# Revisiting Reachability in Timed Automata

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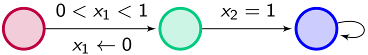
LICS 2017

## Timed Automata (TA)

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# Model Checking Timed Automata & Parametric TCTL

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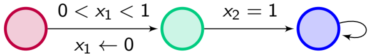


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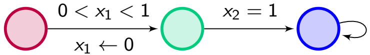


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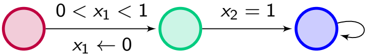


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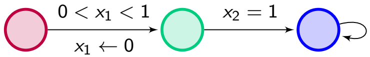


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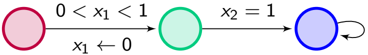


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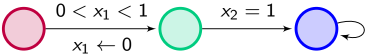


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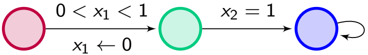


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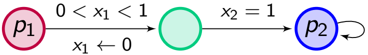


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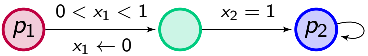


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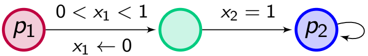


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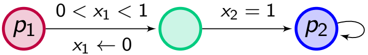


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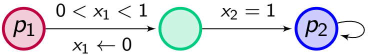


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Classical region-based decision procedures do not work.*

* As opposed to Bruyère et al, 2003/08, where only integer-valued parameters are allowed

# Goal

### Computing the Reachability Relation

**Instance:** A timed automaton and locations $\ell, \ell'$

**Output:** First-order formula $\varphi_{\ell,\ell'}(\mathbf{y}, \mathbf{y}')$ such that $(\mathbf{v}, \mathbf{v}') \models \varphi_{\ell,\ell'}$ iff $(\ell, \mathbf{v}) \to^* (\ell', \mathbf{v}')$.

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### Our Logic

Existential fragment of mix of real-arithmetic and Presburger arithmetic

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- Real-valued variables $\mathbf{r} = (r_1, \ldots, r_n)$

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**Instance:** Locations $\ell, \ell'$, concrete valuation $\mathbf{v}$
**Output:** $\varphi_{\ell,\ell',\mathbf{v}}(\mathbf{z}, \mathbf{r}, \mathbf{z}', \mathbf{r}')$ such that
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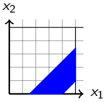


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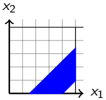


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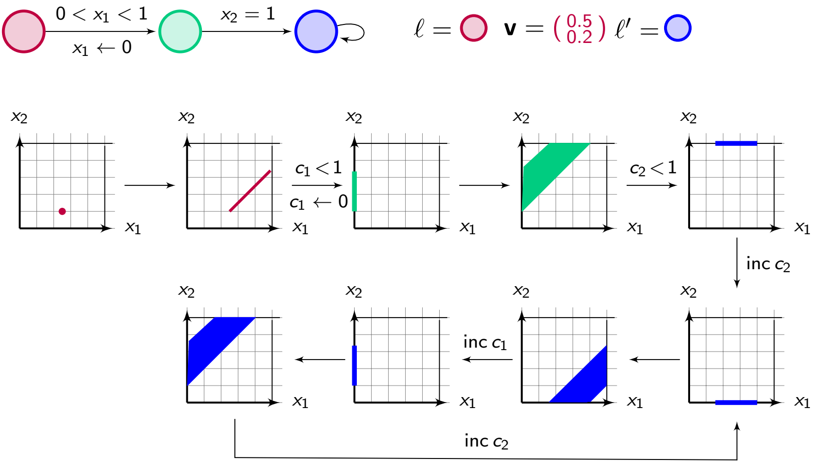
# Solution: Step I, Example



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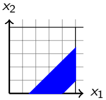


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$$(0.3 < r'_1 \leq 1) \wedge (0 \leq r'_2 < 0.7) \wedge (0.3 < r_1 - r_2 \leq 0.8)$$

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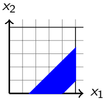


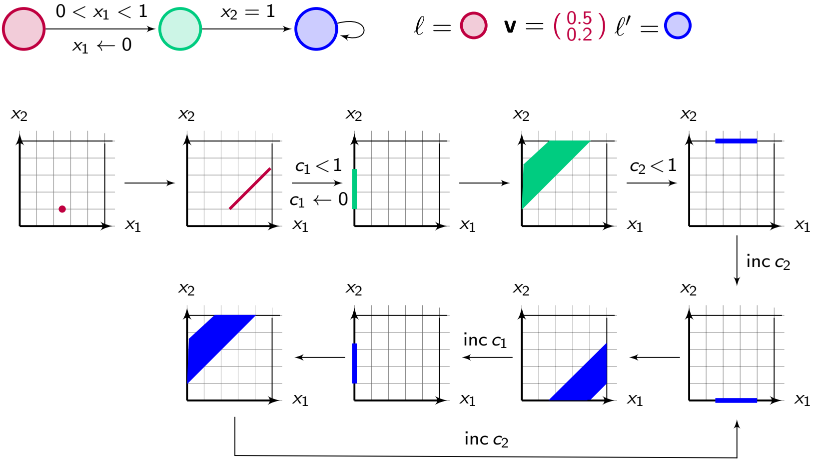

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \left[ \begin{array}{ccc} (\leq, 0) & (<, -0.3) & (\leq, 0) \\ (\leq, 1) & (\leq, 0) & (\leq, 0.8) \\ (<, 0.7) & (<, -0.3) & (\leq, 0) \end{array} \right] \end{array}$$

$$(0.3 < r'_1 \leq 1) \wedge (0 \leq r'_2 < 0.7) \wedge (0.3 < r_1 - r_2 \leq 0.8)$$

- counter $c_i$ stores the integer part of clock $x_i$

The reachability relation of monotonic counter machines is definable in Presburger arithmetic.

# Solution: Step I, Example




$$\varphi_{\ell,\ell',\mathbf{v}} = \big((z_2' - z_1' = 1) \wedge (0.3 < r_1' \leq 1) \wedge (0 \leq r_2' < 0.7) \wedge (0.3 < r_1' - r_2' \leq 0.8)\big)$$
$$\vee \big((z_2' - z_1' = 0) \wedge (0 \leq r_1' \leq 0.8) \wedge (0.3 < r_2' \leq 1) \wedge (0.3 < 1 + r_1' - r_2' \leq 0.8)\big)$$

# Solution: Step I

**Instance:** Locations $\ell, \ell'$, concrete valuation $\mathbf{v}$
**Output:** $\varphi_{\ell,\ell',\mathbf{v}}(\mathbf{z}, \mathbf{r}, \mathbf{z}', \mathbf{r}')$ such that
$(\lfloor \mathbf{v} \rfloor, \mathrm{fr}(\mathbf{v}), \lfloor \mathbf{v}' \rfloor, \mathrm{fr}(\mathbf{v}')) \models \varphi_{\ell,\ell',\mathbf{v}}$ iff $(\ell, \mathbf{v}) \to^* (\ell', \mathbf{v}')$

To obtain $\varphi_{\ell,\ell',\mathbf{v}}$, we construct a monotonic counter machine $M_{\ell,\ell',\mathbf{v}}$

- sound and complete with respect to reachability
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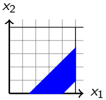


$$\begin{array}{cc} & \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} & \left[\begin{array}{ccc} (\leq, 0) & (<, -0.3) & (\leq, 0) \\ (\leq, 1) & (\leq, 0) & (\leq, 0.8) \\ (<, 0.7) & (<, -0.3) & (\leq, 0) \end{array}\right] \end{array}$$

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- counter $c_i$ stores the integer part of clock $x_i$

The reachability relation of monotonic counter machines is definable in Presburger arithmetic.

# Solution: Step II

**Instance:** Locations $\ell, \ell'$, concrete valuation $\mathbf{v}$
**Output:** $\varphi_{\ell,\ell',\mathsf{type}(\mathbf{u})}(\mathbf{z}, \mathbf{r}, \mathbf{z}', \mathbf{r}')$ such that

$(\lfloor \mathbf{u} \rfloor, \mathsf{fr}(\mathbf{u}), \lfloor \mathbf{v}' \rfloor, \mathsf{fr}(\mathbf{v}')) \models \varphi_{\ell,\ell',\mathsf{type}(\mathbf{v})}$ iff $(\ell, \mathbf{u}) \rightarrow^* (\ell', \mathbf{v}')$

for all $\mathbf{u}$ with $\mathsf{type}(\mathbf{u}) = \mathsf{type}(\mathbf{v})$.

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for all $\mathbf{u}$ with $\mathsf{type}(\mathbf{u}) = \mathsf{type}(\mathbf{v})$.

### Difference Types

$\mathsf{type}(\mathbf{v})$ is collection of formulas of the form

$$c + r_i - r_j \;\leq\; c' + r_{i'} - r_{j'}$$

that are satisfied by $\mathbf{v}$, where $c, c' \in \{0, 1, -1\}$, $i, j, i', j' \in \{0, \ldots, n\}$.

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Example: $r_1 - r_2 \ \leq \ 1 + r_0 - r_1$ is in $\mathsf{type}\left(\begin{smallmatrix} 0.5 \\ 0.2 \end{smallmatrix}\right)$, but it is not in $\mathsf{type}\left(\begin{smallmatrix} 0.7 \\ 0.2 \end{smallmatrix}\right)$

# Solution: Step II

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Example: $r_1 - r_2 \;\leq\; 1 + r_0 - r_1$ is in $\mathrm{type}\binom{0.5}{0.2}$, but it is not in $\mathrm{type}\binom{0.7}{0.2}$

Note: there are only finitely many distinct different types.

# Solution: Step II ctd.

To obtain $\varphi_{\ell,\ell',\mathsf{type}(\mathbf{v})}$, we construct a monotonic counter machine $M_{\ell,\ell',\mathsf{type}(\mathbf{v})}$

- sound and complete with respect to rachability
- control state $q$ is a *parametric DBM* over equivalence classes of difference types (representing the fractional parts of the clock values)

$$\begin{array}{cc} & \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} & \begin{bmatrix} (\leq,[0]) & (\leq,[0-r_1]) & (\leq,[0]) \\ (\leq,[1]) & (\leq,[0]) & (\leq,[1]) \\ (<,[1-r_1]) & (<,[0-r_1]) & (\leq,[0]) \end{bmatrix} \end{array}$$

- counter $c_i$ stores the integer part of clock $x_i$

# Solution: Step II ctd.

To obtain $\varphi_{\ell,\ell',\mathsf{type}(\mathbf{u})}$, we construct a monotonic counter machine $M_{\ell,\ell',\mathsf{type}(\mathbf{v})}$

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$$(r_1 < r_1' \leq 1) \wedge (0 \leq r_2' \leq 1 - r_1) \wedge (-r_1 < r_1 - r_2 \leq 1)$$

- counter $c_i$ stores the integer part of clock $x_i$

The reachability relation of monotonic counter machines is definable in Presburger arithmetic.

# Solution: Final Step

**Instance:** Locations $\ell, \ell'$
**Output:** $\varphi_{\ell,\ell'}(\mathbf{z}, \mathbf{r}, \mathbf{z}', \mathbf{r}')$ such that
$(\lfloor \mathbf{v} \rfloor, \mathrm{fr}(\mathbf{v}), \lfloor \mathbf{v}' \rfloor, \mathrm{fr}(\mathbf{v}')) \models \varphi_{\ell,\ell'}$ iff $(\ell, \mathbf{v}) \to^* (\ell', \mathbf{v}')$

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To obtain $\varphi_{\ell,\ell'}$, we

- construct for each of the finitely many difference types $T$ the corresponding formula $\varphi_{\ell,\ell',T}$, and

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To obtain $\varphi_{\ell,\ell'}$, we

- construct for each of the finitely many difference types $T$ the corresponding formula $\varphi_{\ell,\ell',T}$, and
- define $\varphi_{\ell,\ell'}$ as the disjunction of all these formulas.

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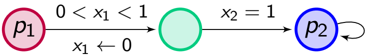
- construct for each of the finitely many difference types $T$ the corresponding formula $\varphi_{\ell,\ell',T}$, and
- define $\varphi_{\ell,\ell'}$ as the disjunction of all these formulas.

Formula $\varphi_{\ell,\ell'}$ can be computed in time exponential in the size of the timed automaton.

# Back to Model Checking

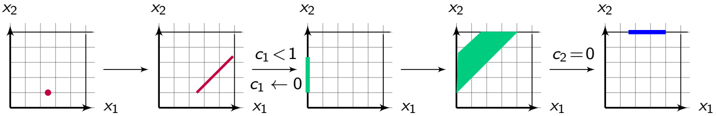
- reachability fragment of parametric TCTL (without until modality)
- we reduce model checking to deciding the truth of formula in a fragment of first-order logic
- the formula contains reachability formulas
- yields a EXPSPACE-decision procedure for model checking

# Model Checking Example

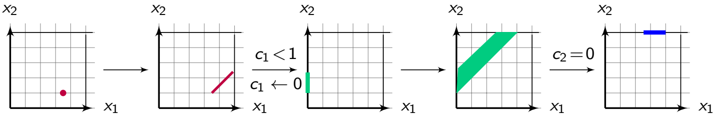



$$\varphi = \exists\theta\, \exists\Diamond_{=\theta} \left(p_1 \wedge \exists\Diamond_{=\theta} p_2\right)$$

$\binom{0.5}{0.2} \models \varphi$




$\binom{0.7}{0.2} \not\models \varphi$



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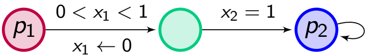


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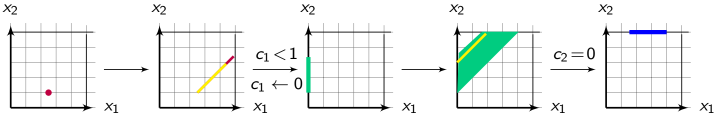


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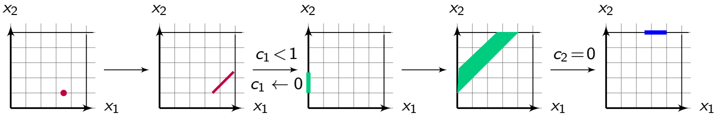

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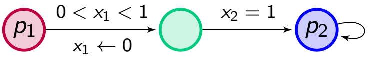


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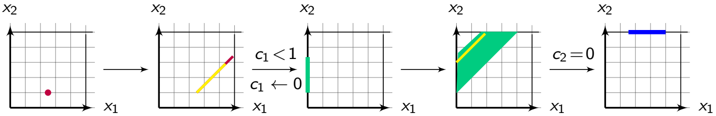


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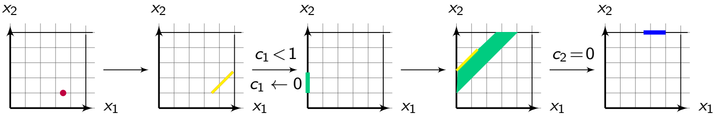

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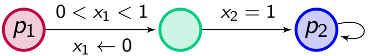


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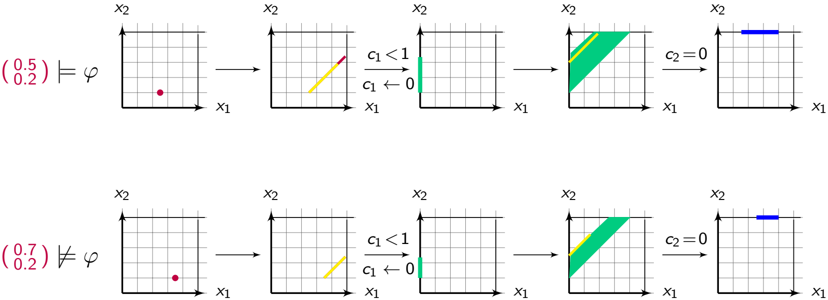


Add a variable $r_3$ measuring the yellow time duration.

# Future Research

- close complexity gap model checking for reachability fragment (NEXPTIME-hard, EXPSPACE-membership)
- model checking full parametric TCTL
- priced timed automata