q) \in S\). The removing of “bad” pairs of locations can be done since by assumption from every such pair there is no run to \(S^1_f \times S^2_f\) anyway. Subsequently, we show that \(\|A\| = \|\mathcal{A}^1\| \circ \|\mathcal{A}^2\|\). We start
by proving that there is a weight-preserving bijective correspondence between the set of successful runs of $A$ and the set of pairs of successful runs of $A_1$ and $A_2$. Let $w \in T \Sigma^+$. Suppose there is a successful run $r = (p_0, q_0)^{a_1,\phi_1}_{a_2,\phi_2} \cdots a_{|w|},\phi_{|w|} (p_{|w|}, q_{|w|})$ of $A$ on $w$. The construction of $A$ implies that there are edges $e_i^1 = (p_{i-1}, a_i, \phi_i^1, p_i) \in E^1$ and $e_i^2 = (q_{i-1}, a_i, \phi_i^2, q_i) \in E^2$ such that $\phi_i^1 \wedge \phi_i^2 = \phi_i$ for every $i \in \{1, \ldots, |w|\}$. Hence, there are successful runs $r_1 = p_0 \xrightarrow{a_1,\phi_1^1} p_1 \xrightarrow{a_2,\phi_2^1} \cdots a_{|w|},\phi_{|w|} p_{|w|}$ and $r_2 = q_0 \xrightarrow{a_1,\phi_1^2} q_1 \xrightarrow{a_2,\phi_2^2} \cdots a_{|w|},\phi_{|w|} q_{|w|}$ of $A_1$ and $A_2$, respectively, on $w$. Using the definition of $C$ and commutativity of $K$, we obtain $\text{rwt}(r) = \text{rwt}(r_1) \cdot \text{rwt}(r_2)$. We can use the same lines of argumentation to show that for every pair of successful runs of $A_1$ and $A_2$ on $w$, there is a successful run of $A$ on $w$ such that the equation $\text{rwt}(r) = \text{rwt}(r_1) \cdot \text{rwt}(r_2)$ also holds. Hence, we have established the weight-preserving bijective correspondence mentioned above, which finally can be used to show $\|A\| = \|A_1\| \circ \|A_2\|$. \hfill \qed

For the other operations, the proofs can be done similarly to the untimed setting.

**Lemma 3.** qWERA-recognizable timed series over $K$ and $F$ w.r.t. $(U, g, C)$ are closed under sum, scalar products, valid renamings and valid inverse renamings.

**Proof.** Sum. The proof can be done similarly to the untimed setting, see the corresponding proof for weighted timed automata [12].

**Scalar products** Let $T : T U^+ \rightarrow K$ be qWERA-recognizable over $K$ and $F$ w.r.t. $(U, f, C)$. Hence, there is a qWERA $A$ over $K$, $\Sigma$ and $F$ for some $\Sigma \subseteq U$ such that $\|A\| = T$. Let $k \in K$. We define a qWERA $A_k = (S, S_0, S_f, E, C)$ over $K$, $\Sigma$ and $F$ by

- $S = \{p_1, p_2\}$
- $S_0 = \{p_1\}$
- $S_f = \{p_2\}$
- $E = \{(p_1, a, \text{true}, p_2), (p_2, a, \text{true}, p_2)|a \in \Sigma\}$
- $C_E(e) = k$ if $e = (p_1, a, \text{true}, p_2)$ for some $a \in \Sigma$, $C_E(e) = 1$ otherwise
- $C_p = 1$ for every $p \in S$

Let $w \in T \Sigma^+$. Then there is exactly one successful run of $A_k$ on $w$ with a running weight of $k$. Hence, $\|A_k\| = k \cdot 1_{TU^+}$. Now, since in general we have $k \cdot T = (k \cdot 1_{T \Sigma^+}) \circ T$, and the fact that $A_k$ is non-interfering with every qWERA over $K$, $\Sigma$ and $F$, in particular with $A$, by Lemma 2 $k \cdot T$ is qWERA-recognizable over $K$ and $F$ w.r.t. $(U, f, C)$.

**Renamings** Let $\Gamma, \Delta \subseteq U$ be finite and $\pi : \Gamma \rightarrow \Delta$ be a renaming valid w.r.t. $(U, f, C)$. Further, let $A = (S, S_0, S_f, E, C)$ be a qWERA over $K$, $\Gamma$ and $F$. Define $E' = \{(s, \pi(a), \pi(\phi), s')|(s, a, \phi, s') \in E\}$, where $\pi(\phi)$ is obtained from $\phi$ by replacing each event-recording clock variable $x_a$ by $x_{\pi(a)}$. Now, define $C_{E'} : E' \rightarrow K$ by

$$C_{E'}(s, a', \phi', s') = \sum_{(s, a, \phi, s') \in E \atop \pi(a) = a', \pi(\phi) = \phi'} C_E((s, a, \phi, s'))$$
and put \( A' = (S, S_0, S_f, E', C') \). Clearly, \( A' \) is a qWERA over \( K, \Delta \) and \( F \).

Next, we show that \( \|A'\| = \pi(\|A\|) \).

Let \( v \in T\Delta^+ \) be of the form \((b_1, t_1)\ldots (b_k, t_k)\) and \( \mathcal{R} \) be the set of successful runs of \( A \) on \( w \in T\Gamma^+ \) such that \( \pi(w) = v \). Let \( r, r' \in \mathcal{R} \) be of the form \( r = s_0 \xrightarrow{a_1,\phi_1} \ldots \xrightarrow{a_{|w|},\phi_{|w|}} s_{|w|} \) and \( r' = s'_0 \xrightarrow{a'_1,\phi'_1} \ldots \xrightarrow{a'_{|w|},\phi'_{|w|}} s'_{|w|} \). We say that \( r \) and \( r' \) are equivalent, written \( r \equiv r' \), if \( s_0 = s'_0, s_i = s'_i \) and \( \pi(\phi_i) = \pi(\phi'_i) \) for \( 1 \leq i \leq |w| \). Notice that \( \pi(a_i) = \pi(a'_i) = b_i \) owing to the assumption that \( r, r' \in \mathcal{R} \). Intuitively, \( r \equiv r' \) if the runs only differ in the labels or the clock variables appearing in the clock constraints of an edge, provided that \( \pi \) maps both of them to the same image. We use \( \mathcal{R}_{/\equiv} \) to denote the set of all equivalence classes induced by \( \equiv \). From the fact that \( \equiv \) induces a partition of \( \mathcal{R} \), we obtain

\[
\sum_{w \in T\Delta^+} (\|A\|, w) = \sum_{X \in \mathcal{R}_{/\equiv}} \sum_{r \in X} rwt(r).
\]

Next, let \( X \in \mathcal{R}_{/\equiv} \) and \( r \in X \) be of the form \( r = s_0 \xrightarrow{a_1,\phi_1} \ldots \xrightarrow{a_{|w|},\phi_{|w|}} s_{|w|} \). We define \( r_X \) to be the sequence that is obtained from \( r \) by replacing each edge \( e_i = (s_{i-1}, a_i, \phi_i, s_i) \in E \) by the corresponding edge \( e'_i = (s_{i-1}, \pi(a_i), \pi(\phi_i), s_i) \in E' \), obtaining the run \( r_X = s_0 \xrightarrow{\pi(a_1),\pi(\phi_1)} \ldots \xrightarrow{\pi(a_{|w|}),\pi(\phi'_{|w|})} s_{|w|} \). In the following, we show that \( r_X \) is a successful run of \( A' \) on \( v \).

We start with proving \( \gamma_i^v \models \pi(\phi_i) \) for every \( i \in \{1, \ldots, |w|\} \). Assume \((x_a \sim c) \in \phi_i \) for some \( i \in \{1, \ldots, |w|\}, a \in \Gamma, c \in \mathbb{N} \) and \( \sim \in \{<,\leq,=,\geq,>\} \). Since \( r \) is a run of \( A \) on \( w \), we have \( \gamma_i^w \models \phi_i \) for each \( i \in \{1, \ldots, |w|\} \). Hence there exists some \( j \) such that \( 1 \leq j < i \) with \( g(a_j) = g(a) \) and \( t_i - t_j \sim c \), and for all \( m \) with \( j < m < i \), we have \( g(a_m) \neq g(a) \). By assumption, \( \pi \) is valid w.r.t. \((U,g,C)\). Thus we can infer \( g(\pi(a_j)) = g(\pi(a)) \) and \( t_i - t_j \sim c \) and we have \( g(\pi(a_m)) \neq g(\pi(a)) \) for all \( m \) with \( j < m < i \). But this immediately implies \( \gamma_i^v \models x_{\pi(a)} \sim c \). Since \( \phi_i \) is a conjunction of clock constraints of the form \( x_a \sim c \), we obtain \( \gamma_i^v \models \pi(\phi_i) \) for every \( i \in \{1, \ldots, |v|\} \). Hence, together with the fact that we still have \( s_0 \in S_0 \) and \( s_{|w|} \in S_f \), it follows that \( r_X \) is a successful run. Moreover, the set of successful runs of \( A' \) on \( v \) is precisely the set of such runs \( r_X \) for each \( X \in \mathcal{R}_{/\equiv} \), i.e.,

\[
(\|A'\|, v) = \sum_{X \in \mathcal{R}_{/\equiv}} rwt(r_X),
\]

where \( r_X \) is the run of \( A' \) on \( v \) obtained from an arbitrary run \( r \in \mathcal{R} \) as described above. Next, we show that for every \( X \in \mathcal{R}_{/\equiv} \), \( rwt(r_X) = \sum_{r \in X} rwt(r) \), which, with the help of the two equations above, leads us to the desired result. Let \( X \in \mathcal{R}_{/\equiv} \) and \( r \in X \) as above. Then, the following equation holds by distributivity
of $K$:

$$\text{rwt}(r_X) = \prod_{1 \leq i \leq |v|} (C'_{s_i-1}(t_i - t_{i-1}) \cdot C_{e_i}(e_i))$$

$$= \prod_{1 \leq i \leq |v|} (C_{s_i-1}(t_i - t_{i-1}) \cdot \sum_{(s, a, \phi, s') \in E} C_{e}(s, a, \phi, s'))$$

$$= \sum_{(s, a, \phi, s') \in E} \prod_{1 \leq i \leq |v|} (C_{s_i-1}(t_i - t_{i-1}) \cdot C_{e}(s, a, \phi, s'))$$

$$= \sum_{r \in X} \text{rwt}(r)$$

Hence, $(\|A'\|, v) = \sum_{X \in K/\pi} \text{rwt}(r_X) = \sum_{X \in K/\pi} \sum_{r \in X} \text{rwt}(r) = \sum_{\pi \in \Gamma^+ (\|A\|, w)}$ and thus $\|A'\| = \pi(\|A\|)$.

**Inverse Renamings** Let $\Gamma, \Delta \subseteq U$ be finite and $\pi : \Gamma \rightarrow \Delta$ be a renaming valid w.r.t. $(U, f, C)$. Further, let $A = (S, S_0, f, E, C)$ be a qWERA over $K$, $\Delta$ and $F$. Define $E' = \{(s, a, \phi, s')|(s, \pi(a), \pi(\phi), s') \in S\}$ and $C'_{\pi}(s, \pi(a), \pi(\phi), s') = C_{\pi}(s, a, \phi, s')$. Then, one can easily check that the behaviour of the qWERA $A' = (S, S_0, f, E', C')$ over $K$, $F$ and $\mathcal{F}$ precisely corresponds to $\pi^{-1}(\|A\|)$.

$\square$

Finally, we investigate closure properties of non-interfering timed series.

**Lemma 4.** Let $T_1, T_2, T_3$ be pairwise non-interfering over $K$ and $F$ w.r.t. $(U, f, C)$. Then the following holds:

1. $T_1 + T_2$ and $T_3$ are non-interfering over $K$ and $F$ w.r.t. $(U, f, C)$.
2. $T_1 \odot T_2$ and $T_3$ are non-interfering over $K$ and $F$ w.r.t. $(U, f, C)$.

**Proof.** We give the proof for 2. Let $T_1, T_2, T_3$ be pairwise non-interfering over $K$ and $F$ w.r.t. $(U, f, C)$. Then, there are qWERA $A^1$, $A^2$ and $A^3$ over $K$, $\Gamma$ and $F$ for some $\Gamma \subseteq U$ such that $\|A^i\| = T_i$ for each $i \in \{1, 2, 3\}$ and $A^i$ and $A^j$ are non-interfering for each $i, j \in \{1, 2, 3\}$ such that $i \neq j$. So let $A^{1,2}$ be the qWERA recognizing $T_1 \odot T_2$ (see the proof of Lemma 2). Now assume there is a location $(s^1, s^2)$ in $A^{1,2}$ and $s^3$ in $A^3$ such that $C(s^1, s^2, s^3) \neq 1$ and $C(s^3) \neq 1$ and in the product qWERA of $A^{1,2}$ and $A^3$ there is a successful run from $(s^1, s^2, s^3)$. Let this successful run be of the form $((s^1, s^2, s^3)^{a_1, \phi_1}) \ldots (s_n^1, s_n^2, s_n^3)^{a_n, \phi_n}$ (for some $n \in N$). By construction of the product qWERA, there must also be successful runs $((s^1, s^2)^{a_1, \phi_1^{1,2}} \ldots a_n^{\phi_1^{1,2}} (s_n^1, s_n^2)^{a_n, \phi_n})$ in $A^{1,2}$ and $s^3 a_1^{\phi_1^{3}} \ldots a_n^{\phi_n} s_n^3$ in $A^3$ such that $\phi_1^{1,2} \land \phi_1^3 = \phi_1$ for each $i \in \{1, \ldots, n\}$. Since $C(s_1, s_2) \neq 1$, we must have either $C_{s_1} \neq 1$ or $C_{s_2} \neq 1$. Without loss of generality, we assume $C_{s_1} \neq 1$. Again, by construction of the product qWERA, there must be a successful run $s^1 a_1^{\phi_1} \ldots a_n^{\phi_n} s_n$ in $A^1$. But then, there is also a successful run of the product qWERA $A^{1,3}$ recognizing $T_1 \odot T_3$ of the form $((s^1, s^3)^{a_1, \phi_1^{1,3}} \ldots a_n^{\phi_n} s_n)$.
where \( \phi^{1,3}_i = \phi^1 \land \phi^3 \) for each \( i \in \{1, \ldots, n\} \). However, we also have \( C^1 \neq 1 \) and \( C^3 \neq 1 \), which contradicts the assumption that \( \mathcal{A}1 \) and \( \mathcal{A}3 \) are non-interfering. Thus, \( \mathcal{A}1,2 \) and \( \mathcal{A}3 \) must be non-interfering.

\[ \square \]

## 2 Weighted Timed MSO Logic

Next, we introduce a weighted timed MSO logic for specifying properties of timed series. Our logic is an extension of the logic \( \text{MSO}_{er}(\Sigma) \) introduced by D’Souza for characterizing the behaviour of event-recording automata [15], which we briefly recall here. Formulas of \( \text{MSO}_{er}(\Sigma) \) are built inductively from atomic formulas \( P_\alpha(x), x = y, x < y, x \in X, \sim_a(x) \sim c \) using the connectives \( \lor, \sim, \land \) and \( \forall \).

Now, we turn to the weighted logic \( \text{MSO}_{er}(\Sigma) \), defined inductively as follows. The atomic formulas are formulas of the form \( P_\alpha(x), x = y, x < y, x \in X, \sim_a(x) \sim c \) and their negations, where \( x, y, X, a, c \) are as above. Additionally, we define weighted atomic formulas to be formulas of the form \( k \in K \) and \( f \in F \). Complex formulas can be built from atomic and weighted atomic formulas using the connectives \( \land, \lor, \exists x, \forall x, \exists X, \forall X \). Notice that
we only allow to apply negation to atomic formulas. This is because for arbitrary semirings it is not clear what the negation of a weighted atomic formula should mean. Let \( \phi \in \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) and \( \mathcal{V} \supseteq \text{Free}(\phi) \). The \( \mathcal{V} \)-semantics of \( \phi \) is a timed series \( \left[ \phi \right]_{\mathcal{V}} : T(\Sigma_{\mathcal{V}})^{+} \rightarrow \mathcal{K} \). Let \( (\bar{a}, \bar{t}) \in T \Sigma^{+} \). If \( \phi \) is a valid \((\mathcal{V}, (\bar{a}, \bar{t}))\)-assignment, \( \left[ \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) \in \mathcal{K} \) is defined inductively as follows:

\[
\left[ \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = 1_{L_{U}(\phi)}((\bar{a}, \bar{t}), \bar{t}) \quad \text{if} \quad \phi \text{ is atomic}
\]

\[
k\left[ \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = k\]

\[
\left[ C_{f}(x) \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = f(t_{\sigma(x)} - t_{\sigma(x-1)})
\]

\[
\left[ \phi \lor \phi' \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = \left[ \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) + \left[ \phi' \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t})
\]

\[
\left[ \phi \land \phi' \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = \left[ \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) \cdot \left[ \phi' \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t})
\]

\[
\left[ \exists x, \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = \sum_{i \in \text{dom}(\bar{a}, \bar{t})} \left[ \phi \right]_{\mathcal{V} \cup \{x\}}((\bar{a}, \sigma[x \leftarrow i]), \bar{t})
\]

\[
\left[ \forall x, \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = \prod_{i \in \text{dom}(\bar{a}, \bar{t})} \left[ \phi \right]_{\mathcal{V} \cup \{x\}}((\bar{a}, \sigma[x \leftarrow i]), \bar{t})
\]

\[
\left[ \exists X, \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = \sum_{I \subseteq \text{dom}(\bar{a}, \bar{t})} \left[ \phi \right]_{\mathcal{V} \cup \{X\}}((\bar{a}, \sigma[X \leftarrow I]), \bar{t})
\]

\[
\left[ \forall X, \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = \prod_{I \subseteq \text{dom}(\bar{a}, \bar{t})} \left[ \phi \right]_{\mathcal{V} \cup \{X\}}((\bar{a}, \sigma[X \leftarrow I]), \bar{t})
\]

For \( \sigma \) not a valid \((\mathcal{V}, (\bar{a}, \bar{t}))\)-assignment, we define \( \left[ \phi \right]_{\mathcal{V}}((\bar{a}, \bar{t}), \bar{t}) = 0 \). We write \( [\phi] \) for \( [\phi]_{\text{Free}(\phi)} \). We say that two formulas \( \phi \) and \( \psi \) are equivalent, written \( \phi \equiv \psi \), if \( [\phi] = [\psi] \).

**Remark 2.** If we let \( \mathcal{K} \) be the Boolean semiring, then \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) corresponds to \( \text{MSO}_{\text{er}}(\Sigma) \) as every formula in \( \text{MSO}_{\text{er}}(\Sigma) \) is equivalent to a formula where negation is applied to atomic subformulas only. Also, every such formula \( \phi \in \text{MSO}_{\text{er}}(\Sigma) \) can be seen to be a formula of our logic.

**Example 2.** Consider the formula \( \phi = \exists x . a_{x} < 2 \) and let \( w = (a, 1.7)(b, 3.0)(a, 3.6)(a, 6.0) \). If we interpret \( \phi \) as an \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \)-formula over the Boolean semiring or, equivalently, as an \( \text{MSO}_{\text{er}}(\Sigma) \)-formula, we have \( [\phi](w) = 1 \). If on the other hand, we let \( \mathcal{K} \) be the semiring over the natural numbers with ordinary addition and multiplication, we have \( [\phi](w) = 2 \), i.e., we count the number of positions \( i \) in \( w \) where \( \gamma_{w}^{a}(x) < 2 \) is satisfied. Counting how often a certain property holds gives rise to interesting applications in the field of verification.

Let \( \mathcal{L} \subseteq \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \). A timed series \( T : T \Sigma^{+} \rightarrow \mathcal{K} \) is called \( \mathcal{L} \)-definable if there is a sentence \( \phi \in \mathcal{L} \) such that \( [\phi] = T \). The goal of this paper is to find a suitable fragment \( \mathcal{L} \) of \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) such that \( \mathcal{L} \)-definable timed series precisely correspond to WERA-recognizable timed series over \( \mathcal{K}, \Sigma \) and \( \mathcal{F} \), i.e., we want to generalize Theorem 1 to the weighted setting. It is not surprising that \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) does not constitute a suitable candidate for \( \mathcal{L} \) since this is
already not the case in the untimed setting [10]. In the next section, we explain
the problems that occur when we do not restrict the logic and step by step
develop solutions resulting in the logic $sRMSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$ for which we are able
to give the following Büchi-type theorem.

**Theorem 2.** A timed series $T : T\Sigma^+ \to K$ is WERA-recognizable over $\mathcal{K}$, $\Sigma$
and $\mathcal{F}$ if and only if $T$ is definable by some sentence in $sRMSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$. The
respective transformations can be done effectively provided that the operations of
$\mathcal{K}$ and $\mathcal{F}$ are given effectively.

### 3 From Logic To Automata

In this section, we want to prove the direction from right to left in Theorem 2
and show that for every sentence $\varphi$ of our weighted timed MSO logic, $\llbracket \varphi \rrbracket$ is a
WERA-recognizable timed series. We do this similarly to the corresponding proof
for the classical setting [27], i.e., by induction over the structure of the logic. As
already mentioned, due to the non-closure of WERA-recognizable timed series
under renaming operations, we adopt the approach proposed by D’Souza [15]
and work with qWERA rather than WERA. However, in the last step, when
there are no free variables left, we will see that the resulting qWERA is nothing
else than a WERA.

Let $n \in \mathbb{N}\setminus\{0\}$. We define $\Sigma^{(n)} = \Sigma \times \{1, \ldots, n\}$. Similarly to timed words
over an extended alphabets of the form $\Sigma_V$ for some finite set of variables $V$,
we write $((\bar{a}, \mu), \bar{t})$ to denote a timed word over $\Sigma^{(n)}$, where $(\bar{a}, \bar{t}) \in T\Sigma^+$ and
$\mu \in \{1, \ldots, n\}^{\text{dom}(\bar{a}, \bar{t})}$. We let $(U, g, C)$ such that $U$ contains all elements of
$\Sigma$, $\Sigma_V$, $\Sigma^{(n)}$ and $(\Sigma^{(n)})_V$ for every finite set of variables $V$ and every $n \in \mathbb{N}\setminus\{0\}$, respectively, and $C = \Sigma$. The function $f$, restricted to $\Sigma, \Sigma_V, \Sigma^{(n)}$ and $(\Sigma^{(n)})_V$, maps every $a \in \Sigma$, $(a, \sigma) \in \Sigma_V$, $(a, \mu) \in \Sigma^{(n)}$ and $(a, \mu, \sigma) \in (\Sigma^{(n)})_V$, respectively, to $a$.

For the induction base, we show that for every atomic formula $\varphi$ in
MSOer$(\mathcal{K}, \Sigma, \mathcal{F})$ there is a qWERA over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$ recognizing
$\llbracket \varphi \rrbracket$. For $\varphi$ of the form $P_a(x), x = y, x < y, x \in X$ and its negations, this can
be done as in the classical setting [27]. For instance, for $\varphi = P_a(x)$ the idea
is to construct a qWERA over $\mathcal{K}$, the extended alphabet $\Sigma_{(x)}$ and $\mathcal{F}$ that (1)
verifies that whenever there is a 1 on the x-row, then an $a$ must be read and
(2) assigns to all edges weight 1 and all locations the cost function $1$. This idea
can easily be extended to the remaining atomic formulas of our logic (see Fig.2).

For $\varphi = c_o(x) \sim c$, we build a qWERA that verifies that whenever an edge is
labeled by a 1 in the x-row, the time distance to the last event that is mapped
by $f$ to $a$ satisfies $\sim c$. We do this by adding a corresponding guard to this edge.
Similarly to the untimed setting, all edges are assigned 1 and all location cost
functions equal 1. The qWERA recognizing the timed series $[k]$ has already been
described formally in the proof of Lemma 3. For $\varphi = C_f(x)$, the qWERA verifies
that whenever an edge is labeled with a 1 in the x-row, the source location of
this edge must be assigned the cost function $f$. All the other locations must have
the cost function 1. The qWERA recognizing the semantics of the negations of these formulas can be constructed in a similar way.

![Diagram](image)

**Figure 2. qWERA with behaviours \([a_n(x) \sim c], [k]\) and \([C_I(x)]\)**

For the induction step, we need to show closure properties of qWERA-recognizable timed series under the constructs of the logic. For disjunction and existential quantification, we give proofs very similar to the classical case [26, 10, 11]. Before, we need to show that for each formula \(\varphi\) of our logic, the semantics for every finite set \(\mathcal{V}\) of variables containing \(\text{Free}(\varphi)\) are consistent with each other.

**Lemma 5.** Let \(\varphi \in \text{MSO}_{\sigma}(\mathcal{K}, \Sigma, \mathcal{F})\) and \(\mathcal{V}\) a finite set of variables containing \(\text{Free}(\varphi)\). Then

\[
[\varphi]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) = [\varphi]((\bar{a}, \sigma|_{\text{Free}(\varphi)}), \bar{t})
\]

for each \(((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^+\) such that \(\sigma\) is a valid \((\mathcal{V}, (\bar{a}, \bar{t}))\)-assignment. In particular, \([\varphi]\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\) if and only if \([\varphi]_{\mathcal{V}}\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\).

**Proof.** The first claim can be shown by induction on the structure of \(\text{MSO}_{\sigma}(\mathcal{K}, \Sigma, \mathcal{F})\). For the second claim, let \(\varphi \in \text{MSO}_{\sigma}(\mathcal{K}, \Sigma, \mathcal{F})\) and \(\mathcal{V} \supseteq \text{Free}(\varphi)\) be a finite set of variables. Consider the projection \(\pi : \Sigma_{\mathcal{V}} \rightarrow \Sigma_{\text{Free}(\varphi)} : (a, \sigma) \mapsto (a, \sigma|_{\text{Free}(\varphi)})\) and notice that this is a valid renaming w.r.t. \((U, f, \Sigma)\). First, assume that \([\varphi]\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\). One can show that \([\varphi]_{\mathcal{V}} = \bar{\pi}^{-1}([\varphi]) \circ 1_{N_{\mathcal{V}}}.\) Recall that \(N_{\mathcal{V}}\) is qERA-recognizable w.r.t. \((U, f, \Sigma)\), and thus, by Lemma 1, \(1_{N_{\mathcal{V}}}\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\). Also, by Lemma 3, \(\bar{\pi}^{-1}([\varphi])\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\). Furthermore, all the conditions for applying Lemma 2 are satisfied, and thus, \([\varphi]_{\mathcal{V}}\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\).

Conversely, let \([\varphi]_{\mathcal{V}}\) be qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\). Define \(F = \{(((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^+ | \forall x, X \in \mathcal{V}\backslash \text{Free}(\varphi), \sigma(x) = 1, \sigma(X) = 1\}\). Clearly, \(F\) is qERA-recognizable w.r.t. \((U, f, \Sigma)\). Furthermore, for each \(((\bar{a}, \sigma'), \bar{t}) \in T(\Sigma_{\text{Free}(\varphi)})^+\) there is a unique \(((\bar{a}, \sigma), \bar{t}) \in F\) such that \(\pi(((\bar{a}, \sigma), \bar{t})) = ((\bar{a}, \sigma'), \bar{t})\). Then, we obtain \([\varphi] = \pi([\varphi]_{\mathcal{V}} \circ 1_F).\) Hence, by Lemmas 1, 2 and 3, \([\varphi]\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\).

\(\square\)

**Lemma 6.** Let \(\varphi, \psi \in \text{MSO}_{\sigma}(\mathcal{K}, \Sigma, \mathcal{F})\). If \([\varphi]\) and \([\psi]\) are qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\), then \([\varphi \lor \psi]\) is qWERA-recognizable over \(\mathcal{K}\) and \(\mathcal{F}\) w.r.t. \((U, f, \Sigma)\).
**Proof.** Let $[\varphi]$ and $[\psi]$ be qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$. Since $\text{Free}(\varphi \lor \psi) = \text{Free}(\varphi) \cup \text{Free}(\psi)$ and thus $\text{Free}(\varphi) \subseteq \text{Free}(\varphi \lor \psi)$, we can apply Lemma 5 and conclude that $[\varphi]_{\text{Free}(\varphi \lor \psi)}$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$. The same holds for $[\psi]_{\text{Free}(\varphi \lor \psi)}$. The semantics of MSO$_{\omega}(\mathcal{K}, \Sigma, \mathcal{F})$ produce the following equation:

$$
[\varphi \lor \psi] = [\varphi \lor \psi]_{\text{Free}(\varphi \lor \psi)} = [\varphi]_{\text{Free}(\varphi \lor \psi)} + [\psi]_{\text{Free}(\varphi \lor \psi)}
$$

By Lemma 3, $[\varphi \lor \psi]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$. \qed

**Lemma 7.** Let $\varphi \in \text{MSO}_{\omega}(\mathcal{K}, \Sigma, \mathcal{F})$. If $[\varphi]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$, then $[\exists x. \varphi]$ and $[\exists X. \varphi]$ are qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$.

**Proof.** Let $\varphi \in \text{MSO}_{\omega}(\mathcal{K}, \Sigma, \mathcal{F})$ be such that $[\varphi]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$. Here, we show that $[\exists X. \varphi]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$. Let $\mathcal{V} = \text{Free}(\exists X. \varphi)$. Consider the projection $\pi : \Sigma_{\mathcal{V}(X)} \to \Sigma_{\mathcal{V}}$ which erases the $X$-row. Notice that $\pi$ is valid w.r.t. $(U, f, \Sigma)$. In the following, with $\overline{\pi}$ we denote the extension of $\pi$ to timed series as defined above.

First, we will show that the projection $\overline{\pi}$ on $[\varphi]$ equals $[\exists X. \varphi]$. Then, we use Lemma 5 and Lemma 3 to conclude that $[\exists X. \varphi]$ is qWERA-recognizable.

Let $((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^{+}$. By definition of $\overline{\pi}$, we obtain

$$
(\overline{\pi}([\varphi]_{\mathcal{V}(X)}), ((\bar{a}, \sigma), \bar{t})) = \sum_{((\bar{a}, \sigma'), \bar{t}) \in T(\Sigma_{\mathcal{V}(X)})^{+}} (\overline{\pi}([\varphi]_{\mathcal{V}(X)}((\bar{a}, \sigma'), \bar{t}))
$$

However, using the equivalences

$$
\pi(((\bar{a}, \sigma'), \bar{t})) = ((\bar{a}, \sigma), \bar{t}) \text{ iff } \sigma' = \sigma|X \to I \text{ for some } I \subseteq \text{dom}((\bar{a}, \bar{t}))
$$

and

$$
\sigma \text{ is a valid } (\mathcal{V}, (\bar{a}, \bar{t})) \text{ iff } \sigma|X \to I \text{ is a valid } (\mathcal{V} \cup \{X\}, (\bar{a}, \bar{t})) - \text{assignment for every } I \subseteq \{1, ..., |\mathcal{V}|\},
$$

we obtain

$$
= \sum_{I \subseteq \text{dom}((\bar{a}, \bar{t}))} ([\varphi]_{\mathcal{V}(X)}((\bar{a}, \sigma|X \to I), \bar{t}))
$$

$$
= [\exists X. \varphi]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t})
$$

Now, as $[\varphi]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$, we obtain qWERA-recognizability over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$ of $[\varphi]_{\mathcal{V}(X)}$ by Lemma 5. By Lemma 3, $\overline{\pi}([\varphi]_{\mathcal{V}(X)})$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$, which finishes the proof. \qed
Altogether, the proofs for disjunction and existential quantification are very similar to the corresponding proofs for the classical setting. However, we will see that for the remaining operators of our logic, we cannot give such easy extensions of the classical proofs.

First of all, in Sect.1 we have seen that recognizable timed series in general are not closed under the Hadamard product. Since the semantics of conjunction is defined using the Hadamard product, we have to restrict the usage of conjunction. More precisely, we either have to require that \( \mathcal{F} \) is such that for all \( f_1, f_2 \in \mathcal{F} \) we have \( f_1 \circ f_2 \in \mathcal{F} \), or we have to formulate a syntactical restriction implying that whenever two formulas \( \varphi_1 \) and \( \varphi_2 \) are combined by a conjunction, then \( [\varphi_1] \) and \( [\varphi_2] \) are non-interfering. We make the following observations. Consider the formula \( C_{f_1}(x_1) \land C_{f_2}(x_2) \) and let \( \mathcal{A}_i \) be a WERA such that \( [\mathcal{A}_i] = [C_{f_i}(x_i)] \) for each \( i = 1, 2 \) (see Fig.2). We use \( s_1 \) (\( s_2 \), resp.) to denote the location in \( \mathcal{A}_1 \) (\( \mathcal{A}_2 \), resp.) with cost function \( f_1 \) (\( f_2 \), resp.). We want to enforce that in the product automaton of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), from the pair \( (s_1, s_2) \) there is no run to a final location. This is the case if from \( s_1 \) and \( s_2 \) no common letter can be read. Observe that from \( s_1 \) (\( s_2 \), resp.) every outgoing edge is labeled with \((a, \sigma)\) such that \( \sigma(x_1) = 1 \) \((\sigma(x_2) = 1\), resp.) for every \( a \in \Sigma \). Hence, in the product automaton every edge from \( (s_1, s_2) \) must be labeled with a letter of the form \((a, \sigma)\) such that \( \sigma(x_1) = \sigma(x_2) = 1 \) for every \( a \in \Sigma \). By requiring \( x_1 \) and \( x_2 \) to refer to different positions in a timed word, we can exclude that reading a letter of this form leads to a final location. This is done by conjoining the formula above with \( \neg(x_1 = x_2) \).

**Lemma 8.** Let \( \varphi_1, \varphi_2 \in \text{MSO}_{cr}(\mathcal{K}, \Sigma, \mathcal{F}) \) such that \( [\varphi_1] \) and \( [\varphi_2] \) are qWERA-recognizable over \( \mathcal{K} \) and \( \mathcal{F} \) w.r.t. \((U, f, C)\). Assume that whenever \( \varphi_1 \) contains the subformula \( C_{f_1}(x_1) \) and \( \varphi_2 \) contains \( C_{f_2}(x_2) \), then \( x_1, x_2 \) are free in both \( \varphi_1 \) and \( \varphi_2 \), and either \( \varphi_1 \) or \( \varphi_2 \) is of the form \( \psi \land \neg(x_1 = x_2) \) for some \( \psi \in \text{MSO}_{cr}(\mathcal{K}, \Sigma, \mathcal{F}) \). Then \( [\varphi_1] \) and \( [\varphi_2] \) are non-interfering over \( \mathcal{K} \) and \( \mathcal{F} \) w.r.t. \((U, f, C)\).

**Proof.** The proof is rather technical and we only present a sketch of it. Let \( \varphi_1, \varphi_2 \in \text{MSO}_{cr}(\mathcal{K}, \Sigma, \mathcal{F}) \) such that \( [\varphi_1] \) and \( [\varphi_2] \) are qWERA-recognizable over \( \mathcal{K} \) and \( \mathcal{F} \) w.r.t. \((U, f, C)\). Assume that whenever \( \varphi_1 \) contains the subformula \( C_{f_1}(x_1) \) and \( \varphi_2 \) contains \( C_{f_2}(x_2) \), then \( x_1, x_2 \) are free in both \( \varphi_1 \) and \( \varphi_2 \), and either \( \varphi_1 \) or \( \varphi_2 \) is of the form \( \psi \land \neg(x_1 = x_2) \) for some \( \psi \in \text{MSO}_{cr}(\mathcal{K}, \Sigma, \mathcal{F}) \).

Assume \( \varphi_2 \) is of the form \( C_{f_2}(x_2) \). Let \( \varphi_1 \) be one of the formulas \( k, P_n(x), x = y, x < y \) or its negation. Then \( [\varphi_1] \) is recognizable by a qWERA over \( \mathcal{K}, \Gamma \) and \( \mathcal{F} \) for some \( \Gamma \subseteq U \) such that for all locations \( s \) we have \( C_s = 1 \). Thus \( [\varphi_1] \) is non-interfering with every qWERA-recognizable timed series, in particular with \( [\varphi_2] \).

Now let \( \varphi_1 = C_{f_1}(x_1) \land \neg(x_1 = x_2) \). In Fig.3, we show the qWERA \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) recognizing \( [\varphi_1] \) and \( [\varphi_2] \), respectively, both over the extended alphabet \( \Sigma_{(x_1, x_2)} \).

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Besides conjunction, we have problems with universal quantification. Examples show that unrestricted application of $\forall x$. and $\forall X$. do not preserve recognizability. For instance, let $\mathcal{K} = (\mathbb{N}, +, \cdot, 0, 1)$ be the semiring of the natural numbers and $\mathcal{F}$.
be the family of constant functions. We consider the formula \( \varphi = \forall y. \exists x. C_2(x) \). Then we have \( \llbracket \varphi \rrbracket(w) = |w|^{|w|} \). However, this cannot be recognized by any qW-ERA over \( K \) and \( F \) as this timed series grows too fast (see [10] for a detailed proof which can also be applied to the timed setting). Similar examples can be given for \( \forall X \). Hence, we need to restrict both the usage of \( \forall x \) and \( \forall X \) in our logic. We adopt the approach of Droste and Gastin [11].

For dealing with \( \forall X \), the idea is to restrict the application of \( \forall X \) to so-called syntactically unambiguous formulas. These are formulas \( \varphi \in \text{MSO}_{\text{er}}(\Sigma) \) such that - even though interpreted over a semiring - the semantics \( \llbracket \varphi \rrbracket \) of \( \varphi \) always equals 0 or 1\(^2\). We define the set of syntactically unambiguous formulas \( \varphi^+ \) and \( \varphi^- \) for \( \varphi \in \text{MSO}_{\text{er}}(\Sigma) \) inductively as follows:

1. If \( \varphi \) is of the form \( P_a(x) \), \( x < y \), \( x = y \), \( x \in X \), \( \xi_a(x) \sim c \), then \( \varphi^+ = \varphi \) and \( \varphi^- = \neg \varphi \).
2. If \( \varphi = \neg \psi \) then \( \varphi^+ = \psi^- \) and \( \varphi^- = \psi^+ \).
3. If \( \varphi = \psi \lor \zeta \) then \( \varphi^+ = \psi^+ \lor (\psi^- \land \zeta^+) \) and \( \varphi^- = \psi^- \land \zeta^- \).
4. If \( \varphi = \exists x. \psi \) then \( \varphi^+ = \exists x. \psi^+ \lor \forall y. (y < x \land \psi(y))^- \) and \( \varphi^- = \forall x. \psi^- \).
5. If \( \varphi = \exists X. \psi \) then \( \varphi^+ = \exists X. \psi^+ \lor \forall Y. (Y < X \land \psi(Y))^- \) and \( \varphi^- = \forall X. \psi^- \).

where \( X < Y = \exists y, y \in Y \land \neg (y \in X) \land \forall z, [z < y \rightarrow (z \in X \iff z \in Y)]^+ \).

Notice that for each \( \varphi \in \text{MSO}_{\text{er}}(\Sigma) \) we have \( \llbracket \varphi^+ \rrbracket = 1_{L(\varphi)} \) and \( \llbracket \varphi^- \rrbracket = 1_{L(\neg \varphi)} \).

**Lemma 10.** Let \( \varphi \) be a syntactically unambiguous formula. Then \( \llbracket \varphi \rrbracket \) is qWERA-recognizable over \( K \) and \( F \) w.r.t. \( (U, f, \Sigma) \).

**Proof.** Let \( \varphi \) be syntactically unambiguous. Hence, \( \varphi \) is of the form \( \psi^+ \) or \( \psi^- \) for some \( \psi \in \text{MSO}_{\text{er}}(\Sigma) \). So assume that \( \varphi = \psi^+ \) for some \( \psi \in \text{MSO}_{\text{er}}(\Sigma) \). Then, we have \( \llbracket \psi^+ \rrbracket = 1_{L(\psi)} \). By Theorem 1, \( L(\psi) \) is qERA-recognizable w.r.t. \( (U, f, \Sigma) \). But then Lemma 1 implies that \( \llbracket \psi^+ \rrbracket \) is qWERA-recognizable over \( K \) and \( F \) w.r.t. \( (U, f, \Sigma) \). Now let \( \varphi \) be of the form \( \psi^- \) for some \( \psi \in \text{MSO}_{\text{er}}(\Sigma) \). Then, we have \( \llbracket \psi^- \rrbracket = (\neg \psi)^+ \) and thus we can reduce this case to the case above.

**Corollary 1.** Let \( \varphi \) be syntactically unambiguous. Then \( \llbracket \forall X. \varphi \rrbracket \) is qWERA-recognizable over \( K \) and \( F \) w.r.t. \( (U, f, \Sigma) \).

**Proof.** Let \( \varphi \) be syntactically unambiguous. Hence, \( \varphi \) is of the form \( \psi^+ \) or \( \psi^- \) for some \( \psi \in \text{MSO}_{\text{er}}(\Sigma) \). So assume that \( \varphi = \psi^+ \) for some \( \psi \in \text{MSO}_{\text{er}}(\Sigma) \). Then, we have \( \forall X. \psi^+ = \llbracket \forall X. (\neg \psi)^- \rrbracket = (\exists x. \neg \psi)^+ = (\exists X. \neg \psi)^+ = (\forall X. \psi)^+ \). Hence, \( \forall X. \varphi \) is syntactically unambiguous. By Lemma 10, \( \llbracket \forall X. \psi \rrbracket \) is qWERA-recognizable over \( K \) and \( F \) w.r.t. \( (U, f, \Sigma) \). If \( \varphi = \psi^- \), we have \( \llbracket \psi^- \rrbracket = (\neg \psi)^+ \) and thus we can reduce this case to the case above.

Next, we explain how to deal with \( \forall x \). The approach used by Droste and Gastin [11] is to restrict the subformula \( \varphi \) in \( \forall x. \varphi \) to so-called almost unambiguous formulas. Formulas of this kind can be transformed into equivalent formulas

\(^2\) Recall that every \( \text{MSO}_{\text{er}}(\Sigma) \)-formula can also be seen as an \( \text{MSO}_{\text{er}}(K, \Sigma, F) \)-formula and may have a semantics different from 0 or 1; see e.g. Ex. 2.
of the form $\bigvee_{1 \leq i \leq n} k_i \land \psi_i$ for some $n \in \mathbb{N}$, $k_i \in K$ and $\psi_i \in \text{MSO}(\Sigma)$ for each $i \in \{1, ..., n\}$. One can easily see that the series corresponding to the semantics of such a formula has a finite image. Moreover, closure properties of recognizable series under sum, Hadamard- and scalar products can be used to prove that the semantics of such a formula is recognizable by a weighted automaton. Finally, this particular form of the formula is the base of an efficient construction of a weighted automaton recognizing $[\forall x. \varphi]$. Here, we use a very similar approach. However, we have to redefine the notion of almost unambiguous formulas a bit in order to include subformulas of the form $C_f(x)$.

Let $x$ be a first-order variable. We say that a formula $\varphi$ is almost unambiguous over $x$ if it is in the disjunctive and conjunctive closure of syntactically unambiguous formulas, constants $k \in K$ and formulas $C_f(x)$ (for $f \in F$), such that $C_f(x)$ may appear at most once in every subformula of $\varphi$ of the form $\varphi_1 \land \varphi_2$.

**Lemma 11.** Let $x$ be a first-order variable and $\varphi \in \text{MSO}_{\text{er}}(K, \Sigma, F)$ be almost unambiguous over $x$. Then $[\varphi]$ is qWERA-recognizable over $K$ and $F$ w.r.t. $(U, f, \Sigma)$.

**Proof.** Let $x$ be a first-order variable and $\varphi \in \text{MSO}_{\text{er}}(K, \Sigma, F)$ be almost unambiguous over $x$. Using similar methods as in [11], one can show every almost unambiguous formula over $x$ can be transformed into a formula $\psi$ of the form $\bigvee_{1 \leq i \leq n} C_{f_i}(x) \land k_i \land \psi_i$ for some $n \in \mathbb{N}$, $k_i \in K$, $f_i \in F$ and $\psi_i \in \text{MSO}_{\text{er}}(\Sigma)$ for every $i \in \{1, ..., n\}$ such that $[\psi] = [\varphi]$. For every $i \in \{1, ..., n\}$, $[C_{f_i}(x)], [k_i]$ and $[\psi_i]$ are (pairwise) non-interfering over $K$ and $F$ w.r.t. $(U, f, \Sigma)$. Hence, by Lemmas 4 and 2, for each $i \in \{1, ..., n\}$, $[C_{f_i}(x) \land k_i \land \psi_i]$ is qWERA-recognizable over $K$ and $F$ w.r.t. $(U, f, \Sigma)$. Then, by Lemma 3, $[\psi]$ (and thus also $[\varphi]$) is qWERA-recognizable over $K$ and $F$ w.r.t. $(U, f, \Sigma)$.

So now assume that $\varphi$ is almost unambiguous. The main challenge of this paper is to prove that $[\forall x. \varphi]$ is recognizable. Before we show how this can be done, we introduce a new normalization technique which will be needed in the proof.

**Lemma 12.** For every qERA-recognizable timed language $L \subseteq TU^+$ w.r.t. $(U, g, C)$ there is an unambiguous qERA $A'$ over $\Gamma$ for some $\Gamma \subseteq U$ such that $L(A') = L$ and for each location $s$ in $A'$ there is a unique $a \in \Gamma$ such that every edge $(s, a', \phi, s')$ in $A'$ satisfies $a' = a$.

**Proof.** Let $L \subseteq TU^+$ be qERA-recognizable w.r.t. $(U, f, C)$. By Prop.1, there is a deterministic qERA $A = (S, \{i\}, S_f, E)$ over $\Gamma$ for some $\Gamma \subseteq U$ such that $L(A) = L$. Define $A' = (S', S'_0, S'_f, E')$ where

- $S' = (S \times \Gamma) \cup \{sf\}$
- $S'_0 = \{i\} \times \Gamma$
- $S'_f = \{sf\}$
- $E' = \{((s, a), a, \phi, (s', a')) | (s, a, \phi, s') \in E, a' \in \Gamma\} \cup \{((s, a), a, \phi, s_f) | (s, a, \phi, s') \in E, s' \in S'_f\}$
First, we prove $L(A') = L(A)$. Let $w \in T\Gamma^+$ and suppose $w \in L(A)$. As $A$ is deterministic, there is exactly one successful run $r = (s_0) a_1 \phi_1 ... a_{|w|} \phi_{|w|} (s_{|w|})$ of $A$ on $w$. By the definition of $A'$ it follows that there are edges $e'_1 = ((s_{i-1}, a_i), a_i, \phi_i, (s_i, a_{i+1})) \in E'$ for each $1 \leq i < |w|$ and there is an edge $e'_{|w|} = ((s_{|w|-1}, a_{|w|}), a_{|w|}, \phi_{|w|}, s_f) \in E'$. Notice that we have $\gamma^w_i \models \phi_i$ for each $1 \leq i \leq |w|$, as we did not change neither the labels $a_i$ nor the clock guards $\phi_i$ of the edges $e_i$. Hence, $r' = (s_0, a_1) a_1 \phi_1 ... a_{|w|} \phi_{|w|} (s_{|w|})$ is a successful run of $A'$ on $w$ and we have $w \in L(A')$. The proof for $L(A') \subseteq L(A)$ is very similar and for this reason left to the reader.

Next, we aim to show that $A'$ is unambiguous. Assume that $r = (s_0, u_0) a_1 \phi_1 ... a_{|w|} \phi_{|w|} (s_{|w|})$ and $r' = (s_0, u'_0) a_1 \phi_1 ... a_{|w|} \phi_{|w|} (s_{|w|})$ both are successful runs of $A'$ on $w$. Thus, we have $((s_{i-1}, u_{i-1}), a_i, \phi_i, (s_i, u_i)) \in E'$ for every $i \in \{1,...,|w|\}$. Moreover, by definition of $E'$, $u_{i-1} = u'_{i-1} = a_i$ for each $i \in \{1,...,|w|\}$. Also, we have $s_0 = s_0' = t$ by definition of $S'_0$. Since both $r$ and $r'$ are successful runs on $w$, it follows that both $\gamma^w_i \models \phi_i$ for every $1 \leq i \leq |w|$. Then, the determinism of $A$ implies that $e_1 = e'_1$ and thus $s_1 = s'_1$. The same lines of argumentation can be used to infer $e_i = e'_i$ and $s_i = s'_i$ for every $i \in \{2,...,|w| - 1\}$. Hence, $r = r'$.

**Remark 3.** Notice that in the construction we included an implicit final-location-normalization, allowing one single final location $(s_f)$ only. This is done to guarantee the uniqueness of the successful runs - and thus the unambiguity - of $A'$; if we set $S'_f = S_f \times \Gamma$, we could no longer assume the locations $(s_{|w|}, u_{|w|})$ and $(s'_{|w|}, u'_{|w|})$ reached after the last edge to be equal, because $u_{|w|}$ and $u'_{|w|}$ is not determined by a next letter as it is the case with $u_i$ and $u'_i$ for $i \in \{1,...,|w| - 1\}$.

**Lemma 13.** Let $\varphi \in MSO_{cr}(\mathcal{K}, \Sigma, \mathcal{F})$ be almost unambiguous. Then $[\forall x. \varphi]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. $(U, f, \Sigma)$.

**Proof.** Let $\varphi \in MSO_{cr}(\mathcal{K}, \Sigma, \mathcal{F})$ be almost unambiguous and $W = \text{Free}(\varphi)$. In the proof of Lemma 11 we have seen that we may assume $\varphi$ to be of the form $\bigwedge_{1 \leq j \leq n} C_j(x) \land k_j \land \psi^j_j$, where $n \in \mathbb{N}$, $k_j \in K$, $f_j \in \mathcal{F}$, $\psi_j \in MSO_{cr}(\Sigma)$ for each $j \in \{1,...,n\}$. Let $L_j = L_W(\psi_j)$ for every $j \in \{1,...,n\}$. We may assume $(L_1,...,L_n)$ to be a partition of $N_W$. Let $V = \text{Free}(\forall x. \varphi) = W \setminus \{x\}$. Recall that for every $((\bar{a}, \sigma), \bar{t}) \in T(\Sigma^*) \setminus N_W$ we have $[\varphi](((\bar{a}, \sigma), \bar{t}) = 0$. We define $\bar{L}$ to be the set of timed words $((\bar{a}, \mu, \sigma), \bar{t})$ in $T(\Sigma^*)$ such that $((\bar{a}, \sigma), \bar{t}) \in N_V$ for all $i \in \text{dom}(\bar{a})$ and $j \in \{1,...,n\}$ we have $\mu(i) = j$ implies $((\bar{a}, \sigma[x \rightarrow i]), \bar{t}) \in L_j$. Notice that for every $((\bar{a}, \sigma), \bar{t}) \in N_V$ there is a unique $\mu$ such that $((\bar{a}, \mu, \sigma), \bar{t}) \in \bar{L}$ since $(L_1, ..., L_n)$ is a partition of $N_W$. Next, we prove that $\bar{L}$ is qERA-recognizable w.r.t. $(U, f, \Sigma)$.

For this, we define for every $\xi \in MSO_{cr}(\Sigma)$ the formula $\bar{\xi} \in MSO_{cr}(\Sigma^*)$ by replacing in $\xi$ every occurrence of $P_{a}(x)$ by $\bigvee_{1 \leq j \leq n} P_{(a,j)}(x)$, and every occurrence of $q_{a}(x) \sim c$ by $\bigvee_{1 \leq j \leq n} q_{(a,j)}(x) \sim c$. One can easily verify that for
every $((\bar{a}, \mu, \sigma), \bar{t}) \in T((\Sigma^{(n)})_\mathcal{U})^+$ with $((\bar{a}, \sigma), \bar{t}) \in \mathcal{N}_\mathcal{U}$ we have $((\bar{a}, \sigma), \bar{t}) \models \xi \iff ((\bar{a}, \mu, \sigma), \bar{t}) \models \xi$ (where $\mathcal{U} \supseteq \text{Free}(\xi)$). Now, define the formula $\zeta \in \text{MSO}_{er}(\Sigma^{(n)})$ as follows:

$$\zeta = \forall x. \bigwedge_{1 \leq j \leq n} \{ P(x, j) \rightarrow \bar{\psi}_j \}.$$ 

Let $((\bar{a}, \mu, \sigma), \bar{t}) \in T((\Sigma^{(n)})_\mathcal{V})^+$ such that $((\bar{a}, \sigma), \bar{t}) \in \mathcal{N}_\mathcal{V}$. Using the semantics of $\text{MSO}_{er}(\Sigma^{(n)})$, one can show that $((\bar{a}, \mu, \sigma), \bar{t}) \models \zeta$ iff for every $i \in \text{dom}((\bar{a}, \bar{t}))$ and $j \in \{1, ..., n\}$ we have $\mu(i) = j$ implies $((\bar{a}, \mu, \sigma[x \rightarrow i]), \bar{t}) \models \bar{\psi}_j$. However, $((\bar{a}, \mu, \sigma[x \rightarrow i]), \bar{t}) \models \bar{\psi}_j$ iff $((\bar{a}, \sigma[x \rightarrow i]), \bar{t}) \models \psi_j$. Thus, $((\bar{a}, \mu, \sigma), \bar{t}) \models \zeta$ iff $((\bar{a}, \mu, \sigma), \bar{t}) \in \bar{L}$. Hence, we have $L(\zeta) = \bar{L}$. By Theorem 1 and Prop.1, there is a deterministic qERA $\bar{\mathcal{A}} = (S, \{1\}, S_f, E)$ over $(\Sigma^{(n)})_\mathcal{V}$ such that $L(\bar{\mathcal{A}}) = L(\zeta)$.

Using a variant of Lemma 12, we construct a qERA $\mathcal{A}' = (S', S'_0, S'_f, E')$ from $\bar{\mathcal{A}}$ such that $L(\mathcal{A}') = L(\bar{\mathcal{A}})$. $\mathcal{A}'$ is unambiguous and the locations of $\mathcal{A}'$ will be of the form $(s, b)$ such that every outgoing edge from $(s, b)$ labeled with $(a, b', c)$ satisfies $b' = b$. For this, we define

$$
\begin{align*}
S' &= (S \times \{1, ..., n\}) \cup \{s_f\} \\
S'_0 &= \{1\} \times \{1, ..., n\} \\
S'_f &= \{s_f\} \\
E' &= \{(s, b, (a, b, c), \phi, (s', b')) \mid (s, (a, b, c), \phi, s') \in E, b' \in \{1, ..., n\}\} \\
&\cup \{(s, b, (a, b, c), \phi, s_f) \mid (s, (a, b, c), \phi, s') \in E, s' \in S_f\}
\end{align*}
$$

The proof for $L(\mathcal{A}') = L(\bar{\mathcal{A}})$ and unambiguity of $\mathcal{A}'$ is along the lines of the proof of Lemma 12. Observe that this construction is crucial for the next step in that without the uniqueness property we could not assign the cost functions to the locations in a proper way: add a cost function $C$ to $\mathcal{A}'$, obtaining the qWERA $\mathcal{A} = (S', S'_0, S'_f, E', C)$ over $K$, $(\Sigma^{(n)})_\mathcal{V}$ and $\mathcal{F}$ as follows: for every edge $((s, b), (a, b, c), \phi, (s', b')) \in E'$, define $C_{\xi}(((s, b), (a, b, c), \phi, (s', b'))) = k_b$, and for every $(s, b) \in S'$, let $C(s, b) = f_b$. Now, let $((\bar{a}, \mu, \sigma), \bar{t}) \in L(\mathcal{A}')$. Since $\mathcal{A}$ is unambiguous, we have

$$
\|\mathcal{A}\|, ((\bar{a}, \mu, \sigma), \bar{t}) = \prod_{i \in \text{dom}((\bar{a}, \bar{t}))} f_{\mu(i)}(t_i - t_{i-1}) \cdot k_{\mu(i)}.
$$

Moreover, we can show that $\|\varphi\|_{W}(((\bar{a}, \sigma[x \rightarrow i]), \bar{t}) = f_{\mu(i)}(t_i - t_{i-1}) \cdot k_{\mu(i)}$. Consider the (valid) renaming $\pi : (\Sigma^{(n)})_\mathcal{V} \rightarrow \Sigma_\mathcal{V}$ defined by $(a, b, c) \mapsto (a, c)$ for each $(a, b, c) \in (\Sigma^{(n)})_\mathcal{V}$. Then, for every $((\bar{a}, \sigma), \bar{t}) \in \mathcal{N}_\mathcal{V}$ and the unique $\mu$ such that $((\bar{a}, \mu, \sigma), \bar{t}) \in \bar{L}$, we have

$$
\left(\tilde{\pi}(\|\mathcal{A}\|), ((\bar{a}, \sigma), \bar{t})\right) = \left(\|\mathcal{A}\|, ((\bar{a}, \mu, \sigma), \bar{t})\right)
= \prod_{i \in \text{dom}((\bar{a}, \bar{t}))} \|\varphi\|_{W}(((\bar{a}, \sigma[x \rightarrow i]), \bar{t})
= \|\forall x. \varphi\|((\bar{a}, \sigma), \bar{t}).
$$

For the case $((\bar{a}, \sigma), \bar{t}) \notin \mathcal{N}_\mathcal{V}$, we obtain 0 for both $\left(\tilde{\pi}(\|\mathcal{A}\|), ((\bar{a}, \sigma), \bar{t})\right)$ and $\|\forall x. \varphi\|((\bar{a}, \sigma), \bar{t})$. Hence we have shown $\tilde{\pi}(\|\mathcal{A}\|) = \|\forall x. \varphi\|$. Then, by Lemma 3,
we obtain that \( [\forall x . \varphi] \) is qWERA-recognizable over \( \mathcal{K} \) and \( \mathcal{F} \) w.r.t. \( (U, f, \Sigma) \).

Finally, we give the definition of the fragment of \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) used in Theorem 2. A formula \( \varphi \in \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) is called syntactically restricted if it satisfies the following conditions:

1. Whenever \( \varphi \) contains a conjunction \( \varphi_1 \land \varphi_2 \) as subformula, \( \varphi_1 \) contains the subformula \( C_{f_1}(x_1) \) and \( \varphi_2 \) contains \( C_{f_2}(x_2) \), then \( x_1, x_2 \) are free in both \( \varphi_1 \) and \( \varphi_2 \), and either \( \varphi_1 \) or \( \varphi_2 \) is of the form \( \psi \land \neg(x_1 = x_2) \) for some \( \psi \in \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \).
2. Whenever \( \varphi \) contains \( \forall x . \psi \) as a subformula, then \( \psi \) is an almost unambiguous formula over \( x \).
3. Whenever \( \varphi \) contains \( \forall X . \psi \) as a subformula, then \( \psi \) is a syntactically unambiguous formula.

We let \( \text{sRMSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) denote the set of all syntactically restricted formulas of \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \). Notice that each of these conditions can be checked for in easy syntax tests. Hence, the logic \( \text{sRMSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) is a decidable fragment, i.e., for each formula in \( \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) we can decide whether it is syntactically restricted or not.

**Remark 4.** We may skip the restriction on \( \mathcal{K} \) being commutative by adding a fourth condition on the formulas in \( \text{sRMSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) concerning the element-wise commuting of weighted formulas (see [11]). For the sake of simplicity, we do not consider this here.

Altogether, we have proved that the semantics \( [\varphi] \) of every \( \varphi \in \text{sRMSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) is qWERA-recognizable over \( \mathcal{K} \) and \( \mathcal{F} \) w.r.t. \( (U, f, \Sigma) \). One further can see that every sentence \( \varphi \in \text{sRMSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F}) \) can be recognized by a qWERA over \( \mathcal{K}, \Sigma \) and \( \mathcal{F} \) - but this is nothing else than a WERA over \( \mathcal{K}, \Sigma \) and \( \mathcal{F} \). Hence, we have proven the direction from right to left in Theorem 2.

## 4 From Automata To Logic

For the implication from left to right of Theorem 2, we extend the proof proposed by Droste and Gastin to the timed setting, briefly explained in the following. Let \( \mathcal{A} = (S, S_0, S, E, C) \) be a WERA. We choose an enumeration \( (e_1, \ldots, e_m) \) of \( E \) with \( m = |E| \) and assume \( e_i = (s_i, a_i, \phi_i, s_i') \). We define a syntactically unambiguous formula \( \psi(X_1, \ldots, X_m) \) without any second-order quantifiers describing the successful runs of \( \mathcal{A} \) (where for each \( i \in \{1, \ldots, m\} \), \( X_i \) stands for the edge \( e_i \)). This can be done similarly to the classical setting [26]. The guards of the edges in \( E \) can be defined by a formula of the form \( \forall x \bigwedge_{1 \leq i \leq m} (x \in X_i \vdash a \in \Sigma \, (x, a, c) \in \phi_i) \) where \( \varphi \vdash \psi \) is an abbreviation for \( \varphi^- \lor (\varphi^+ \land (\psi^+)) \). Then, for every timed word \( (\bar{a}, \bar{t}) \) and valid \( (\{X_1, \ldots, X_m\}, (\bar{a}, \bar{t})) \)-assignment \( \sigma \), we have \( [\psi(X_1, \ldots, X_m)]((\bar{a}, \sigma), \bar{t}) = 1 \), if
there is a successful run of $A$ on $(\bar{a}, \bar{t})$, and $[\psi(X_1, ..., X_m)]((\bar{a}, \sigma), \bar{t}) = 0$, otherwise. Notice that we need to use syntactically unambiguous formulas here in order to avoid getting weights different from 1 or 0. Now, we “add weights” to $\psi$ to obtain a formula $\xi$ whose semantics corresponds to the running weight of a successful run of $A$ on $(\bar{a}, \bar{t})$ as follows:

$$\xi = \psi \land \bigwedge_{e_i \in E} \forall x. \left( \neg (x \in X_i) \lor (x \in X_i \land C_{C_{e_i}}(x) \land C_E(e_i)) \right).$$

Finally, we let $\zeta = \exists X_1...\exists X_m. \xi$, and we obtain $[\zeta] = \|A\|$. Hence, we have shown the second implication, which finishes the proof of Theorem 2.

## 5 Conclusion

We have presented a weighted timed MSO logic, which is - at least to our knowledge - the first MSO logic allowing for the description of both timed and quantitative properties. On the one hand, we provide the real-time-community with a new tool, because sometimes it may be easier to specify properties in terms of logic rather than by automata devices. On the other hand, the coincidence between recognizable and definable timed series, together with a previous work on WERA concerning a Kleene-Schützenberger Theorem [23], shows the robustness of the notion of WERA-recognizable timed series, as they can equivalently be characterized in terms of automata, logics and rational operations. The same applies to timed series recognizable by weighted timed automata, for which we were successful in adapting the proofs presented in this paper using the relative distance logic $L\leftarrow\overrightarrow{d}$ introduced by Wilke and his results concerning timed languages with bounded variability [28, 24]. Notice that our result generalizes corresponding results on ERA-recognizable languages as well as formal power series [15, 11]. Also, we have stated conditions for closure of recognizable timed series under the Hadamard product, which corresponds to the intersection operation in the unweighted setting.

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### References


