Recognizability of Supports of Recognizable Series over the Semiring of the Integers is Undecidable

Daniel Kirsten and Karin Quaas

3rd May 2010

A weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$ is a tuple $\mathcal{A} = (Q, E, \lambda, \varrho, \mu)$, where

- Q is a finite set of states,
- $E \subseteq Q \times \Sigma \times Q$ is a finite set of transitions,
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- $\|\mathcal{A}\|(w) := \sum_{\substack{p,q \in Q \\ \pi \in p \overset{w}{\rightsquigarrow} q}} \lambda(p) \cdot \mu(\pi) \cdot \varrho(q)$ for each $w \in \Sigma^*$

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A series $S : \Sigma^* \to \mathbb{Z}$ is recognizable, if there is a weighted automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = S$.

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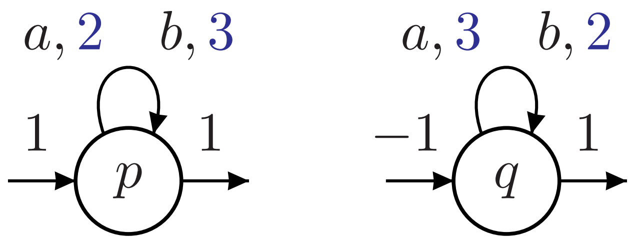

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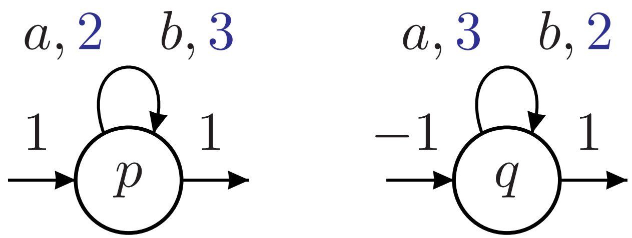

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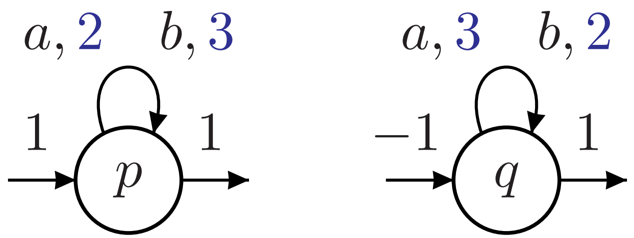

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PROOF IDEA.
Reduction of Post's Correspondence Problem.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

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PROOF IDEA.

- Reduction of Post's Correspondence Problem

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- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

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Example: $\text{num}(001) = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 9 \quad (= 1001)$.

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$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.

(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.

Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$	(since (A, α, β) has no solution)
$\Rightarrow S(w) \neq 0$	(Lemma 1)
$\Rightarrow w \in$ supp(S)	(Definition of support)

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$.

$\Rightarrow S(w) = 1$	(Definition of S)
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$\Rightarrow$ supp$(S) = \Sigma^*$ and thus recognizable.

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By Lemma 1, we have $L(\mathcal{A}) \cap U = \{u\#b^m c^n \mid m \neq n\}$.
Since U and $L(\mathcal{A})$ are recognizable, $L(\mathcal{A}) \cap U$ is recognizable. Contradiction!

Theorem. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

Proof Idea.

- Reduction of Post's Correspondence Problem
- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

- We interprete finite words over $\{0, 1\}$ as binary representation of an integer.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \ \dot{\cup} \ \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \mathsf{num}(\alpha(u)) - \mathsf{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

S is recognizable by a weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$:

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- the series $S'(u) = (2^{2k})^{|u|+m}$ is recognizable for each k and m
- recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ are closed under $+$ and $-$

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Future Work

- Is the recognizability of supports of recognizable series over non-SR-semirings always undecidable?

- What is the complexity of the problem to decide whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is empty?