Recognizability of Supports of Recognizable Series over the Semiring of the Integers is Undecidable

Daniel Kirsten and Karin Quaas

3rd May 2010

A weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$ is a tuple $\mathcal{A} = (Q, E, \lambda, \varrho, \mu)$, where

- Q is a finite set of states,
- $E \subseteq Q \times \Sigma \times Q$ is a finite set of transitions,
- $\lambda, \varrho : Q \to \mathbb{Z}$ are weight functions for entering and leaving a state,
- $\mu : E \to \mathbb{Z}$ is a weight function for taking a transition.

A weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$ is a tuple $\mathcal{A} = (Q, E, \lambda, \varrho, \mu)$, where

- Q is a finite set of states,
- $E \subseteq Q \times \Sigma \times Q$ is a finite set of transitions,
- $\lambda, \varrho : Q \to \mathbb{Z}$ are weight functions for entering and leaving a state,
- $\mu : E \to \mathbb{Z}$ is a weight function for taking a transition.

- μ is extended to a homomorphism $\mu : E^* \to (\mathbb{Z}, \cdot)$.

A weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$ is a tuple $\mathcal{A} = (Q, E, \lambda, \varrho, \mu)$, where

- Q is a finite set of states,
- $E \subseteq Q \times \Sigma \times Q$ is a finite set of transitions,
- $\lambda, \varrho : Q \to \mathbb{Z}$ are weight functions for entering and leaving a state,
- $\mu : E \to \mathbb{Z}$ is a weight function for taking a transition.

- μ is extended to a homomorphism $\mu : E^* \to (\mathbb{Z}, \cdot)$.

The behaviour of $\mathcal{A}$ is a series $\|\mathcal{A}\| : \Sigma^* \to \mathbb{Z}$ defined by

- $\|\mathcal{A}\|(w) := \sum_{\substack{p,q \in Q \\ \pi \in p \overset{w}{\rightsquigarrow} q}} \lambda(p) \cdot \mu(\pi) \cdot \varrho(q)$ for each $w \in \Sigma^*$

A weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$ is a tuple $\mathcal{A} = (Q, E, \lambda, \varrho, \mu)$, where

- Q is a finite set of states,
- $E \subseteq Q \times \Sigma \times Q$ is a finite set of transitions,
- $\lambda, \varrho : Q \to \mathbb{Z}$ are weight functions for entering and leaving a state,
- $\mu : E \to \mathbb{Z}$ is a weight function for taking a transition.

- μ is extended to a homomorphism $\mu : E^* \to (\mathbb{Z}, \cdot)$.

The behaviour of $\mathcal{A}$ is a series $\|\mathcal{A}\| : \Sigma^* \to \mathbb{Z}$ defined by

- $\|\mathcal{A}\|(w) := \sum_{\substack{p,q \in Q \\ \pi \in p \overset{w}{\rightsquigarrow} q}} \lambda(p) \cdot \mu(\pi) \cdot \varrho(q)$ for each $w \in \Sigma^*$

A series $S : \Sigma^* \to \mathbb{Z}$ is recognizable, if there is a weighted automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = S$.

Let $S : \Sigma^* \to \mathbb{Z}$.

The support of S is defined by

$$\mathsf{supp}(S) := \{w \in \Sigma^* \mid S(w) \neq 0\}.$$

Let $S : \Sigma^* \to \mathbb{Z}$.

The support of S is defined by

$$\mathsf{supp}(S) := \{w \in \Sigma^* \mid S(w) \neq 0\}.$$

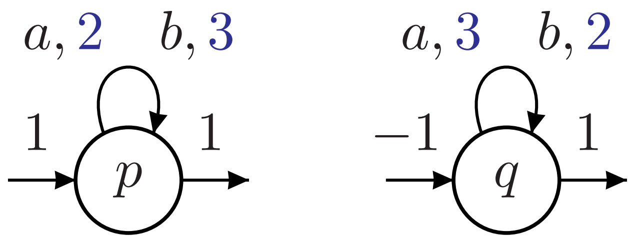
Long known: the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is not necessarily recognizable by a finite automaton.

Let $S : \Sigma^* \to \mathbb{Z}$.

The support of S is defined by

$$\mathsf{supp}(S) := \{w \in \Sigma^* \mid S(w) \neq 0\}.$$

Long known: the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is not necessarily recognizable by a finite automaton. Example:



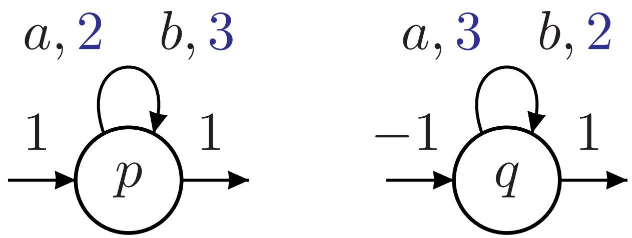
$$\|\mathcal{A}\|(w) = 2^{|w|_a}3^{|w|_b} - 3^{|w|_a}2^{|w|_b} \quad \neq 0 \quad \Leftrightarrow \quad |w|_a \neq |w|_b$$

Let $S : \Sigma^* \to \mathbb{Z}$.

The support of S is defined by

$$\text{supp}(S) := \{w \in \Sigma^* \mid S(w) \neq 0\}.$$

Long known: the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is not necessarily recognizable by a finite automaton. Example:



$$\|\mathcal{A}\|(w) = 2^{|w|_a} 3^{|w|_b} - 3^{|w|_a} 2^{|w|_b} \quad \neq 0 \quad \Leftrightarrow \quad |w|_a \neq |w|_b$$

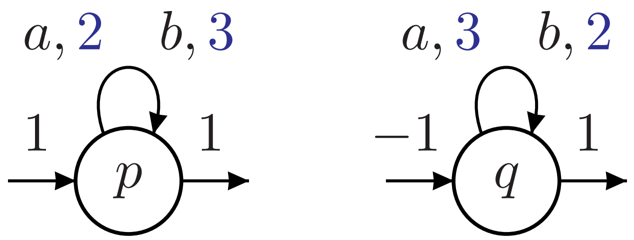
$$\text{supp}(\|\mathcal{A}\|) = \{w \in \Sigma^* \mid |w|_a \neq |w|_b\}$$

Let $S : \Sigma^* \to \mathbb{Z}$.

The support of S is defined by

$$\mathrm{supp}(S) := \{w \in \Sigma^* \mid S(w) \neq 0\}.$$

Long known: the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is not necessarily recognizable by a finite automaton. Example:



$$\|\mathcal{A}\|(w) = 2^{|w|_a} 3^{|w|_b} - 3^{|w|_a} 2^{|w|_b} \quad \neq 0 \quad \Leftrightarrow \quad |w|_a \neq |w|_b$$

$\mathrm{supp}(\|\mathcal{A}\|) = \{w \in \Sigma^* \mid |w|_a \neq |w|_b\}$ and thus not recognizable

Is it decidable whether the support of a recognizable series over $(\mathbb{Z}, +, 0, \cdot, 1)$ is recognizable by a finite automaton?

Is it decidable whether the support of a recognizable series over $(\mathbb{Z}, +, 0, \cdot, 1)$ is recognizable by a finite automaton?

THEOREM. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

Is it decidable whether the support of a recognizable series over $(\mathbb{Z}, +, 0, \cdot, 1)$ is recognizable by a finite automaton?

THEOREM. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

PROOF IDEA.
Reduction of Post's Correspondence Problem.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\begin{array}{lll} \alpha(a) = 1 & \alpha(b) = 10 & \alpha(c) = 011 \\ \beta(a) = 101 & \beta(b) = 00 & \beta(c) = 11 \end{array}$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms. A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\begin{array}{lll} \alpha(a) = 1 & \alpha(b) = 10 & \alpha(c) = 011 \\ \beta(a) = 101 & \beta(b) = 00 & \beta(c) = 11 \end{array}$$

$$\alpha(acbc) =$$
$$\beta(acbc) =$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\alpha(a) = 1 \quad \alpha(b) = 10 \quad \alpha(c) = 011$$
$$\beta(a) = 101 \quad \beta(b) = 00 \quad \beta(c) = 11$$

$$\alpha(acbc) = \quad 1$$
$$\beta(acbc) = \quad 1 \quad 0 \quad 1$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\begin{array}{lll} \alpha(a) = 1 & \alpha(b) = 10 & \alpha(c) = 011 \\ \beta(a) = 101 & \beta(b) = 00 & \beta(c) = 11 \end{array}$$

$$\begin{array}{lccccc} \alpha(acbc) = & 1 & 0 & 1 & 1 & \\ \beta(acbc) = & 1 & 0 & 1 & 1 & 1 \end{array}$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\begin{array}{lll} \alpha(a) = 1 & \alpha(b) = 10 & \alpha(c) = 011 \\ \beta(a) = 101 & \beta(b) = 00 & \beta(c) = 11 \end{array}$$

$$\begin{array}{lccccccc} \alpha(acbc) = & 1 & 0 & 1 & 1 & 1 & 0 & \\ \beta(acbc) = & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms. A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\begin{array}{lll} \alpha(a) = 1 & \alpha(b) = 10 & \alpha(c) = 011 \\ \beta(a) = 101 & \beta(b) = 00 & \beta(c) = 11 \end{array}$$

$$\begin{array}{lccccccccc} \alpha(acbc) = & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ \beta(acbc) = & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
A word $w \in A^+$ is a solution of (A, α, β) if $\alpha(w) = \beta(w)$.

Post's Correspondence Problem

Given	:	A, α, β
?	:	Does (A, α, β) have a solution?

Example: $A = \{a, b, c\}$

$$\begin{array}{lll} \alpha(a) = 1 & \alpha(b) = 10 & \alpha(c) = 011 \\ \beta(a) = 101 & \beta(b) = 00 & \beta(c) = 11 \end{array}$$

$$\begin{array}{lccccccccc} \alpha(acbc) = & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ \beta(acbc) = & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}$$

THEOREM. (Post, 1946)
Post's Correspondence Problem is undecidable.

THEOREM. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

PROOF IDEA.
- Reduction of Post's Correspondence Problem

THEOREM. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

PROOF IDEA.

- Reduction of Post's Correspondence Problem
- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

Theorem. **(Kirsten and Quaas, WATA 2010)**
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

Proof Idea.

- Reduction of Post's Correspondence Problem
- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

- We interprete finite words over $\{0, 1\}$ as binary representation of an integer.

We define the injective function num : $\{0,1\}^* \to \mathbb{Z}$ as follows. For each $w = a_1 ... a_n$, we let

$$\text{num}(w) = \sum_{1 \le i \le n} a_i \cdot 2^{n-i}.$$

We define the injective function num : $\{0,1\}^* \to \mathbb{Z}$ as follows. For each $w = a_1 ... a_n$, we let

$$\mathsf{num}(w) = 2^n + \sum_{1 \leq i \leq n} a_i \cdot 2^{n-i}.$$

We define the injective function num : $\{0,1\}^* \to \mathbb{Z}$ as follows. For each $w = a_1 ... a_n$, we let

$$\text{num}(w) = 2^n + \sum_{1 \leq i \leq n} a_i \cdot 2^{n-i}.$$

Example: $\text{num}(001) = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 9 \quad (= 1001)$.

THEOREM. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

PROOF IDEA.

- Reduction of Post's Correspondence Problem
- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

- We interprete finite words over $\{0, 1\}$ as binary representation of an integer.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms. Let $\Sigma = A \mathbin{\dot\cup} \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \dot{\cup} \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms. Let $\Sigma = A \quad \dot{\cup} \quad \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows. Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \mathsf{num}(\alpha(u)) - \mathsf{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \dot{\cup} \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \mathsf{num}(\alpha(u)) - \mathsf{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms. Let $\Sigma = A \;\dot\cup\; \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows. Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \underbrace{\text{num}(\alpha(u)) - \text{num}(\beta(u))}_{\Delta} \quad + \quad \underbrace{(2^{2k})^{|u|+m} - (2^{2k})^{|u|+n}}_{\Gamma},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

We have $S(u\#b^m c^m) = 0 \quad \Leftrightarrow \quad \Delta = \Gamma = 0$ **or** $\Delta \neq 0$ and $\Delta =$ -Γ.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \ \dot{\cup} \ \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \underbrace{\text{num}(\alpha(u)) - \text{num}(\beta(u))}_{\Delta} \quad + \quad \underbrace{(2^{2k})^{|u|+m} - (2^{2k})^{|u|+n}}_{\Gamma},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+ \# b^+ c^+)$. Then

$$S(w) = 1.$$

We have $S(u\#b^m c^m) = 0 \quad \Leftrightarrow \quad \Delta = \Gamma = 0$ **or** $\Delta \neq 0$ and $\Delta =$ -Γ.

- We can rule out $\Delta \neq 0$ and $\Delta =$ -Γ due to the choice of k.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \;\dot\cup\; \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \underbrace{\text{num}(\alpha(u)) - \text{num}(\beta(u))}_{\Delta} \quad + \quad \underbrace{(2^{2k})^{|u|+m} - (2^{2k})^{|u|+n}}_{\Gamma},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

We have $S(u\#b^m c^m) = 0 \quad \Leftrightarrow \quad \Delta = \Gamma = 0$ **or** $\Delta \neq 0$ and $\Delta =$ -Γ.

- We can rule out $\Delta \neq 0$ and $\Delta =$ -Γ due to the choice of k.
- $\Delta = \Gamma = 0 \Leftrightarrow \text{num}(\alpha(u)) = \text{num}(\beta(u))$ and $m = n$.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \;\dot\cup\; \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \underbrace{\mathsf{num}(\alpha(u)) - \mathsf{num}(\beta(u))}_{\Delta} \quad + \quad \underbrace{(2^{2k})^{|u|+m} - (2^{2k})^{|u|+n}}_{\Gamma},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

We have $S(u\#b^m c^m) = 0 \quad \Leftrightarrow \quad \Delta = \Gamma = 0$ **or** $\Delta \neq 0$ and $\Delta =$ -Γ.

- We can rule out $\Delta \neq 0$ and $\Delta =$ -Γ due to the choice of k.
- $\Delta = \Gamma = 0 \Leftrightarrow \mathsf{num}(\alpha(u)) = \mathsf{num}(\beta(u))$ and $m = n$.
  $\Leftrightarrow \alpha(u) = \beta(u)$ and $m = n$ (by injectivity of num).

Lemma 1.

Let $u \in A^+$ and $m, n \geq 1$. Then we have

$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.
Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.
Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.
$\Rightarrow \alpha(u) \neq \beta(u)$ (since (A, α, β) has no solution)

Lemma 1.

Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.

(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.

Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$ (since (A, α, β) has no solution)

$\Rightarrow S(w) \neq 0$ (Lemma 1)

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.
Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$	(since (A, α, β) has no solution)
$\Rightarrow S(w) \neq 0$	(Lemma 1)
$\Rightarrow w \in$ supp(S)	(Definition of support)

Lemma 1.

Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.

(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.

Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$	(since (A, α, β) has no solution)
$\Rightarrow S(w) \neq 0$	(Lemma 1)
$\Rightarrow w \in$ supp(S)	(Definition of support)

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.
Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$	(since (A, α, β) has no solution)
$\Rightarrow S(w) \neq 0$	(Lemma 1)
$\Rightarrow w \in$ supp(S)	(Definition of support)

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$.

$\Rightarrow S(w) = 1$	(Definition of S)

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

Proof.
Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$	(since (A, α, β) has no solution)
$\Rightarrow S(w) \neq 0$	(Lemma 1)
$\Rightarrow w \in$ supp(S)	(Definition of support)

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$.

$\Rightarrow S(w) = 1$	(Definition of S)
$\Rightarrow w \in$ supp(S)	(Definition of support)

LEMMA 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

LEMMA 2.
(A, α, β) has no solution $\quad \Rightarrow \quad$ supp(S) is recognizable.

PROOF.
Let $w = u\#b^m c^n$ for some $u \in A^+$ and $m, n \geq 1$.

$\Rightarrow \alpha(u) \neq \beta(u)$	(since (A, α, β) has no solution)
$\Rightarrow S(w) \neq 0$	(Lemma 1)
$\Rightarrow w \in$ supp(S)	(Definition of support)

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$.

$\Rightarrow S(w) = 1$	(Definition of S)
$\Rightarrow w \in$ supp(S)	(Definition of support)

$\Rightarrow$ supp$(S) = \Sigma^*$ and thus recognizable.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Proof.
Let $\mathcal{A}$ be a finite automaton such that $L(\mathcal{A}) =$ supp(S).

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Proof.
Let $\mathcal{A}$ be a finite automaton such that $L(\mathcal{A}) =$ supp(S).
Assume by contradiction that (A, α, β) has a solution.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Proof.
Let $\mathcal{A}$ be a finite automaton such that $L(\mathcal{A}) =$ supp(S).
Assume by contradiction that (A, α, β) has a solution.
Then there is some $u \in A^+$ such that $\alpha(u) = \beta(u)$.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Proof.
Let $\mathcal{A}$ be a finite automaton such that $L(\mathcal{A}) = \text{supp}(S)$.
Assume by contradiction that (A, α, β) has a solution.
Then there is some $u \in A^+$ such that $\alpha(u) = \beta(u)$.
Define $U = u\#b^+c^+$.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Proof.
Let $\mathcal{A}$ be a finite automaton such that $L(\mathcal{A}) =$ supp(S).
Assume by contradiction that (A, α, β) has a solution.
Then there is some $u \in A^+$ such that $\alpha(u) = \beta(u)$.
Define $U = u\#b^+c^+$.
By Lemma 1, we have $L(\mathcal{A}) \cap U = \{u\#b^m c^n \mid m \neq n\}$.

Lemma 1.
Let $u \in A^+$ and $m, n \geq 1$. Then we have
$S(u\#b^m c^n) = 0 \quad \Leftrightarrow \quad \alpha(u) = \beta(u)$ and $m = n$.

Lemma 3.
(A, α, β) has no solution $\quad \Leftarrow \quad$ supp(S) is recognizable.

Proof.
Let $\mathcal{A}$ be a finite automaton such that $L(\mathcal{A}) =$ supp(S).
Assume by contradiction that (A, α, β) has a solution.
Then there is some $u \in A^+$ such that $\alpha(u) = \beta(u)$.
Define $U = u\#b^+c^+$.
By Lemma 1, we have $L(\mathcal{A}) \cap U = \{u\#b^m c^n \mid m \neq n\}$.
Since U and $L(\mathcal{A})$ are recognizable, $L(\mathcal{A}) \cap U$ is recognizable. Contradiction!

Theorem. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

Proof Idea.

- Reduction of Post's Correspondence Problem
- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

- We interprete finite words over $\{0, 1\}$ as binary representation of an integer.

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \ \dot{\cup} \ \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \text{num}(\alpha(u)) - \text{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

S is recognizable by a weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$:

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \quad \dot{\cup} \quad \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \mathsf{num}(\alpha(u)) - \mathsf{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+ \# b^+ c^+)$. Then

$$S(w) = 1.$$

S is recognizable by a weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$:
- the series $\mathsf{num} \circ \eta$ is recognizable for each homom. $\eta : A^* \to \{0,1\}^*$

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \;\dot\cup\; \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \mathsf{num}(\alpha(u)) - \mathsf{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \backslash (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

S is recognizable by a weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$:

- the series $\mathsf{num} \circ \eta$ is recognizable for each homom. $\eta : A^* \to \{0,1\}^*$
- the series $S'(u) = (2^{2k})^{|u|+m}$ is recognizable for each k and m

Let A be a finite alphabet and $\alpha, \beta : A^* \to \{0,1\}^*$ be two homomorphisms.
Let $\Sigma = A \ \dot\cup \ \{\#, b, c\}$ and define the series $S : \Sigma^* \to \mathbb{Z}$ as follows.
Let $w = u\#b^m c^n$ for some $u \in A^+, m, n \geq 1$. Then

$$S(w) = \text{num}(\alpha(u)) - \text{num}(\beta(u)) \quad + \quad (2^{2k})^{|u|+m} - (2^{2k})^{|u|+n},$$

where $k \geq 1$ such that $k > |\alpha(a)| + 1$ and $k > |\beta(a)| + 1$ for each $a \in A$.

Let $w \in \Sigma^* \setminus (A^+\#b^+c^+)$. Then

$$S(w) = 1.$$

S is recognizable by a weighted automaton over $(\mathbb{Z}, +, \cdot, 0, 1)$:

- the series num $\circ\, \eta$ is recognizable for each homom. $\eta : A^* \to \{0,1\}^*$
- the series $S'(u) = (2^{2k})^{|u|+m}$ is recognizable for each k and m
- recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ are closed under $+$ and $-$

THEOREM. (Kirsten and Quaas, WATA 2010)
It is not decidable whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is recognizable by a finite automaton.

PROOF IDEA.

- Reduction of Post's Correspondence Problem
- Given A, α, β, we define a recognizable series S over $(\mathbb{Z}, +, \cdot, 0, 1)$ such that

$$(A, \alpha, \beta) \text{ has no solution} \quad \Leftrightarrow \quad \mathsf{supp}(S) \text{ is recognizable}$$

- We interprete finite words over $\{0, 1\}$ as binary representation of an integer.

Future Work

- Is the recognizability of supports of recognizable series over non-SR-semirings always undecidable?

- What is the complexity of the problem to decide whether the support of a recognizable series over $(\mathbb{Z}, +, \cdot, 0, 1)$ is empty?