A Logical Characterization for Weighted Event-Recording Automata

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Abstract. We aim to generalize Büchi's fundamental theorem on the coincidence of recognizable and MSO-definable languages to a weighted timed setting. For this, we investigate subclasses of weighted timed automata and show how we can extend existing timed MSO logics with weights. Here, we focus on the class of weighted event-recording automata and define a weighted extension of the full logic $\mathrm{MSO}_{\mathrm{er}}(\Sigma)$ introduced by D'Souza. We show that every weighted event-recording automaton can effectively be transformed into a corresponding sentence of our logic and vice versa. The methods presented in the paper can be adopted to weighted versions of timed automata and Wilke's logic of relative distance. The results indicate the robustness of weighted timed automata models and may be used for specification purposes.

Introduction

Recently, the model of *weighted timed automata* has received much attention in the real-time community as it can be used to model continuous consumption of resources [2, 3, 6, 5, 12]. The goal of this paper is to generalize Büchi's and Elgot's fundamental theorems about the coincidence of languages *recognizable* by finite automata and languages *definable* by sentences in a monadic second-order (MSO) logic [7, 16] to weighted timed automata. For this, we introduce a weighted timed MSO logic, which may be used for specifying *quantitative* aspects of timed automata, e.g. *how often* a certain property is satisfied by the system.

In this paper, we focus on a weighted version of *event-recording automata*, a subclass of timed automata introduced by Alur et al. [1]. Recent results on event-recording automata include works on alternative characterizations using regular expressions [8] and MSO logic [15], real-time logics [25, 18], and inference/learning [17]. The main advantage of event-recording automata is that they - as opposed to timed automata - always can be determinized. This simplifies some of our constructions compared to the ones necessary for the class of weighted timed automata.

Our work is motivated by recent works on *weighted logics* by Droste and Gastin [9, 11]. The authors introduce a weighted MSO logic for characterizing the behaviour of weighted automata defined over a semiring. They extend classical

MSO logic with formulas of the form k (for k an element of the semiring), which may be used to define the weight of a transition of a weighted automaton. They show that the behaviour of weighted automata coincides with the semantics of sentences of a fragment of the logic. Recently, this result has been generalized to weighted settings of infinite words [13], trees [14], pictures [21], traces [22], texts [19] and nested words [20].

Here, we aim to generalize the result to a weighted timed setting. The basis of our work is the MSO logic $\mathrm{MSO}_{\mathrm{er}}(\Sigma)$ introduced by D'Souza and used for the logical characterization of event-recording automata [15]. We extend it with two kinds of weighted formulas whose semantics correspond to the weights of edges and locations, respectively, in weighted event-recording automata. For proving a Büchi-type theorem we show that for every sentence φ in our logic there is a weighted event-recording automaton whose behaviour corresponds to the semantics of φ and vice versa.

For this, we use parts of the proofs presented by Droste and Gastin [11]. However, in the weighted timed setting we are faced with two new problems. First, due to the weights assigned to locations, the Hadamard product, which is used for defining the semantics of conjunction in our logic, does not preserve recognizability. Second, there are formulas φ such that there are no weighted event-recording automata whose behaviours correspond to the semantics of $\forall x.\varphi$ and $\forall X.\varphi$, respectively. To overcome these problems, we define a suitable fragment of our logic, for which, with the support of some new notions and techniques, we are able to show the result.

1 (Weighted) Event-Recording Automata

Let Σ, $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote an alphabet, the natural numbers and the positive reals, respectively. A *timed word* is a non-empty finite sequence $(a_1, t_1)...(a_k, t_k) \in (\Sigma \times \mathbb{R}_{\geq 0})^+$ such that the sequence $\bar{t} = t_1...t_k$ of timestamps is non-decreasing[1]. Sometimes we denote a timed word as above by $(\bar{a}, \bar{t})$, where $\bar{a} \in \Sigma^+$. We write $T\Sigma^+$ for the set of timed words over Σ. A set $L \subseteq T\Sigma^+$ is called a *timed language*. With Σ we associate a set $C_\Sigma = \{x_a | a \in \Sigma\}$ of *event-recording clock variables* ranging over $\mathbb{R}_{\geq 0}$. The variable x_a measures the time distance between the current event in a timed word and the last occuring a. Formally, given a timed word $w = (a_1, t_1)...(a_k, t_k)$, we let $\mathrm{dom}(w)$ be the set $\{1, ..., k\}$ and define for every $i \in \mathrm{dom}(w)$ a clock valuation function $\gamma_i^w : C_\Sigma \to \mathbb{R}_{\geq 0} \cup \{\perp\}$ by

$$\gamma_i^w(x_a) = \begin{cases} t_i - t_j & \text{if there exists a } j \text{ such that } 1 \leq j < i \text{ and } a_j = a, \\ & \text{and for all } m \text{ with } j < m < i \text{, we have } a_m \neq a \\ \perp & \text{otherwise.} \end{cases}$$

We define *clock constraints* ϕ over C_Σ to be conjunctions of formulas of the form $x = \perp$ or $x \sim c$, where $x \in C_\Sigma$, $c \in \mathbb{N}$, and $\sim \in \{<, \leq, =, \geq, >\}$. We use $\Phi(C_\Sigma)$ to

[1] We assume a timed word to be non-empty for technical simplicity.

denote the set of all clock constraints over C_Σ. A clock valuation γ_i^w satisfies ϕ, written $\gamma_i^w \models \phi$, if ϕ evaluates to true according to the values given by γ_i^w. We further use $|w|$ to denote the length of w. An *event-recording automaton* (ERA) over Σ is a tuple $\mathcal{A} = (S, S_0, S_f, E)$, where

- S is a finite set of locations (states),
- $S_0 \subseteq S$ is a set of initial locations,
- $S_f \subseteq S$ is a set of final locations,
- $E \subseteq S \times \Sigma \times \Phi(C_\Sigma) \times S$ is a finite set of edges.

For w as above, we let a *run* of $\mathcal{A}$ on w be a finite sequence $s_0 \xrightarrow{a_1,\phi_1} s_1 \xrightarrow{a_2,\phi_2} \dots \xrightarrow{a_k,\phi_k} s_k$ of edges $e_i = (s_{i-1}, a_i, \phi_i, s_i) \in E$ such that $\gamma_i^w \models \phi_i$ for all $1 \leq i \leq k$. We say that r is *successful* if $s_0 \in S_0$ and $s_k \in S_f$. We define the timed language $L(\mathcal{A}) = \{w \in T\Sigma^+ |$ there is a successful run of $\mathcal{A}$ on $w\}$. We say that a timed language $L \subseteq T\Sigma^+$ is *ERA-recognizable over* Σ if there is an ERA $\mathcal{A}$ over Σ such that $L(\mathcal{A}) = L$.

Remark 1. The methods presented in this paper can easily be extended to event-clock automata additionally equipped with *event-predicting* clock variables [1] of the form y_a measuring the time distance to the *next* occuring a.

Next, we recall the notion of *quasi*-event-recording automata introduced by D'Souza [15]. First we give the background on why we need this notion. In this paper, we want to give a Büchi-type theorem for the class of weighted event-recording automata. So let us recall how Büchi's theorem is proved in the classical setting (see e.g. Thomas [26]). One part is to show that for every sentence φ in the MSO logic there is a finite automaton recognizing $L(\varphi)$. This is done by induction on the structure of the logic. For the induction base one shows that for every atomic formula of the logic there is a corresponding finite automaton. Recall that such a finite automaton is defined over a so-called *extended alphabet* which encodes the current assignments of the free variables (see Sect.2). In the induction step one shows that recognizable timed languages are closed under the constructs of the logic, i.e., under negation, disjunction and existential quantification. For showing these closure properties, one makes use of projections. First and foremost, existential quantification corresponds to the operation of projection. So closure of recognizable languages under existential quantification follows from the fact that recognizable languages are closed under projection. Unfortunately, this is not the case for the class of ERA-recognizable timed languages [1]. However, D'Souza showed that existential quantification in his logic $\mathrm{MSO}_{\mathrm{er}}(\Sigma)$ corresponds to closure under projection of a *subclass* of ERA-recognizable timed languages, namely the class of quasi-event-recording automata-recognizable timed languages.

Let U be a (possibly infinite) set of letters, called the universe. Consider a finite partition of this universe, given by a function g from U to a finite indexing set C. In the following, we fix (U, g, C) and let $\Gamma \subseteq U$. A *quasi-event-recording automaton* (qERA) over Γ is defined in the same way as ERA except that for every $a \in \Gamma$, the event-recording clock variable x_a records the time to the last event $b \in \Gamma$ such that $g(b) = g(a)$. The definition of the clock valuation function γ is changed accordingly. A timed language $L \subseteq TU^+$ is *qERA-recognizable w.r.t.* (U, g, C) if there is a qERA $\mathcal{A}$ over Γ for some $\Gamma \subseteq U$ such that $L(\mathcal{A}) = L$.

A qERA $\mathcal{A}$ is *deterministic* if $|S_0| = 1$ and whenever (s, a, ϕ_1, s_1) and (s, a, ϕ_2, s_2) are two different edges in $\mathcal{A}$, then for all clock valuations γ we have $\gamma \not\models \phi_1 \wedge \phi_2$. $\mathcal{A}$ is *unambiguous* if for every accepted timed word $w \in L(\mathcal{A})$ there is exactly one successful run of $\mathcal{A}$ on w. A timed language is called *deterministically qERA-recognizable* (*unambiguously qERA-recognizable*, respectively) w.r.t. (U, g, C) if there is a deterministic (unambiguous, respectively) qERA over Γ for some $\Gamma \subseteq U$ recognizing it.

Let $\Gamma, \Delta \subseteq U$ be finite and $\pi : \Gamma \to \Delta$ be a mapping. The *renaming* $\pi(w)$ of a timed word $w \in T\Gamma^+$ is the timed word $w' \in T\Delta^+$ such that $\mathrm{dom}(w') = \mathrm{dom}(w)$, $a'_i = \pi(a_i)$ and $t'_i = t_i$ for all $i \in \mathrm{dom}(w)$. We say that π is *valid* w.r.t. (U, g, C) if for each $a \in \Gamma$ we have $g(a) = g(\pi(a))$.

Proposition 1. *[1, 15] The class of qERA-recognizable timed languages is closed under boolean operations and equal to the class of deterministically qERA-recognizable timed languages. The class of qERA-recognizable timed languages is closed under valid renamings.*

We extend (q)ERA to be equipped with weights taken from a commutative semiring. For this, we let $\mathcal{K}$ be a *commutative semiring*, i.e., an algebraic structure $\mathcal{K} = (K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ and $(K, \cdot, 1)$ are commutative monoids, multiplication distributes over addition and 0 is absorbing. As examples consider the semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$, the Boolean semiring $(\{0, 1\}, \vee, \wedge, 0, 1)$ and the tropical semiring $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)$. Furthermore, we let $\mathcal{F}$ be a family of functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{K}$. For instance, if $\mathcal{K}$ is the tropical semiring, $\mathcal{F}$ may be the family of linear functions of the form $f(\delta) = k \cdot \delta$ mapping every $\delta \in \mathbb{R}_{\geq 0}$ to the value $k \cdot \delta$ in K (for some $k \in \mathbb{R}_{\geq 0}$). Given $f_1, f_2 \in \mathcal{F}$, we define the pointwise product $f_1 \odot f_2$ of f_1 and f_2 by $(f_1 \odot f_2)(\delta) = f_1(\delta) \cdot f_2(\delta)$.

A *weighted event-recording automaton* (WERA) over Σ, $\mathcal{K}$ and $\mathcal{F}$ is a tuple $\mathcal{A} = (S, S_0, S_f, E, C)$ such that (S, S_0, S_f, E) is an ERA over Σ and $C = \{C_\mathcal{E}\} \cup \{C_s | s \in S\}$ is a cost function, where $C_\mathcal{E} : E \to K$ assigns a weight to each edge, and $C_s \in \mathcal{F}$ gives us the weight for staying in location s per time unit for each $s \in S$. Similarly, if $\Sigma \subseteq U$, we define *weighted quasi-event-recording automata* (qWERA) over $\mathcal{K}$, Σ and $\mathcal{F}$. A (q)WERA $\mathcal{A}$ maps to each timed word $w \in T\Sigma^+$ a weight in K as follows: first, we define the *running weight* $rwt(r)$ of a run r as above to be $\prod_{i \in \mathrm{dom}(w)} C_{s_{i-1}}(t_i - t_{i-1}) \cdot C_\mathcal{E}(e_i)$, where $t_0 = 0$. Then, the *behaviour* $\|\mathcal{A}\| : T\Sigma^+ \to K$ of $\mathcal{A}$ is given by $(\|\mathcal{A}\|, w) = \sum\{rwt(r) : r \text{ is a successful run of } \mathcal{A} \text{ on } w\}$. A function $\mathcal{T} : T\Sigma^+ \to K$ is called a *timed series*. A timed series $\mathcal{T}$ is said to be *WERA-recognizable over* $\mathcal{K}$*,* Σ *and* $\mathcal{F}$ if there is a WERA $\mathcal{A}$ over $\mathcal{K}$, Σ and $\mathcal{F}$ such that $\|\mathcal{A}\| = \mathcal{T}$. Equivalently, a timed series $\mathcal{T} : TU^+ \to K$ is said to be *qWERA-recognizable over* $\mathcal{K}$ *and* $\mathcal{F}$ *w.r.t.* (U, g, C) if there is a qWERA $\mathcal{A}$ over $\mathcal{K}$, Σ and $\mathcal{F}$ for some $\Sigma \subseteq U$ such that $\|\mathcal{A}\| = \mathcal{T}$. We define the function

$\mathbb{1} : \mathbb{R}_{\geq 0} \to K$ by $\delta \mapsto 1$ for every $\delta \in \mathbb{R}_{\geq 0}$. In the following, we fix a commutative semiring $\mathcal{K}$ and a family $\mathcal{F}$ of cost functions from $\mathbb{R}_{\geq 0}$ to K containing $\mathbb{1}$.

For $L \subseteq TU^+$, the *characteristic series* 1_L is defined by $(1_L, w) = 1$ if $w \in L$, 0 otherwise. Notice that a qERA $\mathcal{A}$ over Σ can be seen as a qWERA over the Boolean semiring, Σ and the family of constant functions. The timed series recognized by such a qWERA is the characteristic series $1_{L(\mathcal{A})}$. However, due to the determinizability of qERA, $1_{L(\mathcal{A})}$ can also be recognized over arbitrary semirings:

Lemma 1. *If $L \subseteq TU^+$ is qERA-recognizable w.r.t. (U, g, C), then 1_L is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C).*

Proof. Let $L \subseteq TU^+$ be qERA-recognizable w.r.t. (U, g, C). By Prop.1, there is a deterministic qERA $\mathcal{A} = (S, S_0, S_f, E)$ over Σ such that $L(\mathcal{A}) = L$ for some $\Sigma \subseteq U$. We define a cost function C by $C_{\mathcal{E}}(e) = 1 \in K$ for every $e \in E$ and $C_s = \mathbb{1}$ for every $s \in S$. Then for the qWERA $\mathcal{A}' = (S, S_0, S_f, E, C)$ over $\mathcal{K}$, Σ and $\mathcal{F}$ we have $\|\mathcal{A}'\| = 1_L$. Thus, 1_L is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C). $\square$

Given timed series $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ and $k \in K$, we define the *sum* $\mathcal{T}_1 + \mathcal{T}_2$, the *Hadamard product* $\mathcal{T}_1 \odot \mathcal{T}_2$ and the *scalar products* $k \cdot \mathcal{T}$ and $\mathcal{T} \cdot k$ pointwise, i.e., by $(\mathcal{T}_1 + \mathcal{T}_2, w) = (\mathcal{T}_1, w) + (\mathcal{T}_2, w)$, $(\mathcal{T}_1 \odot \mathcal{T}_2, w) = (\mathcal{T}_1, w) \cdot (\mathcal{T}_2, w)$, $(k \cdot \mathcal{T}, w) = k \cdot (\mathcal{T}, w)$ and $(\mathcal{T} \cdot k, w) = (\mathcal{T}, w) \cdot k$ respectively. If $\mathcal{K}$ is the Boolean semiring, then $+$ and $\odot$ correspond to the union and intersection of timed languages, respectively. Let $\pi : \Gamma \to \Delta$ be a renaming and $\mathcal{T} : T\Gamma^+ \to K$ a timed series. We define the renaming $\bar{\pi}(\mathcal{T}) : T\Delta^+ \to K$ of $\mathcal{T}$ by $(\bar{\pi}(\mathcal{T}), u) = \sum_{\pi(w)=u} (\mathcal{T}, w)$ for all $u \in T\Delta^+$. Notice that the sum in the equation is finite. A renaming $\bar{\pi}$ is valid w.r.t. (U, g, C) if the underlying renaming π is valid w.r.t. (U, g, C). For the timed series $\mathcal{T} : T\Delta^+ \to K$ we define the inverse renaming $\bar{\pi}^{-1}(\mathcal{T}) : T\Gamma^+ \to K$ by $(\bar{\pi}^{-1}(\mathcal{T}), w) = (\mathcal{T}, \pi(w))$ for each $w \in T\Gamma^+$. An inverse renaming $\bar{\pi}^{-1}$ is valid w.r.t. (U, g, C) if the underlying renaming π is valid w.r.t. (U, g, C).

Later in the paper, we need closure properties of qERA-recognizable timed series under these operations. We will show that sum, scalar products and valid (inverse) renamings preserve recognizability of timed series. In contrast to this, in general both qWERA- and WERA-recognizable timed series are not closed under the Hadamard product. We illustrate this in the next example for WERA-recognizable timed series, but notice that the same holds for qWERA-recognizable timed series.

Example 1. Let $\mathcal{K} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)$, $\Sigma = \{a\}$ and $\mathcal{F}$ be the family of linear functions of the form $C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. We define two WERA $\mathcal{A}^i$ over $\mathcal{K}$, Σ and $\mathcal{F}$ for each $i = 1, 2$ by $\mathcal{A}^i = (\{p^i, q^i\}, \{p^i\}, \{q^i\}, \{(p^i, a, true, q^i)\}, C^i)$ with $C^i_{\mathcal{E}}((p^i, a, true, q^i)) = 0$, $C^i_{q^i}$ arbitrary, $C^1_{p^1}(\delta) = 2 \cdot \delta$ and $C^2_{p^2}(\delta) = 3 \cdot \delta$ for each $\delta \in \mathbb{R}_{\geq 0}$. Let $w \in T\Sigma^+$. If $w \neq (a, t)$ for some $t \in \mathbb{R}_{\geq 0}$, then $(\|\mathcal{A}^i\|, w) = 0$ for each $i = 1, 2$ and thus $(\|\mathcal{A}^1\| \odot \|\mathcal{A}^2\|, w) = 0$. So let $w = (a, t)$ for some $t \in \mathbb{R}_{\geq 0}$. Then we have $(\|\mathcal{A}^1\| \odot \|\mathcal{A}^2\|, w) = 2 \cdot t + 3 \cdot t = 5 \cdot t$. Clearly, this timed series is WERA-recognizable over the family of linear functions. If $\mathcal{K}$ and $\mathcal{F}$ are as above, for building a WERA recognizing the Hadamard product of the behaviours of two given WERA, we can use the usual product automaton construction together with defining a cost function such that the cost of each edge and location equals the pointwise product of the costs of the two corresponding edges and locations in the original WERA. This can be done since the pointwise product of each pair of linear functions is a linear function and thus in $\mathcal{F}$. However, this is not always the case. For instance, assume that $\mathcal{A}^i$ are WERA over the semiring $(\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$. Then, we have $(\|\mathcal{A}^1\| \odot \|\mathcal{A}^2\|, w) = 2 \cdot t \cdot 3 \cdot t = 6 \cdot t^2$. It can be easily seen that there is no WERA $\mathcal{A}$ over the family $\mathcal{F}$ of **linear** functions such that $\|\mathcal{A}\| = \|\mathcal{A}^1\| \odot \|\mathcal{A}^2\|$.

For this reason, we define the notion of *non-interfering* timed series. So for $i = 1, 2$, let $\mathcal{A}^i = (S^i, S_0^i, S_f^i, E^i, C^i)$ be two qWERA over $\mathcal{K}$, Σ and $\mathcal{F}$ for some $\Sigma \subseteq U$. We say that $\mathcal{A}^1$ and $\mathcal{A}^2$ are *non-interfering* if for all pairs $(s_1, s_2) \in S^1 \times S^2$, whenever there is a run from (s_1, s_2) into $S_f^1 \times S_f^2$, then $C^1_{s_1} = \mathbb{1}$ or $C^2_{s_2} = \mathbb{1}$. Observe that this implies $C^1_{s_1} \odot C^2_{s_2} \in \mathcal{F}$. This enables us to use a product automaton construction for building a qWERA over $\mathcal{K}$, Σ and $\mathcal{F}$ recognizing $\|\mathcal{A}^1\| \odot \|\mathcal{A}^2\|$. Also notice that the premise of the condition is decidable for the whole class of weighted timed automata including qWERA [2]. Two timed series $\mathcal{T}_1, \mathcal{T}_2 : TU^+ \to K$ are non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C) if there are qWERA $\mathcal{A}^1$ and $\mathcal{A}^2$ over $\mathcal{K}$, Σ and $\mathcal{F}$ for some $\Sigma \subseteq U$ such that $\|\mathcal{A}^i\| = \mathcal{T}_i$ for $i = 1, 2$ and $\mathcal{A}^1$ and $\mathcal{A}^2$ are non-interfering.

Lemma 2. *1. If for all $f_1, f_2 \in \mathcal{F}$ we have $f_1 \odot f_2 \in \mathcal{F}$, then qWERA-recognizable timed series over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C) are closed under $\odot$.*
2. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C), then $\mathcal{T}_1 \odot \mathcal{T}_2$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C).

Proof. We show the proof for 2. Let $\mathcal{T}_1, \mathcal{T}_2 : TU^+ \to K$ be two non-interfering series over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C). By definition, there exist two qWERA $\mathcal{A}^i = (S^i, S_0^i, S_f^i, E^i, C^i)$ $(i = 1, 2)$ over $\mathcal{K}$, Σ and $\mathcal{F}$ for some $\Sigma \subseteq U$ such that $\mathcal{A}^i = \mathcal{T}_i$ for $i = 1, 2$ and $\mathcal{A}^1$ and $\mathcal{A}^2$ are non-interfering. Define $S' = \{(s_1, s_2) \in S^1 \times S^2 | C_{s_1} \neq \mathbb{1}$ and $C_{s_2} \neq \mathbb{1}\}$ and put $\mathcal{A} = (S, S_0, S_f, E, C)$, where

- $S = (S^1 \times S^2) \backslash S'$
- $S_0 = (S_0^1 \times S_0^2) \backslash S'$
- $S_0 = (S_f^1 \times S_f^2) \backslash S'$
- $((s_1, s_2), a, \phi_1 \wedge \phi_2, (s_1', s_2')) \in E$ iff $(s_1, a, \phi_1, s_1') \in E^1$, $(s_2, a, \phi_2, s_2') \in E^2$ and $(s_1, s_2), (s_1', s_2') \notin S'$
- $C_{\mathcal{E}}(((s_1, s_2), a, \phi_1 \wedge \phi_2, (s_1', s_2'))) = C^1_{\mathcal{E}}((s_1, a, \phi_1, s_1')) \cdot C^2_{\mathcal{E}}((s_2, a, \phi_2, s_2'))$
- $C_{(s_1, s_2)} = C^1_{s_1} \odot C^2_{s_2}$ for every $(s_1, s_2) \in S$.

Intuitively, $\mathcal{A}$ is the classical product automaton, but we remove all "bad" pairs of locations both of whose cost functions do not equal $\mathbb{1}$. As a consequence, we obtain $C_{(p,q)} \in \mathcal{F}$ for every $(p, q) \in S$. The removing of "bad" pairs of locations can be done since by assumption from every such pair there is no run to $S_f^1 \times S_f^2$ anyway. Subsequently, we show that $\|\mathcal{A}\| = \|\mathcal{A}_1\| \odot \|\mathcal{A}_2\|$. We start by proving that there is a weight-preserving bijective correspondence between the set of successful runs of $\mathcal{A}$ and the set of pairs of successful runs of $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $w \in T\Sigma^+$. Suppose there is a successful run $r = (p_0, q_0) \xrightarrow{a_1,\phi_1} (p_1, q_1) \xrightarrow{a_2,\phi_2} ... \xrightarrow{a_{|w|},\phi_{|w|}} (p_{|w|}, q_{|w|})$ of $\mathcal{A}$ on w. The construction of $\mathcal{A}$ implies that there are edges $e_i^1 = (p_{i-1}, a_i, \phi_i^1, p_i) \in E^1$ and $e_i^2 = (q_{i-1}, a_i, \phi_i^2, q_i) \in E^2$ such that $\phi_i^1 \wedge \phi_i^2 = \phi_i$ for every $i \in \{1, ..., |w|\}$. Hence, there are successful runs $r_1 = p_0 \xrightarrow{a_1,\phi_1^1} p_1 \xrightarrow{a_2,\phi_2^1} ... \xrightarrow{a_{|w|},\phi_{|w|}^1} p_{|w|}$ and $r_2 = q_0 \xrightarrow{a_1,\phi_1^2} q_1 \xrightarrow{a_2,\phi_2^2} ... \xrightarrow{a_{|w|},\phi_{|w|}^2} q_{|w|}$ of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, on w. Using the definition of C and commutativity of $\mathcal{K}$, we obtain $rwt(r) = rwt(r_1) \cdot rwt(r_2)$. We can use the same lines of argumentation to show that for every pair of successful runs of $\mathcal{A}_1$ and $\mathcal{A}_2$ on w, there is a successful run of $\mathcal{A}$ on w such that the equation $rwt(r) = rwt(r_1) \cdot rwt(r_2)$ also holds. Hence, we have established the weight-preserving bijective correspondence mentioned above, which finally can be used to show $\|\mathcal{A}\| = \|\mathcal{A}_1\| \odot \|\mathcal{A}_2\|$. □

For the other operations, the proofs can be done similarly to the untimed setting.

Lemma 3. *qWERA-recognizable timed series over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, g, C) are closed under sum, scalar products, valid renamings and valid inverse renamings.*

Proof. Sum. The proof can be done similarly to the untimed setting, see the corresponding proof for weighted timed automata [12].

Scalar products Let $\mathcal{T} : TU^+ \to K$ be qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C). Hence, there is a qWERA $\mathcal{A}$ over $\mathcal{K}$, Σ and $\mathcal{F}$ for some $\Sigma \subseteq U$ such that $\|\mathcal{A}\| = \mathcal{T}$. Let $k \in K$. We define a qWERA $\mathcal{A}_k = (S, S_0, S_f, E, C)$ over $\mathcal{K}$, Σ and $\mathcal{F}$ by

- $S = \{p_1, p_2\}$
- $S_0 = \{p_1\}$
- $S_f = \{p_2\}$
- $E = \{(p_1, a, true, p_2), (p_2, a, true, p_2) | a \in \Sigma\}$
- $C_\mathcal{E}(e) = k$ if $e = (p_1, a, true, p_2)$ for some $a \in \Sigma$, $C_\mathcal{E}(e) = 1$ otherwise
- $C_p = \mathbb{1}$ for every $p \in S$

Let $w \in T\Sigma^+$. Then there is exactly one sucessful run of $\mathcal{A}_k$ on w with a running weight of k. Hence, $\|\mathcal{A}_k\| = k \cdot 1_{TU^+}$. Now, since in general we have $k \cdot \mathcal{T} = (k \cdot 1_{T\Sigma^+}) \odot \mathcal{T}$, and the fact that $\mathcal{A}_k$ is non-interfering with every qWERA over $\mathcal{K}$, Σ and $\mathcal{F}$, in particular with $\mathcal{A}$, by Lemma 2 $k \cdot \mathcal{T}$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C).

Renamings Let $\Gamma, \Delta \subseteq U$ be finite and $\pi : \Gamma \to \Delta$ be a renaming valid w.r.t. (U, f, C). Further, let $\mathcal{A} = (S, S_0, S_f, E, C)$ be a qWERA over $\mathcal{K}$, Γ and $\mathcal{F}$. Define $E' = \{(s, \pi(a), \pi(\phi), s') | (s, a, \phi, s') \in E\}$, where $\pi(\phi)$ is obtained from ϕ by replacing each event-recording clock variable x_a by $x_{\pi(a)}$. Now, define $C'_\mathcal{E} : E' \to K$ by

$$C'_\mathcal{E}((s, a', \phi', s')) = \sum_{\substack{(s,a,\phi,s') \in E \\ \pi(a)=a', \pi(\phi)=\phi'}} C_\mathcal{E}((s, a, \phi, s'))$$

and put $\mathcal{A}' = (S, S_0, S_f, E', C')$. Clearly, $\mathcal{A}'$ is a qWERA over $\mathcal{K}$, Δ and $\mathcal{F}$. Next, we show that $\|\mathcal{A}'\| = \bar{\pi}(\|\mathcal{A}\|)$.

Let $v \in T\Delta^+$ be of the form $(b_1, t_1)...(b_k, t_k)$ and $\mathcal{R}$ be the set of successful runs of $\mathcal{A}$ on $w \in T\Gamma^+$ such that $\pi(w) = v$. Let $r, r' \in \mathcal{R}$ be of the form $r = s_0 \xrightarrow{a_1,\phi_1} ... \xrightarrow{a_{|w|},\phi_{|w|}} s_{|w|}$ and $r' = s'_0 \xrightarrow{a'_1,\phi'_1} ... \xrightarrow{a'_{|w|},\phi'_{|w|}} s'_{|w|}$. We say that r and r' are equivalent, written $r \equiv r'$, if $s_0 = s'_0$, $s_i = s'_i$ and $\pi(\phi_i) = \pi(\phi'_i)$ for $1 \leq i \leq |w|$. Notice that $\pi(a_i) = \pi(a'_i) = b_i$ owing to the assumption that $r, r' \in \mathcal{R}$. Intuitively, $r \equiv r'$ if the runs only differ in the labels or the clock variables appearing in the clock constraints of an edge, provided that π maps both of them to the same image. We use $\mathcal{R}_{/\equiv}$ to denote the set of all equivalence classes induced by $\equiv$. From the fact that $\equiv$ induces a partition of $\mathcal{R}$, we obtain

$$\sum_{\substack{w \in T\Gamma^+ \\ \pi(w)=v}} (\|\mathcal{A}\|, w) = \sum_{X \in \mathcal{R}_{/\equiv}} \sum_{r \in X} rwt(r).$$

Next, let $X \in \mathcal{R}_{/\equiv}$ and $r \in X$ be of the form $r = s_0 \xrightarrow{a_1,\phi_1} ... \xrightarrow{a_{|w|},\phi_{|w|}} s_{|w|}$. We define r_X to be the sequence that is obtained from r by replacing each edge $e_i = (s_{i-1}, a_i, \phi_i, s_i) \in E$ by the corresponding edge $e'_i = (s_{i-1}, \pi(a_i), \pi(\phi_i), s_i) \in E'$, obtaining the run $r_X = s_0 \xrightarrow{\pi(a_1),\pi(\phi_1)} ... \xrightarrow{\pi(a_{|w|}),\pi(\phi'_{|w|})} s_{|w|}$. In the following, we show that r_X is a successful run of $\mathcal{A}'$ on v.

We start with proving $\gamma_i^v \models \pi(\phi_i)$ for every $i \in \{1, ..., |w|\}$. Assume $(x_a \sim c) \in \phi_i$ for some $i \in \{1, ..., |w|\}$, $a \in \Gamma$, $c \in \mathbb{N}$ and $\sim \in \{<, \leq, =, \geq, >\}$. Since r is a run of $\mathcal{A}$ on w, we have $\gamma_i^w \models \phi_i$ for each $i \in \{1, ..., |w|\}$. Hence there exists some j such that $1 \leq j < i$ with $g(a_j) = g(a)$ and $t_i - t_j \sim c$, and for all m with $j < m < i$, we have $g(a_m) \neq g(a)$. By assumption, π is valid w.r.t. (U, g, C). Thus we can infer $g(\pi(a_j)) = g(\pi(a))$ and $t_i - t_j \sim c$ and we have $g(\pi(a_m)) \neq g(\pi(a))$ for all m with $j < m < i$. But this immediately implies $\gamma_i^v \models x_{\pi(a)} \sim c$. Since ϕ_i is a conjunction of clock constraints of the form $x_a \sim c$, we obtain $\gamma_i^v \models \pi(\phi_i)$ for every $i \in \{1, ..., |v|\}$. Hence, together with the fact that we still have $s_0 \in S_0$ and $s_{|w|} \in S_f$, it follows that r_X is a successful run. Moreover, the set of successful runs of $\mathcal{A}'$ on v is precisely the set of such runs r_X for each $X \in \mathcal{R}_{/\equiv}$, i.e.,

$$(\|\mathcal{A}'\|, v) = \sum_{X \in \mathcal{R}_{/\equiv}} rwt(r_X),$$

where r_X is the run of $\mathcal{A}'$ on v obtained from an arbitrary run $r \in \mathcal{R}$ as described above. Next, we show that for every $X \in \mathcal{R}_{/\equiv}$, $rwt(r_X) = \sum_{r \in X} rwt(r)$, which, with the help of the two equations above, leads us to the desired result. Let $X \in \mathcal{R}_{/\equiv}$ and $r \in X$ as above. Then, the following equation holds by distributivity of $\mathcal{K}$:

$$\begin{aligned} rwt(r_X) &= \prod_{1\leq i\leq |v|} (C'_{s_{i-1}}(t_i - t_{i-1}) \cdot C'_{\mathcal{E}}(e'_i)) \\ &= \prod_{1\leq i\leq |v|} (C_{s_{i-1}}(t_i - t_{i-1}) \cdot \sum_{\substack{(s,a,\phi,s')\in E \\ \pi(a)=a'_i, \pi(\phi)=\phi'_i}} C_{\mathcal{E}}((s,a,\phi,s'))) \\ &= \sum_{\substack{(s,a,\phi,s')\in E \\ \pi(a)=a'_i, \pi(\phi)=\phi'_i}} \prod_{1\leq i\leq |v|} (C_{s_{i-1}}(t_i - t_{i-1}) \cdot C_{\mathcal{E}}((s,a,\phi,s')) \\ &= \sum_{r\in X} rwt(r) \end{aligned}$$

Hence, $(\|\mathcal{A}'\|, v) = \sum_{X\in\mathcal{R}_{/\equiv}} rwt(r_X) = \sum_{X\in\mathcal{R}_{/\equiv}} \sum_{r\in X} rwt(r) = \sum_{\substack{w\in T\Gamma^+ \\ \pi(w)=v}} (\|\mathcal{A}\|, w)$, and thus $\|\mathcal{A}'\| = \bar{\pi}(\|\mathcal{A}\|)$.

Inverse Renamings Let $\Gamma, \Delta \subseteq U$ be finite and $\pi : \Gamma \to \Delta$ be a renaming valid w.r.t. (U, f, C). Further, let $\mathcal{A} = (S, S_0, S_f, E, C)$ be a qWERA over $\mathcal{K}$, Δ and $\mathcal{F}$. Define $E' = \{(s, a, \phi, s')|(s, \pi(a), \pi(\phi), s') \in S\}$ and $C'_{\mathcal{E}}((s, a, \phi, s')) = C_{\mathcal{E}}((s, \pi(a), \pi(\phi), s'))$. Then, one can easily check that the behaviour of the qWERA $\mathcal{A}' = (S, S_0, S_f, E', C')$ over $\mathcal{K}$, Γ and $\mathcal{F}$ precisely corresponds to $\bar{\pi}^{-1}(\|\mathcal{A}\|)$. □

Finally, we investigate closure properties of non-interfering timed series.

Lemma 4. *Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be pairwise non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C). Then the following holds:*

1. *$\mathcal{T}_1 + \mathcal{T}_2$ and $\mathcal{T}_3$ are non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C).*
2. *$\mathcal{T}_1 \odot \mathcal{T}_2$ and $\mathcal{T}_3$ are non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C).*

Proof. We give the proof for 2. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be pairwise non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C). Then, there are qWERA $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ over $\mathcal{K}$, Γ and $\mathcal{F}$ for some $\Gamma \subseteq U$ such that $\|\mathcal{A}^i\| = \mathcal{T}_i$ for each $i \in \{1, 2, 3\}$ and $\mathcal{A}^i$ and $\mathcal{A}^j$ are non-interfering for each $i, j \in \{1, 2, 3\}$ such that $i \neq j$. So let $\mathcal{A}^{1,2}$ be the qWERA recognizing $\mathcal{T}_1 \odot \mathcal{T}_2$ (see the proof of Lemma 2). Now assume there is a location (s^1, s^2) in $\mathcal{A}^{1,2}$ and s^3 in $\mathcal{A}^3$ such that $C_{(s^1,s^2)} \neq \mathbb{1}$ and $C_{s^3} \neq \mathbb{1}$ and in the product qWERA of $\mathcal{A}^{1,2}$ and $\mathcal{A}^3$ there is a successful run from $((s^1, s^2), s^3)$. Let this successful run be of the form $((s^1, s^2), s^3) \xrightarrow{a_1,\phi_1} \ldots \xrightarrow{a_n,\phi_n} ((s^1_n, s^2_n), s^3_n)$ (for some $n \in \mathbb{N}$). By construction of the product qWERA, there must also be successful runs $(s^1, s^2) \xrightarrow{a_1,\phi_1^{1,2}} \ldots \xrightarrow{a_n,\phi_n^{1,2}} (s^1_n, s^2_n)$ in $\mathcal{A}^{1,2}$ and $s^3 \xrightarrow{a_1,\phi_1^3} \ldots \xrightarrow{a_n,\phi_n^3} s^3_n$ in $\mathcal{A}^3$ such that $\phi_i^{1,2} \wedge \phi_i^3 = \phi_i$ for each $i \in \{1, ..., n\}$. Since $C_{(s_1,s_2)} \neq \mathbb{1}$, we must have either $C_{s_1} \neq \mathbb{1}$ or $C_{s_2} \neq \mathbb{1}$. Without loss of generality, we assume $C_{s_1} \neq \mathbb{1}$. Again, by construction of the product qWERA, there must be a successful run $s^1 \xrightarrow{a_1,\phi_1^1} \ldots \xrightarrow{a_n,\phi_n^1} s^1_n$ in $\mathcal{A}^1$. But then, there is also a successful run of the product qWERA $\mathcal{A}^{1,3}$ recognizing $\mathcal{T}_1 \odot \mathcal{T}_3$ of the form $(s^1, s^3) \xrightarrow{a_1,\phi_1^{1,3}} \ldots \xrightarrow{a_n,\phi_n^{1,3}} (s^1_n, s^3_n)$, where $\phi_i^{1,3} = \phi_i^1 \wedge \phi_i^3$ for each $i \in \{1, ..., n\}$. However, we also have $C_{s^1} \neq \mathbb{1}$ and $C_{s^3} \neq \mathbb{1}$, which contradicts the assumption that $\mathcal{A}^1$ and $\mathcal{A}3$ are non-interfering. Thus, $\mathcal{A}^{1,2}$ and $\mathcal{A}^3$ must be non-interfering. □

2 Weighted Timed MSO Logic

Next, we introduce a weighted timed MSO logic for specifying properties of timed series. Our logic is an extension of the logic $\mathrm{MSO_{er}}(\Sigma)$ introduced by D'Souza for characterizing the behaviour of event-recording automata [15], which we briefly recall here. Formulas of $\mathrm{MSO_{er}}(\Sigma)$ are built inductively from atomic formulas $P_a(x)$, $x = y$, $x < y$, $x \in X$, $\lhd_a(x) \sim c$ using the connectives $\vee$, $\neg$, $\exists x.$ and $\exists X.$, where x, y are first-order variables, X is a second-order variable, $a \in \Sigma$, $c \in \mathbb{N}$ and $\sim \in \{<, \leq, =, \geq, >\}$ or $(\sim c) = (= \perp)$. As usual, we may also use $\rightarrow$, $\leftrightarrow$, $\wedge$, $\forall x.$ and $\forall X.$ as abbreviations. Formulas of $\mathrm{MSO_{er}}(\Sigma)$ are interpreted over timed words. For this, we associate with $w = (a_1, t_1)...(a_k, t_k)$ the relational structure consisting of the domain $\mathrm{dom}(w)$ together with the unary relations $P_a = \{i \in \mathrm{dom}(w) | a_i = a\}$ and $\lhd_a(.) \sim c = \{i \in \mathrm{dom}(w) | \gamma_i^w(x_a) \sim c\}$ as well as the usual $<$ and $=$ relations on $\mathrm{dom}(w)$. Now, for $\varphi \in \mathrm{MSO_{er}}(\Sigma)$, let $\mathrm{Free}(\varphi)$ be the set of free variables, $\mathcal{V} \supseteq \mathrm{Free}(\varphi)$ be a finite set of first- and second-order variables, and σ be a $(\mathcal{V}, w)$-assignment mapping first-order (second-order, resp.) variables to elements (subsets, resp.) of $\mathrm{dom}(w)$. For $i \in \mathrm{dom}(w)$, we let $\sigma[x \rightarrow i]$ be the assignment that maps x to i and agrees with σ on every variable $\mathcal{V} \backslash \{x\}$. Similarly, we define $\sigma[X \rightarrow I]$ for any $I \subseteq \mathrm{dom}(w)$. A timed word $(\bar{a}, \bar{t})$ and a $(\mathcal{V}, (\bar{a}, \bar{t}))$-assignment σ is encoded as timed word over the extended alphabet $\Sigma_{\mathcal{V}}$. A timed word over $\Sigma_{\mathcal{V}}$ is written as $((\bar{a}, \sigma), \bar{t})$, where $\bar{a}$ is the projection over Σ and σ is the projection over $\{0, 1\}^{\mathcal{V}}$. Then, σ represents a *valid* assignment over $\mathcal{V}$ if for each first-order variable $x \in \mathcal{V}$, the x-row of σ contains exactly one 1. In this case, σ is identified with the $(\mathcal{V}, (\bar{a}, \bar{t}))$-assignment such that for every first-order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 in the x-row, and for each second-order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions with a 1 in the X-row. We define $N_{\mathcal{V}} = \{((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^+ | \sigma \text{ is a valid } (\mathcal{V}, (\bar{a}, \bar{t}))\text{-assignment}\}$. The definition that $((\bar{a}, \sigma), \bar{t})$ satisfies φ, written $((\bar{a}, \sigma), \bar{t}) \models \varphi$, is as usual. We let $L_{\mathcal{V}}(\varphi) = \{((\bar{a}, \sigma), \bar{t}) \in N_{\mathcal{V}} | ((\bar{a}, \sigma), \bar{t}) \models \varphi\}$. The formula φ *defines* the timed language $L(\varphi) = L_{\mathrm{Free}(\varphi)}(\varphi)$. A timed language $L \subseteq T\Sigma^+$ is $\mathrm{MSO_{er}}(\Sigma)$-*definable* if there exists a sentence $\varphi \in \mathrm{MSO_{er}}(\Sigma)$ such that $L(\varphi) = L$.

Theorem 1. *[15] A timed language $L \subseteq T\Sigma^+$ is $MSO_{er}(\Sigma)$-definable if and only if L is ERA-recognizable over Σ. If $\varphi \in MSO_{er}(\Sigma)$, then $L(\varphi)$ is qERA-recognizable w.r.t. (U, f, C).*

Now, we turn to the weighted logic $\mathrm{MSO_{er}}(\mathcal{K}, \Sigma, \mathcal{F})$, defined inductively as follows. The *atomic* formulas are formulas of the form $P_a(x)$, $x = y$, $x < y$, $x \in X$, $\lhd_a(x) \sim c$ and their negations, where $x, y, X, a, c, \sim$ are as above. Additionaly, we define *weighted atomic* formulas to be formulas of the form k and $C_f(x)$, where $k \in K$ and $f \in \mathcal{F}$. Complex formulas can be built from atomic and weighted atomic formulas using the connectives $\wedge$, $\vee$, $\exists x.$, $\forall x.$, $\exists X.$ and $\forall X.$ Notice that we only allow to apply negation to atomic formulas. This is because for arbitrary semirings it is not clear what the negation of a weighted atomic formula should mean. Let $\varphi \in \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ and $\mathcal{V} \supseteq \text{Free}(\varphi)$. The $\mathcal{V}$-semantics of φ is a timed series $[\![\varphi]\!]_{\mathcal{V}} : T(\Sigma_{\mathcal{V}})^+ \to K$. Let $(\bar{a}, \bar{t}) \in T\Sigma^+$. If σ is a valid $(\mathcal{V}, (\bar{a}, \bar{t}))$-assignment, $[\![\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) \in K$ is defined inductively as follows:

$$
\begin{aligned}
[\![\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= 1_{L_{\mathcal{V}}(\varphi)}((\bar{a}, \sigma), \bar{t}) \text{ if } \varphi \text{ is atomic} \\
[\![k]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= k \\
[\![C_f(x)]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= f(t_{\sigma(x)} - t_{\sigma(x)-1}) \\
[\![\varphi \vee \varphi']\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= [\![\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) + [\![\varphi']\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) \\
[\![\varphi \wedge \varphi']\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= [\![\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) \cdot [\![\varphi']\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) \\
[\![\exists x.\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= \sum_{i \in \text{dom}((\bar{a}, \bar{t}))} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}((\bar{a}, \sigma[x \to i]), \bar{t}) \\
[\![\forall x.\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= \prod_{i \in \text{dom}((\bar{a}, \bar{t}))} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}((\bar{a}, \sigma[x \to i]), \bar{t}) \\
[\![\exists X.\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= \sum_{I \subseteq \text{dom}((\bar{a}, \bar{t}))} [\![\varphi]\!]_{\mathcal{V} \cup \{X\}}((\bar{a}, \sigma[X \to I]), \bar{t}) \\
[\![\forall X.\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) &= \prod_{I \subseteq \text{dom}((\bar{a}, \bar{t}))} [\![\varphi]\!]_{\mathcal{V} \cup \{X\}}((\bar{a}, \sigma[X \to I]), \bar{t})
\end{aligned}
$$

For σ not a valid $(\mathcal{V}, (\bar{a}, \bar{t}))$-assignment, we define $[\![\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) = 0$. We write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\text{Free}(\varphi)}$. We say that two formulas φ and ψ are *equivalent*, written $\varphi \equiv \psi$, if $[\![\varphi]\!] = [\![\psi]\!]$.

Remark 2. If we let $\mathcal{K}$ be the Boolean semiring, then $\text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ corresponds to $\text{MSO}_{\text{er}}(\Sigma)$ as every formula in $\text{MSO}_{\text{er}}(\Sigma)$ is equivalent to a formula where negation is applied to atomic subformulas only. Also, every such formula $\varphi \in \text{MSO}_{\text{er}}(\Sigma)$ can be seen to be a formula of our logic.

Example 2. Consider the formula $\varphi = \exists x. \lhd_a (x) < 2$ and let $w = (a, 1.7)(b, 3.0)(a, 3.6)(a, 6.0)$. If we interpret φ as an $\text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$-formula over the Boolean semiring or, equivalently, as an $\text{MSO}_{\text{er}}(\Sigma)$-formula, we have $[\![\varphi]\!](w) = 1$. If on the other hand, we let $\mathcal{K}$ be the semiring over the natural numbers with ordinary addition and multiplication, we have $[\![\varphi]\!](w) = 2$, i.e., we *count* the number of positions i in w where $\gamma_i^w(x_a) < 2$ is satisfied. Counting how often a certain property holds gives rise to interesting applications in the field of verification.

Let $\mathcal{L} \subseteq \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$. A timed series $\mathcal{T} : T\Sigma^+ \to \mathcal{K}$ is called $\mathcal{L}$-*definable* if there is a sentence $\varphi \in \mathcal{L}$ such that $[\![\varphi]\!] = \mathcal{T}$. The goal of this paper is to find a suitable fragment $\mathcal{L}$ of $\text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ such that $\mathcal{L}$-definable timed series precisely correspond to WERA-recognizable timed series over $\mathcal{K}$, Σ and $\mathcal{F}$, i.e., we want to generalize Theorem 1 to the weighted setting. It is not surprising that $\text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ does not constitute a suitable candidate for $\mathcal{L}$ since this is already not the case in the untimed setting [10]. In the next section, we explain the problems that occur when we do not restrict the logic and step by step develop solutions resulting in the logic $\mathrm{sRMSO_{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ for which we are able to give the following Büchi-type theorem.

Theorem 2. *A timed series $\mathcal{T} : T\Sigma^+ \to K$ is WERA-recognizable over $\mathcal{K}$, Σ and $\mathcal{F}$ if and only if $\mathcal{T}$ is definable by some sentence in $sRMSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$. The respective transformations can be done effectively provided that the operations of $\mathcal{K}$ and $\mathcal{F}$ are given effectively.*

3 From Logic To Automata

In this section, we want to prove the direction from right to left in Theorem 2 and show that for every sentence φ of our weighted timed MSO logic, $[\![\varphi]\!]$ is a WERA-recognizable timed series. We do this similarly to the corresponding proof for the classical setting [27], i.e., by induction over the structure of the logic. As already mentioned, due to the non-closure of WERA-recognizable timed series under renaming operations, we adopt the approach proposed by D'Souza [15] and work with qWERA rather than WERA. However, in the last step, when there are no free variables left, we will see that the resulting qWERA is nothing else than a WERA.

Let $n \in \mathbb{N}\backslash\{0\}$. We define $\Sigma^{(n)} = \Sigma \times \{1, ..., n\}$. Similarly to timed words over an extended alphabets of the form $\Sigma_\mathcal{V}$ for some finite set of variables $\mathcal{V}$, we write $((\bar{a}, \mu), \bar{t})$ to denote a timed word over $\Sigma^{(n)}$, where $(\bar{a}, \bar{t}) \in T\Sigma^+$ and $\mu \in \{1, ..., n\}^{\mathrm{dom}((\bar{a}, \bar{t}))}$. We let (U, g, C) such that U contains all elements of Σ, $\Sigma_\mathcal{V}$, $\Sigma^{(n)}$ and $(\Sigma^{(n)})_\mathcal{V}$ for every finite set of variables $\mathcal{V}$ and every $n \in \mathbb{N}\backslash\{0\}$, respectively, and $C = \Sigma$. The function f, restricted to $\Sigma, \Sigma_\mathcal{V}$, $\Sigma^{(n)}$ and $(\Sigma^{(n)})_\mathcal{V}$, maps every $a \in \Sigma$, $(a, \sigma) \in \Sigma_\mathcal{V}$, $(a, \mu) \in \Sigma^{(n)}$ and $(a, \mu, \sigma) \in (\Sigma^{(n)})_\mathcal{V}$, respectively, to a.

For the induction base, we show that for every atomic formula φ in $\mathrm{MSO_{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ there is a qWERA over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ) recognizing $[\![\varphi]\!]$. For φ of the form $P_a(x)$, $x = y$, $x < y$, $x \in X$ and its negations, this can be done as in the classical setting [27]. For instance, for $\varphi = P_a(x)$ the idea is to construct a qWERA over $\mathcal{K}$, the extended alphabet $\Sigma_{\{x\}}$ and $\mathcal{F}$ that (1) verifies that whenever there is a 1 on the x-row, then an a must be read and (2) assigns to all edges weight 1 and all locations the cost function $\mathbb{1}$. This idea can easily be extended to the remaining atomic formulas of our logic (see Fig.2). For $\varphi = \triangleleft_a(x) \sim c$, we build a qWERA that verifies that whenever an edge is labeled by a 1 in the x-row, the time distance to the last event that is mapped by f to a satisfies $\sim c$. We do this by adding a corresponding guard to this edge. Similarly to the untimed setting, all edges are assigned 1 and all location cost functions equal $\mathbb{1}$. The qWERA recognizing the timed series $[\![k]\!]$ has already been described formally in the proof of Lemma 3. For $\varphi = C_f(x)$, the qWERA verifies that whenever an edge is labeled with a 1 in the x-row, the source location of this edge must be assigned the cost function f. All the other locations must have the cost function $\mathbb{1}$. The qWERA recognizing the semantics of the negations of these formulas can be constructed in a similar way.

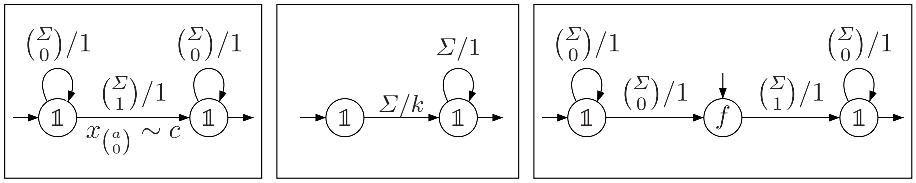

Figure 2. qWERA with behaviours $[\![\triangleleft_a(x) \sim c]\!]$, $[\![k]\!]$ and $[\![C_f(x)]\!]$

For the induction step, we need to show closure properties of qWERA-recognizable timed series under the constructs of the logic. For disjunction and existential quantification, we give proofs very similar to the classical case [26, 10, 11]. Before, we need to show that for each formula φ of our logic, the semantics for every finite set $\mathcal{V}$ of variables containing $\mathrm{Free}(\varphi)$ are consistent with each other.

Lemma 5. *Let $\varphi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$ and $\mathcal{V}$ a finite set of variables containing $Free(\varphi)$. Then*

$$[\![\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t}) = [\![\varphi]\!]((\bar{a}, \sigma_{|Free(\varphi)}), \bar{t})$$

for each $((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^+$ such that σ is a valid $(\mathcal{V}, (\bar{a}, \bar{t}))$-assignment. In particular, $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ) if and only if $[\![\varphi]\!]_{\mathcal{V}}$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).

Proof. The first claim can be shown by induction on the structure of $\mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$. For the second claim, let $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ and $\mathcal{V} \supseteq \mathrm{Free}(\varphi)$ be a finite set of variables. Consider the projection $\pi : \Sigma_{\mathcal{V}} \to \Sigma_{\mathrm{Free}(\varphi)} : (a, \sigma) \mapsto (a, \sigma_{|\mathrm{Free}(\varphi)})$ and notice that this is a valid renaming w.r.t. (U, f, Σ). First, assume that $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). One can show that $[\![\varphi]\!]_{\mathcal{V}} = \bar{\pi}^{-1}([\![\varphi]\!]) \odot 1_{N_{\mathcal{V}}}$. Recall that $N_{\mathcal{V}}$ is qERA-recognizable w.r.t. (U, f, Σ), and thus, by Lemma 1, $1_{N_{\mathcal{V}}}$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Also, by Lemma 3, $\bar{\pi}^{-1}([\![\varphi]\!])$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Furthermore, all the conditions for applying Lemma 2 are satisfied, and thus, $[\![\varphi]\!]_{\mathcal{V}}$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).

Conversely, let $[\![\varphi]\!]_{\mathcal{V}}$ be qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Define $F = \{((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^+ | \forall x, X \in \mathcal{V} \backslash \mathrm{Free}(\varphi). \sigma(x) = 1, \sigma(X) = 1\}$. Clearly, F is qERA-recognizable w.r.t. (U, f, Σ). Furthermore, for each $((\bar{a}, \sigma'), \bar{t}) \in T(\Sigma_{\mathrm{Free}(\varphi)})^+$ there is a unique $((\bar{a}, \sigma), \bar{t}) \in F$ such that $\pi(((\bar{a}, \sigma), \bar{t})) = ((\bar{a}, \sigma'), \bar{t})$. Then, we obtain $[\![\varphi]\!] = \pi([\![\varphi]\!]_{\mathcal{V}} \odot 1_F)$. Hence, by Lemmas 1, 2 and 3, $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). $\square$

Lemma 6. *Let $\varphi, \psi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$. If $[\![\varphi]\!]$ and $[\![\psi]\!]$ are qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ), then $[\![\varphi \vee \psi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).*

Proof. Let $[\![\varphi]\!]$ and $[\![\psi]\!]$ be qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Since $\text{Free}(\varphi \vee \psi) = \text{Free}(\varphi) \cup \text{Free}(\psi)$ and thus $\text{Free}(\varphi) \subseteq \text{Free}(\varphi \vee \psi)$, we can apply Lemma 5 and conclude that $[\![\varphi]\!]_{\text{Free}(\varphi\vee\psi)}$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). The same holds for $[\![\psi]\!]_{\text{Free}(\varphi\vee\psi)}$. The semantics of $\text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ produce the following equation:

$$\begin{aligned}[\![\varphi \vee \psi]\!] &= [\![\varphi \vee \psi]\!]_{\text{Free}(\varphi\vee\psi)}\\ &= [\![\varphi]\!]_{\text{Free}(\varphi\vee\psi)} + [\![\psi]\!]_{\text{Free}(\varphi\vee\psi)}\end{aligned}$$

By Lemma 3, $[\![\varphi \vee \psi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). □

Lemma 7. *Let $\varphi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$. If $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ), then $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).*

Proof. Let $\varphi \in \text{MSO}_{\text{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ be such that $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Here, we show that $[\![\exists X.\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Let $\mathcal{V} = \text{Free}(\exists X.\varphi)$. Consider the projection $\pi : \Sigma_{\mathcal{V}\cup\{X\}} \to \Sigma_{\mathcal{V}}$ which erases the X-row. Notice that π is valid w.r.t. (U, f, Σ). In the following, with $\bar{\pi}$ we denote the extension of π to timed series as defined above.

First, we will show that the projection $\bar{\pi}$ on $[\![\varphi]\!]$ equals $[\![\exists X.\varphi]\!]$. Then, we use Lemma 5 and Lemma 3 to conclude that $[\![\exists X.\varphi]\!]$ is qWERA-recognizable.

Let $((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{V}})^+$. By definition of $\bar{\pi}$, we obtain

$$\big(\bar{\pi}([\![\varphi]\!]_{\mathcal{V}\cup\{X\}}), ((\bar{a}, \sigma), \bar{t})\big) = \sum_{\substack{((\bar{a},\sigma'),\bar{t})\in T(\Sigma_{\mathcal{V}\cup\{X\}})^+\\ \pi(((\bar{a},\sigma'),\bar{t}))=((\bar{a},\sigma),\bar{t})}} \big([\![\varphi]\!]_{\mathcal{V}\cup\{X\}}((\bar{a}, \sigma'), \bar{t})\big)$$

However, using the equivalences

$$\pi(((\bar{a}, \sigma'), \bar{t})) = ((\bar{a}, \sigma), \bar{t}) \text{ iff } \sigma' = \sigma[X \to I] \text{ for some } I \subseteq \text{dom}((\bar{a}, \bar{t}))$$

and

$$\begin{aligned}\sigma \text{ is a valid } (\mathcal{V}, (\bar{a}, \bar{t})) \text{ iff } \sigma[X \to I] &\text{ is a valid } (V \cup \{X\}, (\bar{a}, \bar{t}))\\ &\text{-assignment for every } I \subseteq \{1, ..., |w|\},\end{aligned}$$

we obtain

$$\begin{aligned}&= \sum_{I\subseteq\text{dom}((\bar{a},\bar{t}))} \big([\![\varphi]\!]_{\mathcal{V}\cup\{X\}}((\bar{a}, \sigma[X \to I]), \bar{t})\big)\\ &= [\![\exists X.\varphi]\!]_{\mathcal{V}}((\bar{a}, \sigma), \bar{t})\end{aligned}$$

Now, as $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ), we obtain qWERA-recognizability over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ) of $[\![\varphi]\!]_{\mathcal{V}\cup\{X\}}$ by Lemma 5. By Lemma 3, $\bar{\pi}([\![\varphi]\!]_{\mathcal{V}\cup\{X\}})$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ), which finishes the proof. □

Altogether, the proofs for disjunction and existential quantification are very similar to the corresponding proofs for the classical setting. However, we will see that for the remaining operators of our logic, we cannot give such easy extensions of the classical proofs.

First of all, in Sect.1 we have seen that recognizable timed series in general are not closed under the Hadamard product. Since the semantics of conjunction is defined using the Hadamard product, we have to restrict the usage of conjunction. More precisely, we either have to require that $\mathcal{F}$ is such that for all $f_1, f_2 \in \mathcal{F}$ we have $f_1 \odot f_2 \in \mathcal{F}$, or we have to formulate a syntactical restriction implying that whenever two formulas φ_1 and φ_2 are combined by a conjunction, then $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$ are non-interfering. We make the following observations. Consider the formula $C_{f_1}(x_1) \wedge C_{f_2}(x_2)$ and let $\mathcal{A}_i$ be a WERA such that $\|\mathcal{A}_i\| = [\![C_{f_i}(x_i)]\!]$ for each $i = 1, 2$ (see Fig.2). We use s_1 (s_2, resp.) to denote the location in $\mathcal{A}_1$ ($\mathcal{A}_2$, resp.) with cost function f_1 (f_2, resp.). We want to enforce that in the product automaton of $\mathcal{A}_1$ and $\mathcal{A}_2$, from the pair (s_1, s_2) there is no run to a final location. This is the case if from s_1 and s_2 no common letter can be read. Observe that from s_1 (s_2, resp.) every outgoing edge is labeled with (a, σ) such that $\sigma(x_1) = 1$ ($\sigma(x_2) = 1$, resp.) for every $a \in \Sigma$. Hence, in the product automaton every edge from (s_1, s_2) must be labeled with a letter of the form (a, σ) such that $\sigma(x_1) = \sigma(x_2) = 1$ for every $a \in \Sigma$. By requiring x_1 and x_2 to refer to different positions in a timed word, we can exclude that reading a letter of this form leads to a final location. This is done by conjoining the formula above with $\neg(x_1 = x_2)$.

Lemma 8. *Let $\varphi_1, \varphi_2 \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$ such that $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$ are qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C). Assume that whenever φ_1 contains the subformula $C_{f_1}(x_1)$ and φ_2 contains $C_{f_2}(x_2)$, then x_1, x_2 are free in both φ_1 and φ_2, and either φ_1 or φ_2 is of the form $\psi \wedge \neg(x_1 = x_2)$ for some $\psi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$. Then $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$ are non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C).*

Proof. The proof is rather technical and we only present a sketch of it. Let $\varphi_1, \varphi_2 \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ such that $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$ are qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C). Assume that whenever φ_1 contains the subformula $C_{f_1}(x_1)$ and φ_2 contains $C_{f_2}(x_2)$, then x_1, x_2 are free in both φ_1 and φ_2, and either φ_1 or φ_2 is of the form $\psi \wedge \neg(x_1 = x_2)$ for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$.

Assume φ_2 is of the form $C_{f_2}(x_2)$. Let φ_1 be one of the formulas k, $P_a(x)$, $x = y$, $x < y$ or its negation. Then $[\![\varphi_1]\!]$ is recognizable by a qWERA over $\mathcal{K}$, Γ and $\mathcal{F}$ for some $\Gamma \subseteq U$ such that for all locations s we have $C_s = \mathbb{1}$. Thus $[\![\varphi_1]\!]$ is non-interfering with every qWERA-recognizable timed series, in particular with $[\![\varphi_2]\!]$.

Now let $\varphi_1 = C_{f_1}(x_1) \wedge \neg(x_1 = x_2)$. In Fig.3, we show the qWERA $\mathcal{A}_1$ and $\mathcal{A}_2$ recognizing $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$, respectively, both over the extended alphabet $\Sigma_{\{x_1,x_2\}}$.

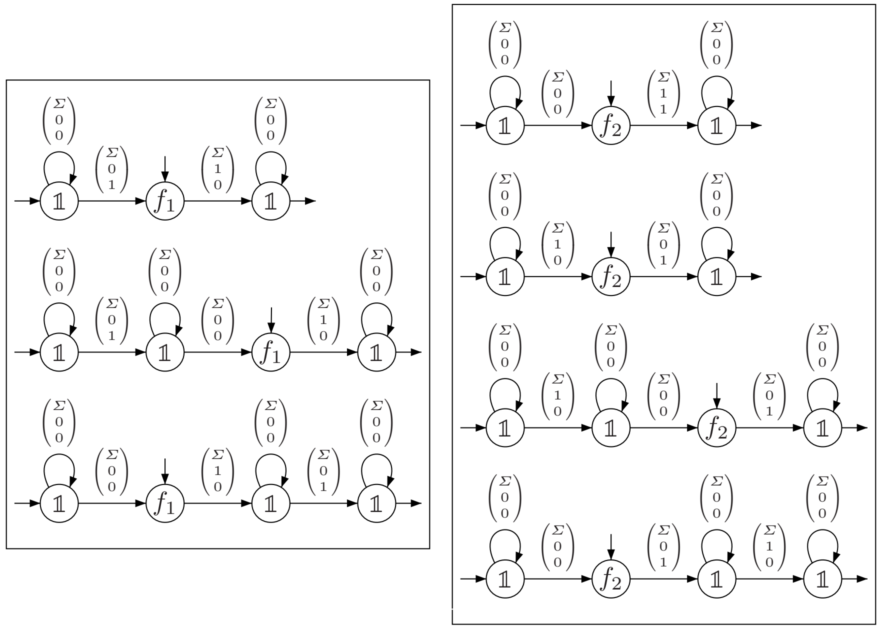

Fig. 3. qWERA over $\Sigma_{\{x_1,x_2\}}$ with behaviours $[\![C_{f_1}(x_1) \wedge \neg(x_1 = x_2)]\!]$ and $[\![C_{f_2}(x_2)]\!]$

Clearly, $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-interfering: from each location s_1 in $\mathcal{A}_1$ such that $C^1_{s_1} = f_1$ there are only edges labeled with $(a, 1, 0)$ for every $a \in \Sigma$, whereas from every location s_2 in $\mathcal{A}_2$ such that $C^2_{s_2} = f_2$ there are only edges labeled with either $(a, 1, 1)$ or $(a, 0, 1)$ for every $a \in \Sigma$. Hence, in the product automaton of $\mathcal{A}_1$ and $\mathcal{A}_2$, there is no successful run starting in (s_1, s_2).

Now assume $\varphi_1 = \psi_1 \wedge \psi_2$ such that both $[\![\psi_1]\!]$ and $[\![\psi_2]\!]$ are non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ) with $[\![\varphi_2]\!]$. Then by Lemma 4 $[\![\psi_1 \wedge \psi_2]\!]$ is non-interfering with $[\![\varphi_2]\!]$, too. The same applies if $\varphi_1 = \psi_1 \vee \psi_2$. Now observe that if $\varphi_1 = \exists x.\psi$ or $\varphi_1 = \forall x.\psi$ such that $[\![\psi]\!]$ is non-interfering with $[\![\varphi_2]\!]$, we must have $x \neq x_1, x_2$. This implies that the qWERA recognizing $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$ do not change in the crucial parts that we used for proving non-interference. The same holds for φ_1 to be of the form $\forall X.\psi$ or $\exists X.\psi$. □

Lemma 9. *Let $\varphi_1, \varphi_2 \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$ such that $[\![\varphi_1]\!]$ and $[\![\varphi_2]\!]$ are qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C). Assume that whenever φ_1 contains the subformula $C_{f_1}(x_1)$ and φ_2 contains $C_{f_2}(x_2)$, then x_1, x_2 are free in both φ_1 and φ_2, and either φ_1 or φ_2 is of the form $\psi \wedge \neg(x_1 = x_2)$ for some $\psi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$. Then $[\![\varphi_1 \wedge \varphi_2]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, C).*

Proof. The proof can be done similarly to the proof of Lemma 6 using Lemmas 8 and 2. □

Besides conjunction, we have problems with universal quantification. Examples show that unrestricted application of $\forall x.$ and $\forall X.$ do not preserve recognizability. For instance, let $\mathcal{K} = (\mathbb{N}, +, \cdot, 0, 1)$ be the semiring of the natural numbers and $\mathcal{F}$

be the family of constant functions. We consider the formula $\varphi = \forall y.\exists x.C_{\mathbb{1}}(x)$. Then we have $[\![\varphi]\!](w) = |w|^{|w|}$. However, this cannot be recognized by any qWERA over $\mathcal{K}$ and $\mathcal{F}$ as this timed series grows too fast (see [10] for a detailed proof which can also be applied to the timed setting). Similar examples can be given for $\forall X$. Hence, we need to restrict both the usage of $\forall x.$ and $\forall X.$ in our logic. We adopt the approach of Droste and Gastin [11].

For dealing with $\forall X.$, the idea is to restrict the application of $\forall X.$ to so-called *syntactically unambiguous* formulas. These are formulas $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$ such that - even though interpreted over a semiring - the semantics $[\![\varphi]\!]$ of φ always equals 0 or 1[2]. We define the set of syntactically unambiguous formulas φ^+ and φ^- for $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$ inductively as follows:

1. If φ is of the form $P_a(x)$, $x < y$, $x = y$, $x \in X$, $\lhd_a(x) \sim c$, then $\varphi^+ = \varphi$ and $\varphi^- = \neg\varphi$.
2. If $\varphi = \neg\psi$ then $\varphi^+ = \psi^-$ and $\varphi^- = \psi^+$.
3. If $\varphi = \psi \vee \zeta$ then $\varphi^+ = \psi^+ \vee (\psi^- \wedge \zeta^+)$ and $\varphi^- = \psi^- \wedge \zeta^-$
4. If $\varphi = \exists x.\psi$ then $\varphi^+ = \exists x.\psi^+ \wedge \forall y.(y < x \wedge \psi(y))^-$ and $\varphi^- = \forall x.\psi^-$
5. If $\varphi = \exists X.\psi$ then $\varphi^+ = \exists X.\psi^+ \wedge \forall Y.(Y < X \wedge \psi(Y))^-$ and $\varphi^- = \forall X.\psi^-$

where $X < Y = \exists y.y \in Y \wedge \neg(y \in X) \wedge \forall z.[z < y \longrightarrow (z \in X \longleftrightarrow z \in Y)]^+$. Notice that for each $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$ we have $[\![\varphi^+]\!] = 1_{L(\varphi)}$ and $[\![\varphi^-]\!] = 1_{L(\neg\varphi)}$.

Lemma 10. *Let φ be a syntactically unambiguous formula. Then $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).*

Proof. Let φ be syntactically unambiguous. Hence, φ is of the form ψ^+ or ψ^- for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$. So assume that $\varphi = \psi^+$ for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$. Then, we have $[\![\psi^+]\!] = 1_{L(\psi)}$. By Theorem 1, $L(\psi)$ is qERA-recognizable w.r.t. (U, f, Σ). But then Lemma 1 implies that $[\![\psi^+]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Now let φ be of the form ψ^- for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$. Then, we have $(\psi)^- = (\neg\psi)^+$ and thus we can reduce this case to the case above. □

Corollary 1. *Let φ be syntactically unambiguous. Then $[\![\forall X.\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).*

Proof. Let φ be syntactically unambiguous. Hence, φ is of the form ψ^+ or ψ^- for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$. So assume that $\varphi = \psi^+$ for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$. Then, we have $\forall X.\psi^+ \equiv \forall X.(\neg\psi)^- \equiv (\exists X.\neg\psi)^- \equiv (\neg\exists X.\neg\psi)^+ \equiv (\forall X.\psi)^+$. Hence, $\forall X.\varphi$ is syntacically unambiguous. By Lemma 10, $[\![\forall X.\psi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). If $\varphi = \psi^-$, we have $(\psi)^- \equiv (\neg\psi)^+$ and thus we can reduce this case to the case above. □

Next, we explain how to deal with $\forall x$. The approach used by Droste and Gastin [11] is to restrict the subformula φ in $\forall x.\varphi$ to so-called *almost unambiguous* formulas. Formulas of this kind can be transformed into equivalent formulas

[2] Recall that every $\mathrm{MSO}_{\mathrm{er}}(\Sigma)$-formula can also be seen as an $\mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$-formula and may have a semantics different from 0 or 1; see e.g. Ex. 2

of the form $\bigvee_{1\leq i\leq n} k_i \wedge \psi_i^+$ for some $n \in \mathbb{N}$, $k_i \in K$ and $\psi_i \in \mathrm{MSO}(\Sigma)$ for each $i \in \{1, ..., n\}$. One can easily see that the series corresponding to the semantics of such a formula has a finite image. Moreover, closure properties of recognizable series under sum, Hadamard- and scalar products can be used to prove that the semantics of such a formula is recognizable by a weighted automaton. Finally, this particular form of the formula is the base of an efficient construction of a weighted automaton recognizing $[\![\forall x.\varphi]\!]$. Here, we use a very similar approach. However, we have to redefine the notion of almost unambiguous formulas a bit in order to include subformulas of the form $C_f(x)$.

Let x be a first-order variable. We say that a formula φ is *almost unambiguous over x* if it is in the disjunctive and conjunctive closure of syntactically unambiguous formulas, constants $k \in K$ and formulas $C_f(x)$ (for $f \in \mathcal{F}$), such that $C_f(x)$ may appear at most once in every subformula of φ of the form $\varphi_1 \wedge \varphi_2$.

Lemma 11. *Let x be a first-order variable and $\varphi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$ be almost unambiguous over x. Then $[\![\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).*

Proof. Let x be a first-order variable and $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ be almost unambiguous over x. Using similar methods as in [11], one can show every almost unambiguous formala over x can be transformed into a formula ψ of the form $\bigvee_{1\leq i\leq n} C_{f_i}(x) \wedge k_i \wedge \psi_i^+$ for some $n \in \mathbb{N}$, $k_i \in K$, $f_i \in \mathcal{F}$ and $\psi_i \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$ for every $i \in \{1, ..., n\}$ such that $[\![\psi]\!] = [\![\varphi]\!]$. For every $i \in \{1, ..., n\}$, $[\![C_{f_i})(x)]\!]$, $[\![k_i]\!]$ and $[\![\psi_i^+]\!]$ are (pairwise) non-interfering over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Hence, by Lemmas 4 and 2, for each $i \in \{1, ..., n\}$, $[\![C_{f_i}(x) \wedge k_i \wedge \psi_i^+]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). Then, by Lemma 3, $[\![\psi]\!]$ (and thus also $[\![\varphi]\!]$) is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). □

So now assume that φ is almost unambiguous. The main challenge of this paper is to prove that $[\![\forall x.\varphi]\!]$ is recognizable. Before we show how this can be done, we introduce a new normalization technique which will be needed in the proof.

Lemma 12. *For every qERA-recognizable timed language $L \subseteq TU^+$ w.r.t. (U, g, C) there is an unambiguous qERA $\mathcal{A}'$ over Γ for some $\Gamma \subseteq U$ such that $L(\mathcal{A}') = L$ and for each location s in $\mathcal{A}'$ there is a unique $a \in \Gamma$ such that every edge (s, a', ϕ, s') in $\mathcal{A}'$ satisfies $a' = a$.*

Proof. Let $L \subseteq TU^+$ be qERA-recognizable w.r.t. (U, f, C). By Prop.1, there is a deterministic qERA $\mathcal{A} = (S, \{\iota\}, S_f, E)$ over Γ for some $\Gamma \subseteq U$ such that $L(\mathcal{A}) = L$. Define $\mathcal{A}' = (S', S'_0, S'_f, E')$ where

- $S' = (S \times \Gamma) \cup \{s_f\}$
- $S'_0 = \{\iota\} \times \Gamma$
- $S'_f = \{s_f\}$
- $E' = \{((s, a), a, \phi, (s', a')) \,|(s, a, \phi, s') \in E, a' \in \Gamma\} \cup \{((s, a), a, \phi, s_f) \,|(s, a, \phi, s') \in E, s' \in S_f\}$

First, we prove $L(\mathcal{A}') = L(\mathcal{A})$. Let $w \in T\Gamma^+$ and suppose $w \in L(\mathcal{A})$. As $\mathcal{A}$ is deterministic, there is exactly one successful run $r = \iota \xrightarrow{a_1,\phi_1} ... \xrightarrow{a_{|w|},\phi_{|w|}} s_{|w|}$ of $\mathcal{A}$ on w. By the definition of $\mathcal{A}'$ it follows that there are edges $e'_i = ((s_{i-1}, a_i), a_i, \phi_i, (s_i, a_{i+1})) \in E'$ for each $1 \leq i < |w|$ and there is an edge $e'_{|w|} = \big((s_{|w|-1}, a_{|w|}), a_{|w|}, \phi_{|w|}, s_f\big) \in E'$. Notice that we have $\gamma_i^w \models \phi_i$ for each $1 \leq i \leq |w|$, as we did not change neither the labels a_i nor the clock guards ϕ_i of the edges e_i. Hence, $r' = (s_0, a_1) \xrightarrow{a_1,\phi_1} (s_1, a_2) \xrightarrow{a_2,\phi_2} ... \xrightarrow{a_{|w|-1},\phi_{|w|-1}} (s_{|w|-1}, a_{|w|}) \xrightarrow{a_{|w|},\phi_{|w|}} s_f$ is a successful run of $\mathcal{A}'$ on w and we have $w \in L(\mathcal{A}')$. The proof for $L(\mathcal{A}') \subseteq L(\mathcal{A})$ is very similar and for this reason left to the reader.

Next, we aim to show that $\mathcal{A}'$ is unambiguous. Assume that $r = (s_0, u_0) \xrightarrow{a_1,\phi_1} (s_1, u_1) \xrightarrow{a_2,\phi_2} ... \xrightarrow{a_{|w|},\phi_{|w|}} s_f$ and $r' = (s'_0, u'_0) \xrightarrow{a_1,\phi'_1} (s'_1, u'_1) \xrightarrow{a_2,\phi'_2} ... \xrightarrow{a_{|w|},\phi'_{|w|}} s_f$ both are successful runs of $\mathcal{A}'$ on w. Thus, we have $((s_{i-1}, u_{i-1}), a_i, \phi_i, (s_i, u_i)) \in E'$ and $((s'_{i-1}, u'_{i-1}), a_i, \phi'_i, (s'_i, u'_i)) \in E'$ for every $i \in \{1, ..., |w|\}$. Moreover, by definition of E', $u_{i-1} = u'_{i-1} = a_i$ for each $i \in \{1, ..., |w|\}$. Also, we have $s_0 = s'_0 = \iota$ by definition of S'_0. Since both r and r' are successful runs on w, it follows that both $\gamma_1^w \models \phi_1$ and $\gamma_1^w \models \phi'_1$. Then, the determinism of $\mathcal{A}$ implies that $e_1 = e'_1$ and thus $s_1 = s'_1$. The same lines of argumentation can be used to infer $e_i = e'_i$ and $s_i = s'_i$ for every $i \in \{2, ..., |w| - 1\}$ and $e_{|w|} = e'_{|w|}$, respectively. Hence, $r = r'$. □

Remark 3. Notice that in the construction we included an implicit final-location-normalization, allowing one single final location (s_f) only. This is done to guarantee the uniqueness of the successful runs - and thus the unambiguity - of $\mathcal{A}'$: if we set $S'_f = S_f \times \Gamma$, we could no longer assume the locations $(s_{|w|}, u_{|w|})$ and $(s'_{|w|}, u'_{|w|})$ reached after the last edge to be equal, because $u_{|w|}$ and $u'_{|w|}$ is not determined by a next letter as it is the case with u_i and u'_i for $i \in \{1, ..., |w|-1\}$.

Lemma 13. *Let $\varphi \in MSO_{er}(\mathcal{K}, \Sigma, \mathcal{F})$ be almost unambiguous. Then $[\![\forall x.\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ).*

Proof. Let $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ be almost unambiguous and $\mathcal{W} = \mathrm{Free}(\varphi)$. In the proof of Lemma 11 we have seen that we may assume φ to be of the form $\bigvee_{1 \leq j \leq n} C_{f_j}(x) \wedge k_j \wedge \psi_j^+$, where $n \in \mathbb{N}$, $k_j \in K$, $f_j \in \mathcal{F}$, $\psi_j \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$ for each $j \in \{1, ..., n\}$. Let $L_j = L_{\mathcal{W}}(\psi_j)$ for every $j \in \{1, ..., n\}$. We may assume $(L_1, ..., L_n)$ to be a partition of $N_{\mathcal{W}}$. Let $\mathcal{V} = \mathrm{Free}(\forall x.\varphi) = \mathcal{W} \backslash \{x\}$. Recall that for every $((\bar{a}, \sigma), \bar{t}) \in T(\Sigma_{\mathcal{W}})^+ \backslash N_{\mathcal{W}}$ we have $[\![\varphi]\!]((\bar{a}, \sigma), \bar{t}) = 0$. We define $\widetilde{L}$ to be the set of timed words $((\bar{a}, \mu, \sigma), \bar{t})$ in $T((\Sigma^{(n)})_{\mathcal{V}})^+$ such that $((\bar{a}, \sigma), \bar{t}) \in N_{\mathcal{V}}$ and for all $i \in \mathrm{dom}((\bar{a}, \bar{t}))$ and $j \in \{1, ..., n\}$ we have $\mu(i) = j$ implies $((\bar{a}, \sigma[x \rightarrow i]), \bar{t}) \in L_j$. Notice that for every $((\bar{a}, \sigma), \bar{t}) \in N_{\mathcal{V}}$ there is a unique μ such that $((\bar{a}, \mu, \sigma), \bar{t}) \in \widetilde{L}$ since $(L_1, ..., L_n)$ is a partition of $N_{\mathcal{W}}$. Next, we prove that $\widetilde{L}$ is qERA-recognizable w.r.t. (U, f, Σ).

For this, we define for every $\xi \in \mathrm{MSO}_{\mathrm{er}}(\Sigma)$ the formula $\widetilde{\xi} \in \mathrm{MSO}_{\mathrm{er}}(\Sigma^{(n)})$ by replacing in ξ every occurence of $P_a(x)$ by $\bigvee_{1 \leq j \leq n} P_{(a,j)}(x)$, and every occurrence of $\triangleleft_a(x) \sim c$ by $\bigvee_{1 \leq j \leq n} \triangleleft_{(a,j)}(x) \sim c$. One can easily verify that for every $((\bar{a},\mu,\sigma),\bar{t}) \in T((\Sigma^{(n)})_{\mathcal{U}})^+$ with $((\bar{a},\sigma),\bar{t}) \in N_{\mathcal{U}}$ we have $((\bar{a},\sigma),\bar{t}) \models \xi \Leftrightarrow ((\bar{a},\mu,\sigma),\bar{t}) \models \widetilde{\xi}$ (where $\mathcal{U} \supseteq \mathrm{Free}(\xi)$). Now, define the formula $\zeta \in \mathrm{MSO}_{\mathrm{er}}(\Sigma^{(n)})$ as follows:

$$\zeta = \forall x. \bigwedge_{1\le j\le n} \bigwedge_{a\in\Sigma} \left(P_{(a,j)}(x) \longrightarrow \widetilde{\psi_j}\right).$$

Let $((\bar{a},\mu,\sigma),\bar{t}) \in T((\Sigma^{(n)})_{\mathcal{V}})^+$ such that $((\bar{a},\sigma),\bar{t}) \in N_{\mathcal{V}}$. Using the semantics of $\mathrm{MSO}_{\mathrm{er}}(\Sigma^{(n)})$, one can show that $((\bar{a},\mu,\sigma),\bar{t}) \models \zeta$ iff for every $i \in \mathrm{dom}((\bar{a},\bar{t}))$ and $j \in \{1,...,n\}$ we have $\mu(i) = j$ implies $((\bar{a},\mu,\sigma[x \to i]),\bar{t}) \models \widetilde{\psi_j}$. However, $((\bar{a},\mu,\sigma[x \to i]),\bar{t}) \models \widetilde{\psi_j}$ iff $((\bar{a},\sigma[x \to i]),\bar{t}) \models \psi_j$. Thus, $((\bar{a},\mu,\sigma),\bar{t}) \models \zeta$ iff $((\bar{a},\mu,\sigma),\bar{t}) \in \widetilde{L}$. Hence, we have $L(\zeta) = \widetilde{L}$. By Theorem 1 and Prop.1, there is a deterministic qERA $\widetilde{\mathcal{A}} = (S,\{\iota\},S_f,E)$ over $(\Sigma^{(n)})_{\mathcal{V}}$ such that $L(\widetilde{\mathcal{A}}) = L(\zeta)$. Using a variant of Lemma 12, we construct a qERA $\mathcal{A}' = (S',S'_0,S'_f,E')$ from $\widetilde{\mathcal{A}}$ such that $L(\mathcal{A}') = L(\widetilde{\mathcal{A}})$, $\mathcal{A}'$ is unambiguous and the locations of $\mathcal{A}'$ will be of the form (s,b) such that every outgoing edge from (s,b) labeled with (a,b',c) satisfies $b' = b$. For this, we define

- $S' = (S \times \{1,...,n\}) \cup \{s_f\}$
- $S'_0 = \{\iota\} \times \{1,...,n\}$
- $S'_f = \{s_f\}$
- $E' = \{((s,b),(a,b,c),\phi,(s',b'))\,|(s,(a,b,c),\phi,s') \in E, b' \in \{1,...,n\}\} \cup \{((s,b),(a,b,c),\phi,s_f)\,|(s,(a,b,c),\phi,s') \in E, s' \in S_f\}$

The proof for $L(\mathcal{A}') = L(\widetilde{\mathcal{A}})$ and unambiguity of $\mathcal{A}'$ is along the lines of the proof of Lemma 12. Observe that this construction is crucial for the next step in that without the uniqueness property we could not assign the cost functions to the locations in a proper way: add a cost function C to $\mathcal{A}'$, obtaining the qWERA $\mathcal{A} = (S',S'_0,S'_f,E',C)$ over $\mathcal{K}$, $(\Sigma^{(n)})_{\mathcal{V}}$ and $\mathcal{F}$ as follows: for every edge $((s,b),(a,b,c),\phi,(s',b')) \in E'$, define $C_{\mathcal{E}}(((s,b),(a,b,c),\phi,(s',b'))) = k_b$, and for every $(s,b) \in S'$, let $C_{(s,b)} = f_b$. Now, let $((\bar{a},\mu,\sigma),\bar{t}) \in L(\mathcal{A}')$. Since $\mathcal{A}$ is unambiguous, we have

$$\big(\|\mathcal{A}\|,((\bar{a},\mu,\sigma),\bar{t})\big) = \prod_{i\in\mathrm{dom}((\bar{a},\bar{t}))} f_{\mu(i)}(t_i - t_{i-1}) \cdot k_{\mu(i)}.$$

Moreover, we can show that $[\![\varphi]\!]_{\mathcal{W}}((\bar{a},\sigma[x \to i]),\bar{t}) = f_{\mu(i)}(t_i - t_{i-1}) \cdot k_{\mu(i)}$. Consider the (valid) renaming $\pi : (\Sigma^{(n)})_{\mathcal{V}} \to \Sigma_{\mathcal{V}}$ defined by $(a,b,c) \mapsto (a,c)$ for each $(a,b,c) \in (\Sigma^{(n)})_{\mathcal{V}}$. Then, for every $((\bar{a},\sigma),\bar{t}) \in N_{\mathcal{V}}$ and the unique μ such that $((\bar{a},\mu,\sigma),\bar{t}) \in \widetilde{L}$, we have

$$\begin{aligned}\big(\bar{\pi}(\|\mathcal{A}\|),((\bar{a},\sigma),\bar{t})\big) &= \big(\|\mathcal{A}\|,((\bar{a},\mu,\sigma),\bar{t})\big)\\ &= \prod_{i\in\mathrm{dom}((\bar{a},\bar{t}))} [\![\varphi]\!]_{\mathcal{W}}((\bar{a},\sigma[x \to i]),\bar{t})\\ &= [\![\forall x.\varphi]\!]((\bar{a},\sigma),\bar{t}).\end{aligned}$$

For the case $((\bar{a},\sigma),\bar{t}) \notin N_{\mathcal{V}}$, we obtain 0 for both $\big(\bar{\pi}(\|\mathcal{A}\|),((\bar{a},\sigma),\bar{t})\big)$ and $[\![\forall x.\varphi]\!]((\bar{a},\sigma),\bar{t})$. Hence we have shown $\bar{\pi}(\|\mathcal{A}\|) = [\![\forall x.\varphi]\!]$. Then, by Lemma 3, we obtain that $[\![\forall x.\varphi]\!]$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). □

Finally, we give the definition of the fragment of $\mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ used in Theorem 2. A formula $\varphi \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ is called *syntactically restricted* if it satisfies the following conditions:

1. Whenever φ contains a conjunction $\varphi_1 \wedge \varphi_2$ as subformula, φ_1 contains the subformula $C_{f_1}(x_1)$ and φ_2 contains $C_{f_2}(x_2)$, then x_1, x_2 are free in both φ_1 and φ_2, and either φ_1 or φ_2 is of the form $\psi \wedge \neg(x_1 = x_2)$ for some $\psi \in \mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$.
2. Whenever φ contains $\forall x.\psi$ as a subformula, then ψ is an almost unambiguous formula over x.
3. Whenever φ contains $\forall X.\psi$ as a subformula, then ψ is a syntactically unambiguous formula.

We let $\mathrm{sRMSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ denote the set of all syntactically restricted formulas of $\mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$. Notice that each of these conditions can be checked for in easy syntax tests. Hence, the logic $\mathrm{sRMSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ is a *decidable* fragment, i.e., for each formula in $\mathrm{MSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ we can decide whether it is syntactically restricted or not.

Remark 4. We may skip the restriction on $\mathcal{K}$ being commutative by adding a fourth condition on the formulas in $\mathrm{sRMSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ concerning the element-wise commuting of weighted formulas (see [11]). For the sake of simplicity, we do not consider this here.

Altogether, we have proved that the semantics $[\![\varphi]\!]$ of every $\varphi \in \mathrm{sRMSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ is qWERA-recognizable over $\mathcal{K}$ and $\mathcal{F}$ w.r.t. (U, f, Σ). One further can see that every *sentence* $\varphi \in \mathrm{sRMSO}_{\mathrm{er}}(\mathcal{K}, \Sigma, \mathcal{F})$ can be recognized by a qWERA over $\mathcal{K}$, Σ and $\mathcal{F}$ - but this is nothing else than a WERA over $\mathcal{K}$, Σ and $\mathcal{F}$. Hence, we have proven the direction from right to left in Theorem 2.

4 From Automata To Logic

For the implication from left to right of Theorem 2, we extend the proof proposed by Droste and Gastin to the timed setting, briefly explained in the following. Let $\mathcal{A} = (S, S_0, S, E, C)$ be a WERA. We choose an enumeration $(e_1, ..., e_m)$ of E with $m = |E|$ and assume $e_i = (s_i, a_i, \phi_i, s_i')$. We define a syntactically unambiguous formula $\psi(X_1, ..., X_m)$ without any second-order quantifiers describing the successful runs of $\mathcal{A}$ (where for each $i \in \{1, ..., m\}$, X_i stands for the edge e_i). This can be done similarly to the classical setting [26]. The guards of the edges in E can be defined by a formula of the form $\forall x. \bigwedge_{1 \leq i \leq m} \big(x \in X_i \xrightarrow{+} \bigwedge_{a \in \Sigma} (\bigwedge_{(x_a \sim c) \in \phi_i} \lhd_a(x) \sim c)\big)$ where $\varphi \xrightarrow{+} \psi$ is an abbreviation for $\varphi^- \vee (\varphi^+ \wedge \psi^+)$. Then, for every timed word $(\bar{a}, \bar{t})$ and valid $(\{X_1, ..., X_m\}, (\bar{a}, \bar{t}))$-assignment σ, we have $[\![\psi(X_1, ..., X_m)]\!]((\bar{a}, \sigma), \bar{t}) = 1$, if there is a successful run of $\mathcal{A}$ on $(\bar{a}, \bar{t})$, and $[\![\psi(X_1, ..., X_m)]\!]((\bar{a}, \sigma), \bar{t}) = 0$, otherwise. Notice that we need to use syntactically unambiguous formulas here in order to avoid getting weights different from 1 or 0. Now, we "add weights" to ψ to obtain a formula ξ whose semantics corresponds to the running weight of a successful run of $\mathcal{A}$ on $(\bar{a}, \bar{t})$ as follows:

$$\xi = \psi \wedge \bigwedge_{e_i \in E} \forall x. \left(\neg(x \in X_i) \vee [x \in X_i \wedge C_{C_{s_i}}(x) \wedge C_{\mathcal{E}}(e_i)]\right).$$

Finally, we let $\zeta = \exists X_1 ... \exists X_m . \xi$, and we obtain $[\![\zeta]\!] = \|\mathcal{A}\|$. Hence, we have shown the second implication, which finishes the proof of Theorem 2.

5 Conclusion

We have presented a weighted timed MSO logic, which is - at least to our knowledge - the first MSO logic allowing for the description of both timed and quantitative properties. On the one hand, we provide the real-time-community with a new tool, because sometimes it may be easier to specify properties in terms of logic rather than by automata devices. On the other hand, the coincidence between recognizable and definable timed series, together with a previous work on WERA concerning a Kleene-Schützenberger Theorem [23], shows the robustness of the notion of WERA-recognizable timed series, as they can equivalently be characterized in terms of automata, logics and rational operations. The same applies to timed series recognizable by weighted timed automata, for which we were successful in adapting the proofs presented in this paper using the relative distance logic $\mathcal{L}\overleftarrow{d}$ introduced by Wilke and his results concerning timed languages with bounded variability [28, 24]. Notice that our result generalizes corresponding results on ERA-recognizable languages as well as formal power series [15, 11]. Also, we have stated conditions for closure of recognizable timed series under the Hadamard product, which corresponds to the intersection operation in the unweighted setting.

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