# A Kleene-Schützenberger Theorem for Weighted Event-Clock Automata

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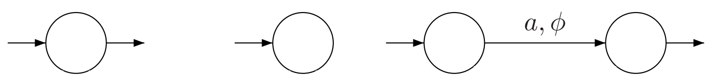
**Abstract.** We present a Kleene-Schützenberger-Theorem for weighted event-clock automata, i.e., we show that the class of *recognizable* timed series coincides with the class of *rational* timed series. The result generalizes Kleene's famous theorem and its extensions to the classes of weighted automata and event-clock automata. For proving the theorem, we use the method of a recent work on the class of weighted timed automata, a proper superclass of weighted event-clock automata, and define a *clock semantics* which allows for a natural definition of the concatenation operation. We show that for every rational clock series there is a weighted event-clock automaton recognizing the same clock series and vice versa. Finally we obtain a Kleene-Schützenberger-Theorem for the classical semantics by proving that rational (recognizable, respectively) timed series are the projection of rational (recognizable, respectively) clock series.

## 1 Introduction

Kleene's fundamental theorem on the coincidence of *recognizable* and *rational* languages is a cornerstone in the theory of automata and formal languages. Consequently, it has been extended to many other classes of automata. For weighted automata, Schützenberger has shown that the set of recognizable formal power series (corresponding to the behaviour of weighted automata) coincide with the set of rational formal power series [12]. Also, there have been several proposals of Kleene-type theorems for the class of timed automata, including the papers by Bouyer and Petit [4, 5], Asarin, Caspi and Maler [2], and Asarin and Dima [3]. Recently, the results of Bouyer and Petit as well as Schützenberger have been extended to the class of weighted timed automata [7]. The goal of this report is to give a Kleene-Schützenberger-Theorem for the class of weighted event-clock automata, a proper subclass of weighted timed automata.

Event-clock automata, introduced by Alur et al. [1], are an interesting subclass of timed automata, since they - as opposed to timed automata - allow for a determinization and thus have a decidable complementation problem. None the less, they have sufficent power to express interesting real-time properties. Consequently, there has been much research on event-clock automata, e.g. including work on real-time logics [10, 11], inference/learning [9] or a logical characterization via a monadic second-oder logic [8]. Also, there has been a Kleene-type theorem for event-clock automata, proposed by Dima [6]. For this, Dima defines rational expressions built starting from atomic expressions of the form $\varepsilon$, $\emptyset$ and $(a, \phi)$, where $a \in \Sigma$ and $\phi$ is a clock constraint, and the usual rational operations $+$, ; and $^*$. The natural idea is to define the semantics of rational expressions similarly to the classical case, i.e., such that the semantics of atomic expressions correspond to the language accepted by the basic event-clock automata pictured below, and the semantics of more complex expressions are compositional. However, the following example shows that this does not work for the class of event-clock automata.




$\mathcal{A}_\varepsilon$ $\mathcal{A}_\emptyset$ $\mathcal{A}_{(a,\phi)}$

Consider the expression $(a, y_a = 2 \wedge x_a = \bot)(a, x_a = 2 \wedge y_a = \bot)$. Clearly, we expect the semantics to be the set of timed words of the form $(a, t_1)(a, t_2)$ such that $t_2 - t_1 = 2$. Unfortunately, the sets of timed words recognized by the event-clock automata corresponding to $(a, y_a = 2 \wedge x_a = \bot)$ and $(a, x_a = 2 \wedge y_a = \bot)$ both are empty due to the constraint $y_a = 2$ and $x_a = 2$, respectively. Dima solves this problem by introducing a new semantics for event-clock automata. He considers so-called *limited observation timed words* of the form $(w, [\tau_1, \tau_2])$, where $w$ is a timed word, and $[\tau_1, \tau_2]$ is an interval over the positive reals restricting the attention to the subword whose timestamps lie within the interval. Plainly put, an event-clock automaton recognizes the limited timed observation word $(w, [\tau_1, \tau_2])$, if there is a successful run for the subword within $[\tau_1, \tau_2]$. For instance, the event-clock automaton corresponding to the expression $(a, y_a = 2 \wedge x_a = \bot)$ recognizes a limited observation timed word of the form $((a, t_1)(a, t_2), [\delta_1, \delta_2])$ if $t_1 \in [\delta_1, \delta_2]$, $t_2 \notin [\delta_1, \delta_2]$ and $t_2 - t_1 = 2$. Two limited observation timed words $(w, [\delta_1, \delta_2])$ and $(w', [\delta'_1, \delta'_2])$ are compatible if $w = w'$ and $\delta'_1 = \delta_2$. In this case, the concatenation of $(w, [\delta_1, \delta_2])$ and $(w', [\delta'_1, \delta'_2])$ is the limited observation timed word $(w, [\delta_1, \delta'_2])$. With this definition, Dima is able to present a Kleene theorem for the limited observation timed word-semantics. Moreover, he shows that an easy projection induces the classical semantics in terms of timed words (resulting in a second Kleene theorem).

Although it is presumably possible to extend Dima's ideas to the weighted setting, we adopt the approach introduced by Bouyer and Petit and use the so-called *clock semantics* [5]. Similarly to the case of event-clock automata, timed words recognized by timed automata do not allow for a natural concatenation operation. Therefore, Bouyer and Petit use clock words, which - as opposed to timed words - contain information concerning the actual values of the clock variables after an event has taken place. Additionally, they consist of an initial global time as well as initial values of the clock variables. This enables the authors to give a compositional semantics to rational expressions in terms of clock words. Furthermore, similarly to the approach of Dima, they show that an easy projection operation on both recognizable and rational clock words results in recognizable and rational timed words, respectively. The clock semantics is successfully carried over to *weighted* timed automata and is the crucial step for providing a Kleene-Schützenberger-Theorem for the class of weighted timed automata [7]. For weighted event-clock automata, the clock semantics must be refined owing to the difference between the interpretation of clock variables in timed automata and event-recording/event-predicting clock variables in event-clock automata. None the less, we can reuse most of the constructions given for weighted timed automata. Thus, this report concentrates on defining the clock semantics for the class of weighted event-clock automata and investigating the relation to the usual timed semantics.

## 2 Weighted Event-Clock Automata

Let $\Sigma$, $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote a finite alphabet, the natural numbers and the positive reals, respectively.

**Timed Words** A *timed word* is a finite sequence $w = (a_1, t_1)...(a_n, t_n) \in (\Sigma \times \mathbb{R}_{\geq 0})^*$, where the sequence $t_1...t_k$ of timestamps is non-decreasing. Intuitively, $t_i$ gives the time of occurence of the event $a_i$. We denote the set of all timed words over $\Sigma$ by $T\Sigma^*$. A set $L \subseteq T\Sigma^*$ of timed words is called a *timed language*. With $\Sigma$ we associate a set $C_\Sigma = \{x_a, y_a | a \in \Sigma\}$ of clock variables ranging over $\mathbb{R}_{\geq 0}$. Clock variables of the form $x_a$ are called *event-recording clock variables* and measure the distance between the current event in a timed word $w$ and the last occuring $a$. On the other hand, clock variables of the form $y_a$ are called *event-predicting clock variables*. They indicate the distance to the next occuring event $a$. Formally, given a timed word $w$ as above, we let $\mathrm{dom}(w)$ be the set $\{1, ..., n\}$ and define for every $i \in \mathrm{dom}(w)$ a clock valuation function $\gamma_i^w : \mathrm{dom}(w) \times C_\Sigma \to \mathbb{R}_{\geq 0} \cup \{\bot\}$ by

$$\gamma_i^w(x_a) = \begin{cases} t_i - t_j & \text{if there exists a } j \text{ such that } 1 \leq j < i \text{ and } a_j = a \\ & \text{and for all } m \text{ with } j < m < i \text{, we have } a_m \neq a \\ \bot & \text{otherwise} \end{cases}$$

$$\gamma_i^w(y_a) = \begin{cases} t_j - t_i & \text{if there exists a } j \text{ such that } i < j \leq n \text{ and } a_j = a \\ & \text{and for all } m \text{ with } i < m < j \text{, we have } a_m \neq a \\ \bot & \text{otherwise} \end{cases}$$

We define *clock constraints* $\phi$ over $C_\Sigma$ to be conjunctions of constraint formulas of the form $x = \bot$ or $x \sim c$, where $x \in C_\Sigma$, $c \in \mathbb{N}$ and $\sim \in \{<, \leq, =, \geq, >\}$. A *clock constraint* $\phi$ over $C_\Sigma$ is a finite conjunction of clock constraints. We use $\Phi(C_\Sigma)$ to denote the set of all clock constraints over $C_\Sigma$. A clock valuation $\gamma_i^w$

satisfies $\phi$, written $\gamma_i^w \models \phi$, if $\phi$ evaluates to true according to the values given by $\gamma_i^w$.

**Clock Words** A *clock word* is a finite sequence $w = (t_0, \nu_0)(a_1, t_1, \nu_1)...(a_n, t_n, \nu_n) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{C_\Sigma})(\Sigma \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{C_\Sigma})^*$, where $(a_1, t_1)...(a_n, t_n) \in T\Sigma^*$ is a timed word and $\nu_i$ is a function from $C_\Sigma$ to $\mathbb{R}_{\geq 0}$ assigning a value to every clock variable in $C_\Sigma$ for each $i \in \{1, ..., n\}$. An *empty clock word* is of the form $(t_0, \nu_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{C_\Sigma}$. We use $C\Sigma^*$ to denote the set of clock words over $\Sigma$. Let $w = (t_0, \nu_0)(a_1, t_1, \nu_1)...(a_m, t_m, \nu_m), w' = (t'_0, \nu'_0)(a'_1, t'_1, \nu'_1)...(a'_n, t'_n, \nu'_n) \in C\Sigma^*$. We say that $w$ and $w'$ are compatible if $(t'_0, \nu'_0) = (t_m, \bar{\nu}_m)$, where $\bar{\nu}_m$ is defined by $\bar{\nu}_m(x_{a_m}) = 0$ and $\bar{\nu}_m(c) = \nu_m(c)$ for all $c \in C_\Sigma \backslash \{x_{a_m}\}$. In this case, we define the concatenation $w; w'$ of $w$ and $w'$ to be the clock word $(t_0, \nu_0)(a_1, t_1, \nu_1)...(a_m, t_m, \nu_m)(a'_1, t'_1, \nu'_1)...(a'_n, t'_n, \nu'_n)$.

**Semirings** Let $\mathcal{K}$ be a *semiring*, i.e., an algebraic structure $\mathcal{K} = (K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition and 0 is absorbing. As examples consider the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers with the usual addition and multiplication, the Boolean semiring $(\{0, 1\}, \vee, \wedge, 0, 1)$ and the tropical semiring $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)$. Furthermore, we let $\mathcal{F}$ be a family of functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{K}$.

**Weighted Event-Clock Automata** A *weighted event-clock automaton* (WECA) over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ is a tuple $\mathcal{A} = (S, S_0, S_f, E, C)$, where

- $S$ is a finite set of locations (states)
- $S_0 \subseteq S$ the set of initial locations
- $S_f \subseteq S$ the set of final locations
- $E \subseteq S \times \Sigma \times \Phi(C_\Sigma) \times S$ is a finite set of edges
- $C = \{C_\mathcal{E}\} \cup \{C_s | s \in S\}$ is a cost function, where $C_\mathcal{E} : E \to K$ gives the weight for taking an edge, and $C_s \in \mathcal{F}$ determines the weight that arises when letting time pass while being in a location $s$ for each $s \in S$.

**Timed Semantics** Let $w = (a_1, t_1)...(a_n, t_n)$ be a timed word. A *run* of $\mathcal{A}$ on $w$ is a finite sequence $s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} ... \xrightarrow{e_n} s_n$ of locations $s_0, s_i \in S$ and edges $e_i = (s_{i-1}, a_i, \phi_i, s_i) \in E$ such that $\gamma_i^w \models \phi_i$ for every $1 \leq i \leq n$. We say that a run is *successful* if $s_0 \in S_0$ and $s_n \in S_f$. We define the *running weight* $rwt(r)$ of a run $r$ to be $\prod_{1 \leq i \leq n} C_{s_{i-1}}(t_i - t_{i-1}) \cdot C_\mathcal{E}(e_i)$, where $t_0 = 0$. The running weight of the empty run is defined to be $1 \in K$. The *timed behaviour* $\|\mathcal{A}\|^T : T\Sigma^* \to K$ of $\mathcal{A}$ is given by $(\|\mathcal{A}\|^T, w) = \sum\{rwt(r) : r \text{ is a successful run of } \mathcal{A} \text{ on } w\}$. A function $\mathcal{T} : T\Sigma^* \to K$ is called a *timed series*. A timed series $\mathcal{T}$ is *recognizable* over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ if there is a WECA $\mathcal{A}$ over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ with $\|\mathcal{A}\|^T = \mathcal{T}$.

**Clock Semantics** A clock run of a WECA $\mathcal{A}$ from the initial conditions $(t_0, \nu_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{C_\Sigma}$ is a sequence of the form $(s_0, t_0, \nu_0) \xrightarrow{\delta_1} \xrightarrow{a_1} (s_1, t_1, \nu_1) \xrightarrow{\delta_2} \xrightarrow{a_2} ... \xrightarrow{\delta_n} \xrightarrow{a_n} (s_n, t_n, \nu_n)$ satisfying the following conditions:

- there in an edge $e_i = (s_{i-1}, a_i, \phi_i, s_i) \in E$ for $1 \leq i \leq n$
- $t_i = t_{i-1} + \delta_i$ for every $1 \leq i \leq n$
- $\nu_i \models \phi_i$ for every $1 \leq i \leq n$
- $\nu_1(x_a) = \begin{cases} \bot & \text{if } \nu_0(x_a) = \bot \\ \nu_0(x_a) + \delta_1 & \text{otherwise} \end{cases}$ for each $a \in \Sigma$
- $\nu_i(x_a) = \begin{cases} \nu_{i-1}(x_a) + \delta_i & \text{if } a_{i-1} \neq a \\ \delta_i & \text{otherwise} \end{cases}$ for every $2 \leq i \leq n$
- $\nu_0(y_a) = \begin{cases} t_j - t_0 & \text{if there is a } j \text{ such that } 1 \leq j \leq n \text{ and } a_j = a \text{ and for all } k \text{ with } 1 \leq k < j \text{ we have } a_k \neq a \\ \bot \text{ or } m \text{ such that } m \geq t_n - t_0 & \text{if for all } j \text{ with } 1 \leq j \leq n \text{ we have } a_j \neq a \end{cases}$
- $\nu_i(y_a) = \begin{cases} \nu_{i-1}(y_a) - \delta_i & \text{if } a_{i-1} \neq a \\ t_j - t_i & \text{if } a_i = a \text{ and there is a } j \text{ such that } i < j \leq n \text{ and } a_j = a \text{ and for all } k \text{ with } i < k < j \text{ we have } a_k \neq a \\ \bot \text{ or } m \text{ such that } m \geq t_n - t_i & \text{if } a_i = a \text{ and for all } j \text{ with } i < j \leq n \text{ we have } a_j \neq a \end{cases}$

  for every $1 \leq i \leq n$.

With a clock run as above, we associate the clock word $(t_0, \nu_0)(a_1, t_1, \nu_1)...(a_n, t_n, \nu_n)$. A clock run is successful if $s_0 \in S_0$ and $s_n \in S_f$. The running weight of a clock run is defined to be $\prod_{1 \leq i \leq n} C_{s_{i-1}}(\delta_i) \cdot C_{\mathcal{E}}(e_i)$. The *clock behaviour* $\|\mathcal{A}\|^C : C\Sigma^* \to K$ of $\mathcal{A}$ is given by $(\|\mathcal{A}\|^C, w) = \sum\{rwt(r) : r \text{ is a successful clock run of } \mathcal{A} \text{ on } w\}$. A function $\mathcal{T} : C\Sigma^* \to K$ is called a *clock series*. A clock series $\mathcal{T}$ is *recognizable* over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ if there is a WECA $\mathcal{A}$ over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ with $\|\mathcal{A}\|^C = \mathcal{T}$.

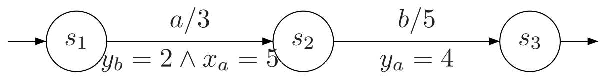
Intuitively, clock words and clock runs allow for a more relaxed clock valuation function than timed words. For instance, in a successful timed run as above, we must have $\gamma_1^w(x_a) = \bot$ for every $a \in \Sigma$. In a successful clock run, the initial clock valuation $\nu_0$ of $x_a$ may be different from $\bot$. Moreover, even though there may be no more $a$ appearing in a clock word $w$ after position $i$, we allow $\nu_i(y_a)$ to be different from $\bot$ to indicate that there may be an $a$ in a future time after $t_n$. This relaxation is crucial for showing closure of weighted event-clock automata under the concatenation operation. However, the choice of a new value especially for event-predicting clock variables must be restricted as specified above.

*Example 1.* Consider the WECA in the figure below, where for every location $s$, $C_s$ maps every $\delta \in \mathbb{R}_{\geq 0}$ to the constant $1 \in K$. Clearly, owing to the guard $\phi$ in the edge between $s_1$ and $s_2$ containing the clock constraint $x_a = 5$, there is no timed word $w \in T\Sigma^*$ such that there is a run $r$ of $\mathcal{A}$ on $w$ with $\gamma_1^w \models \phi$. The clock constraint $y_a = 4$ in the edge between $s_2$ and $s_3$ causes similar problems. Hence,

$(\|\mathcal{A}\|^T, w) = 0$ for every timed word $w \in T\Sigma^*$. In contrast to this, consider the clock word $w = (1.0, \nu_0)(a, 5.5, \nu_1)(b, 7.5, \nu_2)$, where

- $\nu_0(x_a) = 0.5, \nu_0(x_b) = \bot, \nu_0(y_a) = 4.5, \nu_0(y_b) = 6.5$,
- $\nu_1(x_a) = 5.0, \nu_1(x_b) = \bot, \nu_1(y_a) = 6.0, \nu_1(y_b) = 2.0$,
- $\nu_2(x_a) = 2.0, \nu_2(x_b) = \bot, \nu_2(y_a) = 4.0, \nu_2(y_b) = \bot$,

There is a unique run $r = (s_1, t_0, \nu_0) \xrightarrow{4.5} \xrightarrow{e_1} (s_2, t_1, \nu_1) \xrightarrow{2.0} \xrightarrow{e_2} (s_3, t_2, t_2)$ of $\mathcal{A}$ on $w$, and hence $(\|\mathcal{A}\|^{\mathcal{C}}, w) = rwt(r) = 3 \cdot 5$.




In the following, we use $Rec(\mathcal{K}, \Sigma, \mathcal{F})$ to denote the set of clock series recognizable over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$.

## 3 Rational Clock Series

In this section, we introduce another class of clock series, called *rational clock series*, which can be inductively built from the "atomic" clock series $\mathbb{1}_\varepsilon$, $\mathbb{0}$ and so-called monomials (corresponding to the empty word $\varepsilon$, the empty set $\emptyset$ and $a \in \Sigma$ in the classical setting) and operations $+$, ; and $^*$.

Let $\mathcal{K}$ be a semiring, $\Sigma$ an alphabet and $\mathcal{F}$ a family of functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{K}$. We define the clock series $\mathbb{1}_\varepsilon$ by $(\mathbb{1}_\varepsilon, w) = 1$ if $w \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{C_\Sigma})$ and 0 otherwise, as well as $\mathbb{0}$ by $(\mathbb{0}, w) = 0$ for every $w \in C\Sigma^*$. Let $\mu \in \mathcal{F}$, $k \in \mathcal{K}$, $a \in \Sigma$ and $\phi \in \Phi(C_\Sigma)$. A monomial over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ is a clock series $\langle \mu, k, a, \phi \rangle : C\Sigma^* \to K$ defined by $(\langle \mu, k, a, \phi \rangle, w) = \mu(t_1 - t_0) \cdot k$ if $w = (t_0, \nu_0)(a_1, t_1, \nu_1)$ such that

- $a_1 = a$,
- $\nu_1 \models \phi$,
- $\nu_1(x_b) = \nu_0(x_b) + (t_1 - t_0)$ for all $b \in \Sigma$,
- $\nu_0(y_a) = t_1 - t_0$,
- $\nu_0(y_b) = \bot$ or $\nu_0(y_b) \geq t_1 - t_0$ for each, $b \in \Sigma \backslash \{a\}$,
- $\nu_1(y_a) = \bot$ or $\nu_1(y_a) \geq 0$ and
- $\nu_1(y_b) = \nu_0(y_b) - (t_1 - t_0)$ for each $b \in \Sigma \backslash \{a\}$.

Otherwise, $(\langle \mu, k, a, \phi \rangle, w) = 0$. On the set of all clock series, we define the sum $\mathcal{T}_1 + \mathcal{T}_2$ pointwise, i.e., $(\mathcal{T}_1 + \mathcal{T}_2, w) = (\mathcal{T}_1, w) + (\mathcal{T}_2, w)$. We define the Cauchy product $\mathcal{T}_1 \cdot \mathcal{T}_2$ by $(\mathcal{T}_1 \cdot \mathcal{T}_2, w) = \sum_{u;v=w} (\mathcal{T}_1, u) \cdot (\mathcal{T}_2, v)$. For a clock series $\mathcal{T}$, we let $\mathcal{T}^0 = \mathbb{1}_\varepsilon$, and, inductively $\mathcal{T}^k = \mathcal{T} \cdot \mathcal{T}^{k-1}$ be the $k$-th power of $\mathcal{T}$ for $k \geq 1$. The clock series $\mathcal{T}$ is called *proper*, if $(\mathcal{T}, \varepsilon) = 0$ for every $\varepsilon \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{C_\Sigma})$. For a proper clock series $\mathcal{T}$, we define the Kleene star iteration

$\mathcal{T}^*$ by $(\mathcal{T}^*, w) = \sum_{k \geq 0} (\mathcal{T}^k, w)$. Notice that the the sum is finite, because we have $(\mathcal{T}^k, w) = 0$ for every $k > |w|$ if $\mathcal{T}$ is proper.

A clock series is *rational over* $\mathcal{K}$*,* $\Sigma$ *and* $\mathcal{F}$ if it can be defined starting from finitely many monomials or the clock series $\mathbb{1}_\varepsilon$ and $\mathbb{0}$ by means of a finite number of applications of $+$, $\cdot$ and $^*$, where the latter may only be applied to proper clock series. We use $Rat(\mathcal{K}, \Sigma, \mathcal{F})$ to denote the set of clock series being rational over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$.

In the next section, we show that the class of rational clock series coincides with the class of recognizable clock series.

## 4 The Kleene-Schützenberger Theorem

Next, we present the main theorem.

**Theorem 1.** *Let* $\mathcal{K}$ *be a semiring,* $\Sigma$ *be an alphabet and* $\mathcal{F}$ *be a family of functions from* $\mathbb{R}_{\geq 0}$ *to* $\mathcal{K}$*. Then the class of rational clock series coincides with the class of recognizable clock series:*

$$Rat(\mathcal{K}, \Sigma, \mathcal{F}) = Rec(\mathcal{K}, \Sigma, \mathcal{F})$$

The direction from right to left, i.e., that every recognizable clock series is rational, can be shown by applying the method of solving equations: every WECA induces a system of linear equations, whose unique solution corresponds to the rational clock series and can be computed effectively. The complete proof can be done in the same manner as for weighted timed automata [7]. For the other inclusion, the crucial part is to show that recognizable clock series are closed under $+$, ; and $^*$. However, again we can refer to the constructions given in [7], which can be carried over to the case of WECA without any difficulties. Moreover, we would like to mention that all the constructions are effective.

## 5 From Clock Series to Timed Series

Lastly, we would like to give a Kleene-Schützenberger-Theorem with respect to the *timed* semantics rather than the clock semantics. For this, we introduce a fourth operation as follows: define the partial function $\pi : C\Sigma^* \to T\Sigma^*$ by $\pi((t_0, \nu_0)(a_1, t_1, \nu_1)...(a_n, t_n, \nu_n)) = (a_1, t_1)...(a_n, t_n)$ if

- $t_0 = 0$
- $\nu_0 = \nu_1$
- $\nu_i = \gamma_i^w$ for every $i \in \{1, ..., n\}$

Otherwise, $\pi((t_0, \nu_0)(a_1, t_1, \nu_1)...(a_n, t_n, \nu_n))$ is undefined. One can easily see that for every timed word $w^T$ there is exactly one clock word $w^C$ such that $\pi(w^C) = w^T$. Hence we can write $w^C = \pi^{-1}(w^T)$. We extend $\pi$ to a function from the set of clock series to the set of timed series by putting $(\bar{\pi}(\mathcal{T}), w^T) = (\mathcal{T}, \pi^{-1}(w^T))$.

A timed series $\mathcal{T}$ is *rational over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$* if it is defined by a single application of $\bar{\pi}$ to a rational clock series $R$ over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$, i.e., $\mathcal{T} = \bar{\pi}(R)$. In the following lemma we prove that the relation between *recognizable* timed series and clock series is the same.

**Lemma 1.** *Let $\mathcal{A}$ be a WECA and $w^T \in T\Sigma^*$. Then $\|\mathcal{A}\|^T = \bar{\pi}(\|\mathcal{A}\|^C)$.*

*Proof.* First, we show that for every successful run $r$ of $\mathcal{A}$ on $w_T$ there is a successful clock run $r'$ of $\mathcal{A}$ on $\pi^{-1}(w_T)$ such that $rwt(r') = rwt(r)$. Let $w_T = (a_1, t_1)...(a_n, t_n)$ be a timed word and $r = s_1 \xrightarrow{e_1} ... \xrightarrow{e_n} s_n$ be a successful run of $\mathcal{A}$ on $w_T$. Define $r' = (s_1, t_0, \nu_0) \xrightarrow{\delta_1}\xrightarrow{a_1} ... \xrightarrow{\delta_n}\xrightarrow{a_n} (s_n, t_n, \nu_n)$, where

- $t_0 = 0$
- $\delta_i = t_i - t_{i-1}$ for every $i \in \{1, ..., n\}$
- $\nu_0 = \nu_1$
- $\nu_i = \gamma_i^w$ for every $i \in \{1, ..., n\}$

Clearly, $r'$ is a succcessful clock run of $\mathcal{A}$ on $\pi^{-1}(w_T)$. Moreover, we have $rwt(run') = rwt(r)$.

Second, we prove that for every successful clock run $r'$ of $\mathcal{A}$ on some clock word $w_C$ such that there is some timed word $w_T$ with $\pi(w_C) = w_T$, there is a successful run $r$ of $\mathcal{A}$ on $w_T$ such that $rwt(r) = rwt(r')$. Therefore, let $w_C = (t_0, \nu_0)(a_1, t_1, \nu_1)...(a_n, t_n, \nu_n)$ be a clock word such that there is some timed word $w_T$ with $\pi(w_C) = w_T$. Let $r' = (s_1, t_0, \nu_0) \xrightarrow{\delta_1}\xrightarrow{a_1} ... \xrightarrow{\delta_n}\xrightarrow{a_n} (s_n, t_n, \nu_n)$ be the successful clock run of $\mathcal{A}$ on $w_C$. Hence, there is an edge $(s_{i-1}, a_i, \phi_i, s_i) \in E$ for every $i \in \{1, ..., n\}$. We define $r = s_1 \xrightarrow{e_1} ... \xrightarrow{e_n} s_n$ and show that $r$ is a successful run of $\mathcal{A}$ on $\pi(w_C)$. Therefore, we need to prove $\gamma_i^{\pi(w_C)} \models \phi_i$ for every $i \in \{1, ..., n\}$, but this is clearly the case since by definition of $\pi$ we have $\nu_i = \gamma_i^{\pi(w_C)}$. Also, one can easily see that $rwt(r) = rwt(r')$.

In the following, we use these two facts to show

$$\begin{aligned}(\|\mathcal{A}\|^T, w_T) &= \sum\{rwt(r) | r \text{ is a successful timed run of } \mathcal{A} \text{ on } w_T\} \\ &= \sum\{rwt(r') | r' \text{ is a successful clock run of } \mathcal{A} \text{ on } \pi^{-1}(w_T)\} \\ &= (\|\mathcal{A}\|^C, \pi^{-1}(w_T)) \\ &= (\pi(\|\mathcal{A}\|^C), w_T)\end{aligned}$$

which finishes the proof.

This and the main theorem imply the following Kleene-Schützenberger theorem on timed series:

**Corollary 1.** *Let $\mathcal{K}$ be a semiring, $\Sigma$ be an alphabet and $\mathcal{F}$ be a family of functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{K}$. The class of timed series recognizable by a WECA over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$ coincides with the class of rational timed series over $\mathcal{K}$, $\Sigma$ and $\mathcal{F}$.*

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