# Synchronizing Data Words for Register Automata

Parvaneh Babari [1], Karin Quaas [1], Mahsa Shirmohammadi [2]

[1] Universität Leipzig, [2] University of Oxford

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# Data Words and Data Languages

$\Sigma$...a finite alphabet
$D$...an infinite data domain, e.g., $\mathbb{N}$, $\mathbb{R}_{\geq 0}$, `ASCII strings`

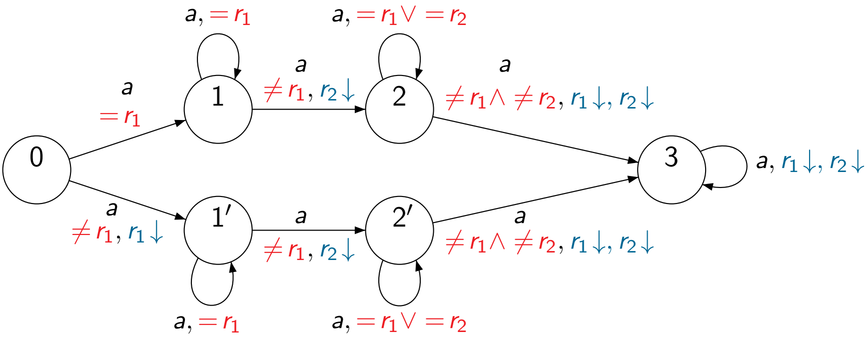
A **data word** is a finite sequence

$$(a_1, d_1)(a_2, d_2) \ldots (a_n, d_n)$$

over $\Sigma \times D$.

A **data language** is a subset of $(\Sigma \times D)^*$.

# Register Automata (RA)



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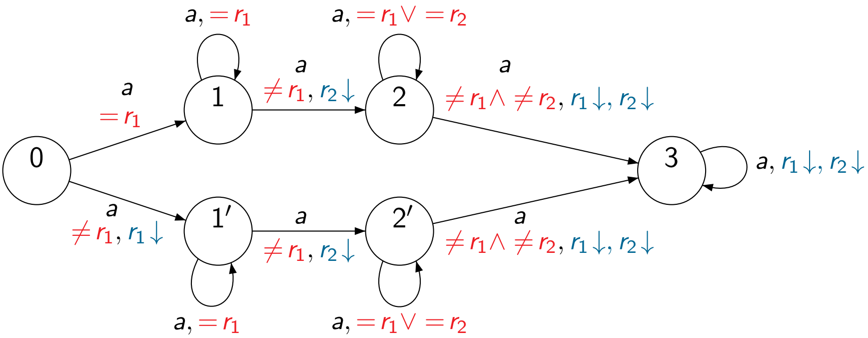


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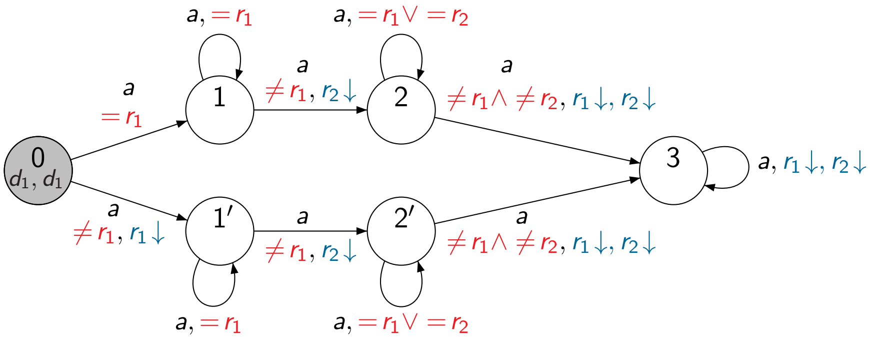


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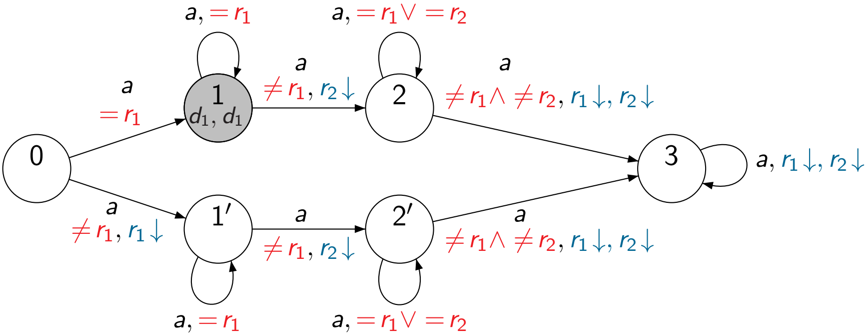


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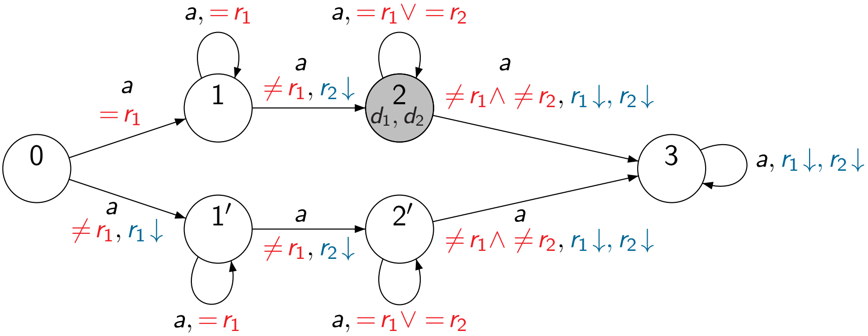


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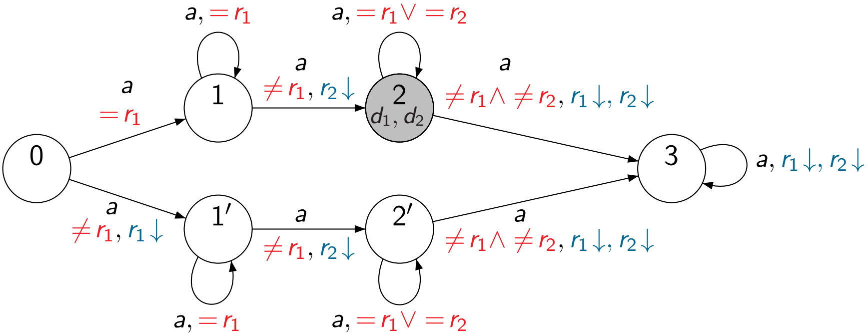


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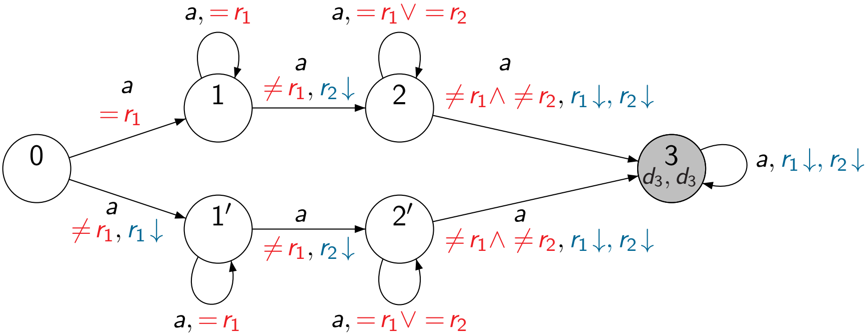


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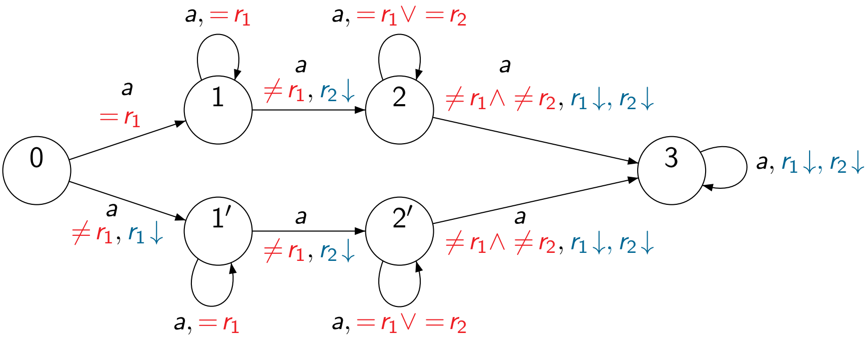


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# Facts about Register Automata

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- decidable for NRA with one register (1-NRA), Ackermann-complete (Demri & Lazic, 2008)

# Synchronizing Data Words for Register Automata

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Let $\mathcal{A}$ be a RA with $k$ registers.

A data word $\mathbf{w}$ is called **synchronizing for** $\mathcal{A}$ if there exists a configuration $(\mathbf{s}, \mathbf{v_1}, \ldots, \mathbf{v_k})$ such that $\mathcal{A}$ is in $(\mathbf{s}, \mathbf{v_1}, \ldots, \mathbf{v_k})$ after processing $\mathbf{w}$,
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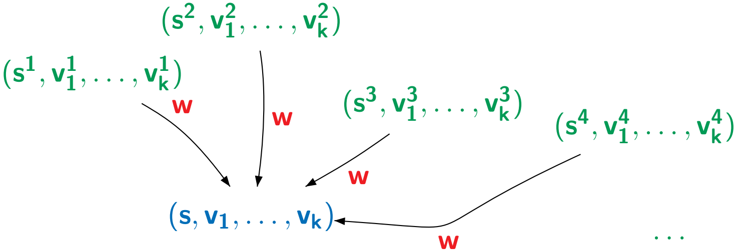

# Example Synchronizing Data Words

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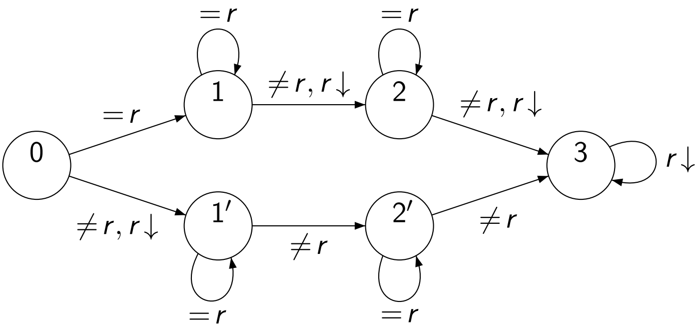


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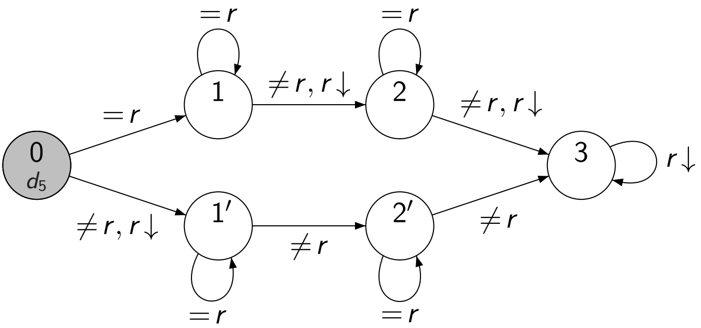


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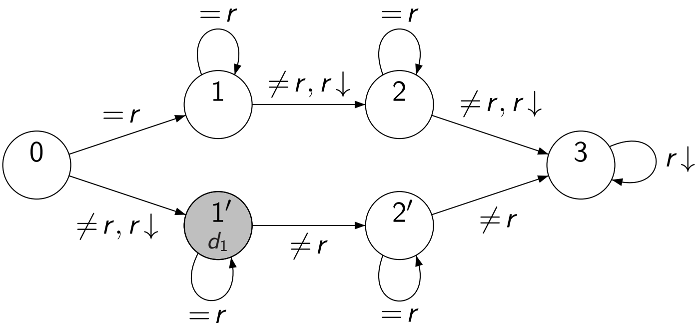


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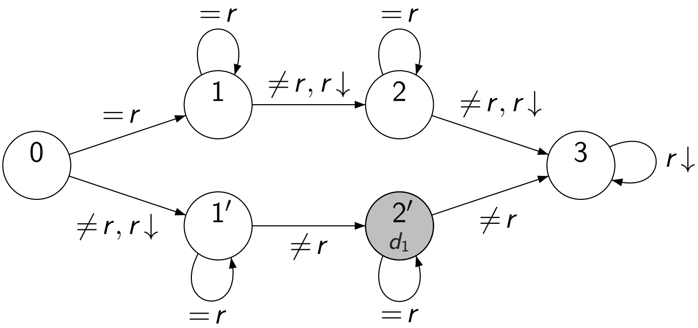


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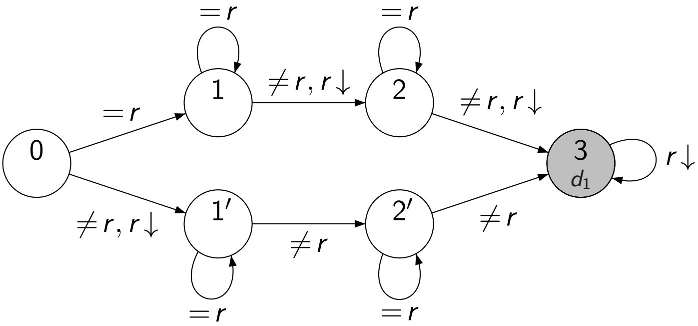


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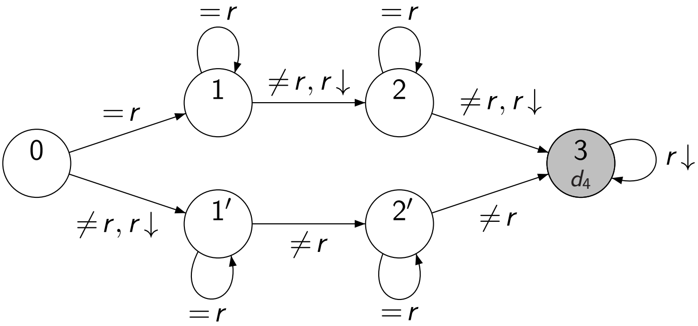


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# How to prove the Synchronizing Property

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\begin{array}{c}
\{0, 1, 2, 1', 2', 3\} \times D \\
\Big\downarrow (a, d_1) \\
\{(1, d_1), (1', d_1), (2, d_1), (3, d_1), (2', d_1)\} \cup (\{2', 3\} \times D \backslash \{d_1\}) \\
\Big\downarrow (a, d_2) \\
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# Synchronizing Problem for Register Automata

**The Synchronizing Problem**

Instance: A register automaton $\mathcal{A}$.

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- Overview article by Volkov, 2008.

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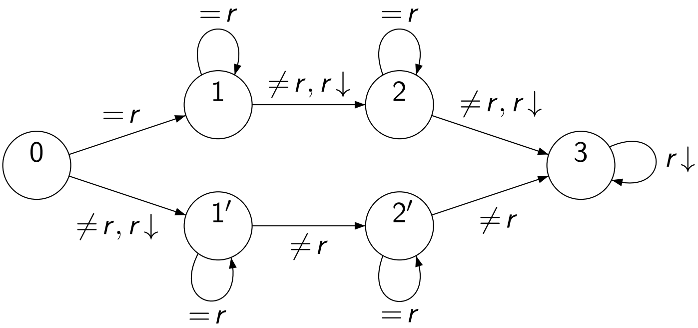


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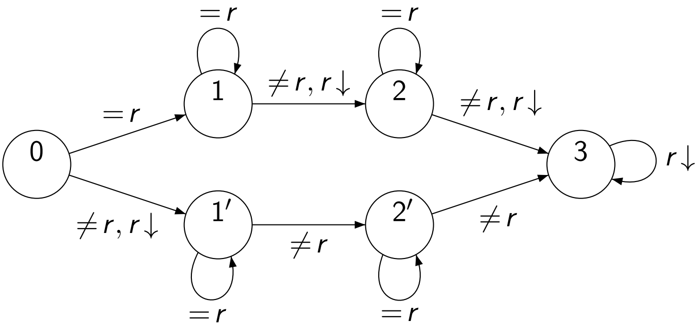


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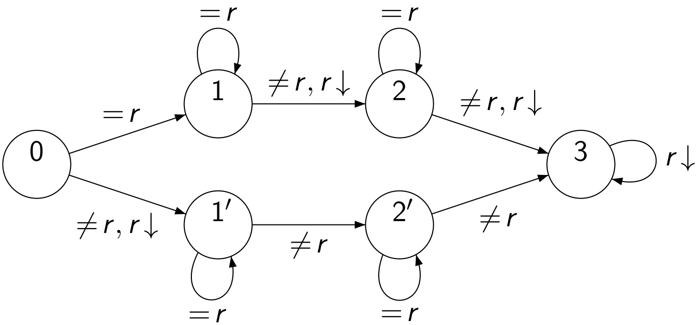


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The data word $(a, d_1)(a, d_1)(a, d_1)(a, d_2)(a, d_1)(a, d_2)(a, d_1)$ is also synchronizing for $\mathcal{A}$.

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# Synchronizing Data Words for DRA

**Lemma**

*If for a $k$-DRA $\mathcal{A}$ there exists some synchronizing data word, then there exists also some synchronizing data word for $\mathcal{A}$ with at most $2k + 1$ distinct data values.*

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**Theorem**

*The synchronizing problem for DRA is* PSPACE*-complete.*

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*The synchronizing problem for NRA is undecidable.*

**Proof Sketch.**

By reduction of the non-universality problem for NRA (preserving the number of registers). $\square$

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*For* 1-*NRA the synchronizing problem is* Ackermann-*complete.*

**Proof Sketch.**

By reduction to the non-universality problem for 1-NRA. □

### The Length-Bounded Synchronizing Problem

Instance: An NRA $\mathcal{A}$, a natural number $N \in \mathbb{N}$.

Question: Does there exist a synchronizing data word for $\mathcal{A}$ with length at most $N$ ?

# Length-Bounded Synchronizing Data Words for NRA

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**Proof Sketch.**

NEXPTIME-hardness by reduction from the bounded universality problem for **regular-like expressions** (built from atomic expressions and $\cdot$, $+$, and $^2$). $\square$

# Conclusion and Open Problems

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Open: Precise complexity for the length-bounded synchronizing problem for DRA.