Kleene Algebras and Semimodules for Energy Functions

Zoltán Ésik, Uli Fahrenberg, Axel Legay, Karin Quaas
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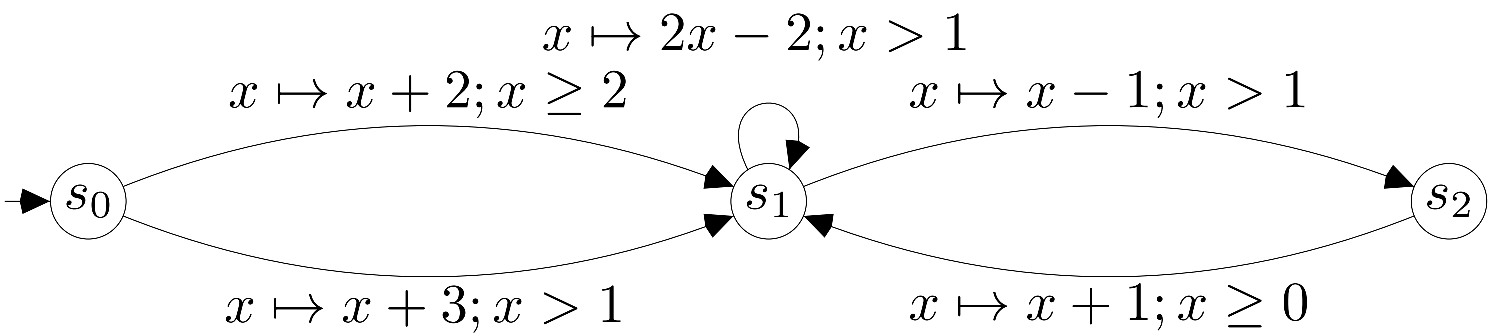
Introduction

- Energy/resource consumption problems:
  - system where certain tasks can be repeatedly accomplished
  - the system should never run out of energy (resources)

- Energy problems for, e.g.,
  - weighted timed automata [Bouyer et al, 2008,2010,2012],
  - multiweighted automata [Fahrenberg et al, 2011]
  - vector addition systems with states [Brázdil et al, 2010]

- We introduce energy automata in order to unify these approaches
- Energy automata are semiring-weighted automata
- Close connection between energy problems and reachability and Büchi acceptance problems for semiring-weighted automata

Energy Automata



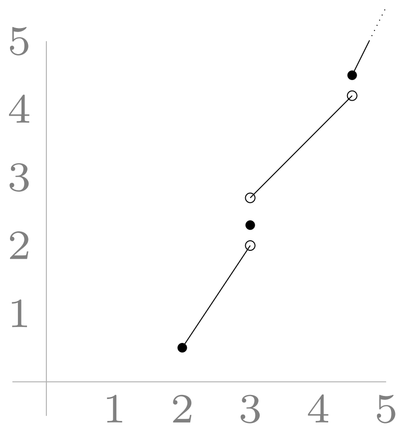
An energy automaton is a pair (S, T), where

- S is a finite set of states,
- $T \subseteq S \times \mathcal{E} \times S$ is a finite set of transitions labelled with energy functions.

Energy Functions

- Let $[0,\infty]_\bot = \mathbb{R}_{\geq 0} \cup \{\infty, \bot\}$.
- Extend the standard order on $\mathbb{R}_{\geq 0}$ by $\bot < x < \infty$ for all $x \in \mathbb{R}_{\geq 0}$.

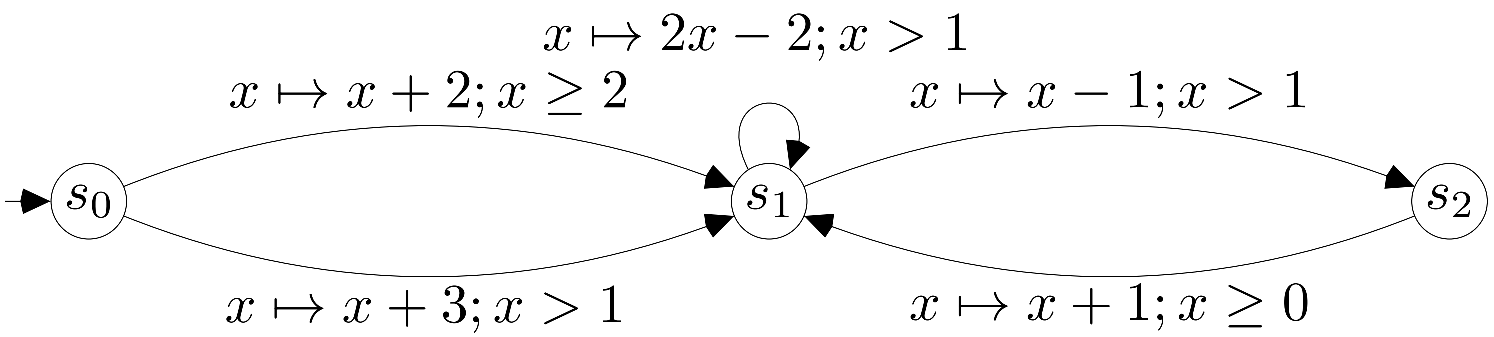
- An energy function is a mapping $f : [0,\infty]_\bot \to [0,\infty]_\bot$ satisfying
  - $f(\bot) = \bot$,
  - $f(x_2) - f(x_1) \geq x_2 - x_1$ for all $x_1 \leq x_2$,
  - $f(\infty) = \infty$ (unless $f(x) = \bot$ for all $x \in [0,\infty]_\bot$)



$$f(x) = \begin{cases} .5 & (x = 2) \\ 1.5\,x - 2.5 & (2 < x < 3) \\ 2.3 & (x = 3) \\ x - .3 & (3 < x < 4.5) \\ 4.5 & (x = 4.5) \\ 2\,x - 4.5 & (x > 4.5) \end{cases}$$

- We use $\mathcal{E}$ to denote the set of energy functions

Energy Automata



An energy automaton is a pair (S, T), where
- S is a finite set of states,
- $T \subseteq S \times \mathcal{E} \times S$ is a finite set of transitions labelled with energy functions.

An example run:
$(s_0, 1.5) \to (s_1, 4.5) \to (s_1, 7) \to (s_2, 6) \to \ldots$

Problems

Let $\mathcal{E}' \subseteq \mathcal{E}$ be a subset of computable energy functions.

The Reachability Problem

Instance: $\mathcal{E}'$-automaton (S, T), $s_0 \in S$, $F \subseteq S$, computable $x_0 \in \mathrm{R}_{\geq 0}$.

Question: Is there some finite run of (S, T) from (s_0, x_0) that ends in some state in F?

$\text{Reach}_{\mathcal{E}}(S, T)(s_0, F, x_0)$

The Büchi Acceptance Problem

Instance: $\mathcal{E}'$-automaton (S, T), $s_0 \in S$, $F \subseteq S$, computable $x_0 \in \mathrm{R}_{\geq 0}$.

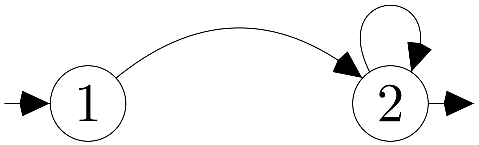
Question: Is there some infinite run of (S, T) from (s_0, x_0) that visits states in F infinitely often?

$\text{Buchi}_{\mathcal{E}}(S, T)(s_0, F, x_0)$

Background

Reachability Problem for Finite Directed Graphs

- For finite graphs, Reach$(S,T)(s_0,F)$ is in NL



- Algebraic approach:

  There is a finite run from 1 that ends in 2 $\Leftrightarrow$ $\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}^*\begin{bmatrix}0\\1\end{bmatrix} = 1$

- M is the $|S| \times |S|$-adjacency matrix of T over Boolean algebra
- Compute M^* using Warshall's algorithm
- Warshall's algorithm extended for computing * of matrices over other structures, e.g., closed semirings [Lehmann 1977]
- For, e.g., Conway semirings, * can be defined inductively by decomposition of matrices:

  $$M = \begin{bmatrix}a & b\\c & d\end{bmatrix} \quad M^* = \begin{bmatrix}(a \vee bd^*c)^* & (a \vee bd^*c)^*bd^*\\(d \vee ca^*b)^*ca^* & (d \vee ca^*b)^*\end{bmatrix}$$

The Algebra of Energy Functions

The Algebra of Energy Functions (1)

- $(\mathcal{E}, \vee, \circ, \underline{\perp}, \mathsf{id})$ is an idempotent semiring with natural partial order $\leq$
  - $f \leq g$ iff $f(x) \leq g(x)$,
  - $(f \vee g)(x) = \max(f(x), g(x))$,
  - $(f \circ g)(x) = f(g(x))$,
  - $\underline{\perp}(x) = \perp$,
  - $\mathsf{id}(x) = x$.

- Define * operation on $\mathcal{E}$:

$$f^*(x) = \begin{cases} x & \text{if } f(x) \leq x \\ \infty & \text{if } f(x) > x \end{cases}$$

- $(\mathcal{E}, \vee, \circ, ^*, \underline{\perp}, \mathsf{id})$ is a star-continuous Kleene algebra: $gf^*h = \sup_{n \in \mathbb{N}} gf^n h$

The Algebra of Energy Functions (2)

- $(\mathcal{E}^{(n\times n)}, \vee, \circ, Z, I)$ is an idempotent semiring with natural partial order $\leq$
  - $A \leq B$ iff $A_{i,j} \leq B_{i,j}$ for all $1 \leq i, j \leq n$,
  - $(A \vee B)_{i,j} = A_{i,j} \vee B_{i,j}$,
  - $(A \circ B)_{i,j} = \bigvee_{k=1}^{n} A_{i,k} \circ B_{k,j}$,
  - Z is the zero $(n \times n)$-matrix,
  - I is the identity $(n \times n)$-matrix

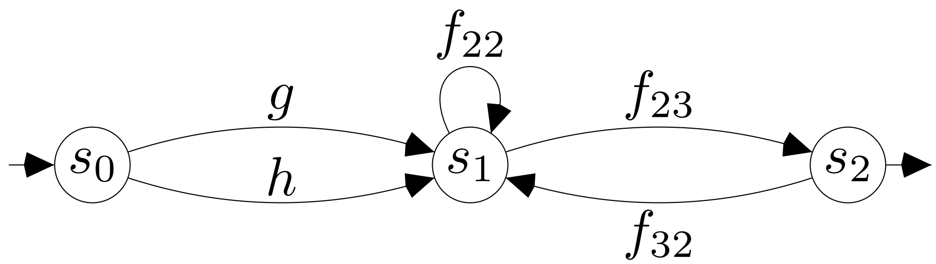
- The matrix semiring of a star-continuous Kleene algebra is also a star-continuous Kleene algebra: $MN^*O = \sup_{n\in\mathbb{N}} MN^nO$ [DKV 2009]

- Every star-continuous Kleene algebra is a Conway semiring [DKV 2009]
- Thus: M^* can be computed by matrix decomposition:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^*bd^* \\ (d \vee ca^*b)^*ca^* & (d \vee ca^*b)^* \end{bmatrix}$$

Reachability Problem

- Let $\mathcal{E}' \subseteq \mathcal{E}$ be a subalgebra so that $f^* \in \mathcal{E}'$ for each $f \in \mathcal{E}'$.
- Let (S, T) with $|S| = n$ be an $\mathcal{E}'$-automaton.
- Goal: apply Kleene algebra framework to solve $\text{Reach}_{\mathcal{E}'}(S, T)(s_0, F, x_0)$



There is a finite run of (S, T) from (s_0, x_0) that ends in s_2 $\Leftrightarrow$ $\begin{bmatrix} \text{id} & \perp & \perp \end{bmatrix} \begin{bmatrix} \perp & \max\{g, h\} & \perp \\ \perp & f_{22} & f_{23} \\ \perp & f_{32} & \perp \end{bmatrix}^* \begin{bmatrix} \perp \\ \perp \\ \text{id} \end{bmatrix} (x_0) \neq \perp$

- Note: We can compute M^* because $\mathcal{E}^{(n \times n)}$ is a Conway semiring.
- Note: The algorithm is **static**.

Reachability Problem

- Let $\mathcal{E}' \subseteq \mathcal{E}$ be a subalgebra so that $f^* \in \mathcal{E}'$ for each $f \in \mathcal{E}'$.
- Let (S, T) with $|S| = n$ be an $\mathcal{E}'$-automaton.
- Goal: apply Kleene algebra framework to solve Reach$_{\mathcal{E}'}(S, T)(s_0, F, x_0)$

Theorem: There is a finite run of (S, T) from (s_0, x_0) to some state in F if, and only if,

$$I^{s_0} T^* F(x_0) \neq \bot.$$

- Note: Proof is based on the fact that $\mathcal{E}^{(n \times n)}$ is a star-continuous Kleene algebra: $NM^*O = \sup_{n \in \mathbb{N}}(NM^n O)$

The Algebra of Energy Functions (3)

- $(\mathcal{E}, \vee, \circ, ^*, \perp, \mathrm{id})$ is a star-continuous Kleene algebra
- $(\mathcal{E}, \leq)$ is a complete lattice: $\vee$ defined for infinite sets of energy functions
- Define $\circ$ operation for infinite sequences $f_0, f_1, f_2, \ldots$ of energy functions:

$$(\prod_{i=0}^{\infty} f_i)(x_0) = \begin{cases} \texttt{false} & \text{if } \exists n \in \mathbb{N} . x_n = \perp \\ \texttt{true} & \text{if } \forall n \in \mathbb{N} . x_n \neq \perp \end{cases}$$

where $x_0 \in [0, \infty]_\perp$ and $x_{n+1} = f_n(x_n)$.

- Define $^\omega$ operation on energy functions:

$$f^\omega(x) = \begin{cases} \texttt{false} & \text{if } x = \perp \text{ or } f(x) < x \\ \texttt{true} & \text{if } x \neq \perp \text{ and } f(x) \geq x \end{cases}, \text{ where } x \in [0, \infty]_\perp$$

- Note that $\circ$ and $^\omega$ do not map to $[0, \infty]_\perp$, but $\{\texttt{true}, \texttt{false}\}$

The Algebra of Energy Functions (4)

- Let $\mathbb{B} = \{\texttt{true}, \texttt{false}\}$ be the Boolean algebra, with $\texttt{false} < \texttt{true}$.
- Define $\mathcal{V}$ be the set of mappings $u : [0, \infty]_\perp \to \mathbb{B}$ satisfying
  - $u(\perp) = \texttt{false}$
  - $x_1 \leq x_2 \Rightarrow u(x_1) \leq u(x_2)$

- $(\mathcal{E}, \mathcal{V})$ is a Conway semiring-semimodule pair [Ésik & Kuich 2005]
  - with action $\mathcal{V} \times \mathcal{E} \to \mathcal{V} : (u, f) \mapsto uf$

- $(\mathcal{E}^{(n \times n)}, \mathcal{V}^n)$ is a Conway semiring-semimodule pair
  - with action $\mathcal{E}^{(n \times n)} \times \mathcal{V}^n \to \mathcal{V}^n$ similar to matrix multiplication
  - ${}^{\omega_k} : \mathcal{E}^{(n \times n)} \to \mathcal{V}^n$ is defined inductively by matrix decomposition.

Büchi Acceptance Problem

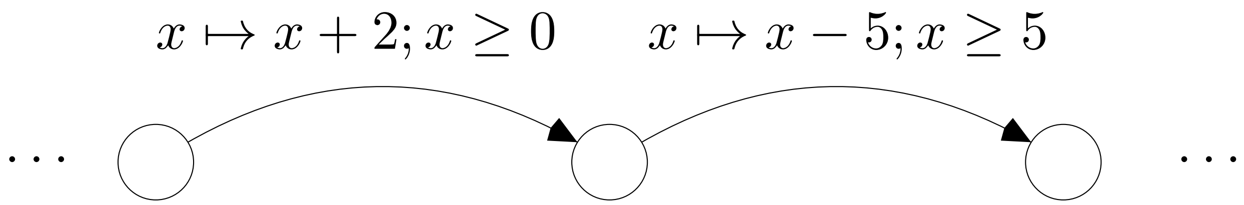
- Let $\mathcal{E}' \subseteq \mathcal{E}$ be a subalgebra so that $f^* \in \mathcal{E}'$ for each $f \in \mathcal{E}'$.
- Let (S, T) with $|S| = n$ be an $\mathcal{E}'$-automaton.
- Goal: apply Kleene algebra framework to solve $\text{Buchi}_{\mathcal{E}'}(S, T)(s_0, F, x_0)$

Theorem:
There is an infinite run of (S, T) from (s_0, x_0) visiting states in F infinitely often
if, and only if,
$T^\omega F(x_0) = \texttt{true}$.

Applications

Application: Integer Update Functions

- Used in integer weighted automata and VASS

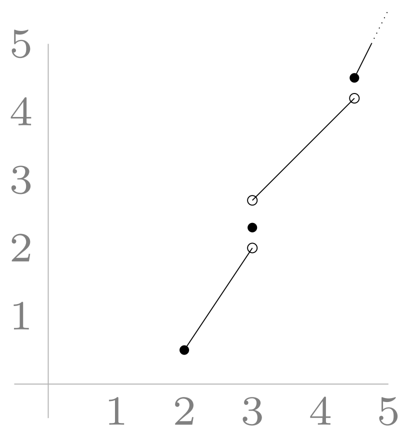


- The class $\mathcal{E}_{\mathsf{int}}$ of integer update functions forms a subalgebra of $\mathcal{E}$

Theorem: For $\mathcal{E}_{\mathsf{int}}$-automata, reachability and Büchi acceptance are decidable in PTIME.

Application: Piecewise Affine Functions

- Used in the reduction to show decidability of energy problems for single-clock weighted timed automata [Bouyer et al 2010]



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- The class $\mathcal{E}_{\text{pw}}$ of piecewise affine functions forms a subalgebra of $\mathcal{E}$.

Theorem: For $\mathcal{E}_{\text{pw}}$-automata, reachability and Büchi acceptance are decidable in EXPTIME.

Extensions

Extension to Multidimensional Energy Automata

- Multiple energy variables

- The **static** decision algorithms from our algebraic approach cannot be applied (counterexample)

- The class of n-dimensional integer piecewise affine $\mathcal{E}^n_{\text{pwi}}$-automata are upward-compatible well-structured transition systems.

Theorem: For $\mathcal{E}^n_{\text{pwi}}$-automata, reachability is decidable.

- Büchi acceptance problem: open.

Extension to Flat Energy Automata

- we lift the requirement $f(x_2) - f(x_1) \geq x_2 - x_1$ for each $x_2 \geq x_1$

- flat energy functions
  - still require f to be strictly increasing
  - derivative may be less than 1

- we conjecture that the algebraic approach can be extended to flat energy functions (for one energy variable)

Theorem: For flat $\mathcal{E}^4_{\text{pw}}$-automata, reachability is undecidable.

Extension to Energy Games

- Let (S, T) be an n-dimensional energy automaton such that
  - $S = S_A \cup S_B$ forms a partition, and
  - $T \subset (S_A \times \mathcal{E} \times S_B) \cup (S_B \times \mathcal{E} \times S_A)$.

- Then (S, T) induces an n-dimensional energy game.

Theorem: Whether A wins the reachability game in $\mathcal{E}_{\text{int}}$-automata (i.e., 1-dimensional VASS) is decidable. For $\mathcal{E}^2_{\text{int}}$-automata the same problem is undecidable.[Brázdil et al 2010].

Theorem: Whether A wins the reachability game in flat $\mathcal{E}_{\text{pw}}$-automata is undecidable.

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