Universität Leipzig
Institut für Informatik
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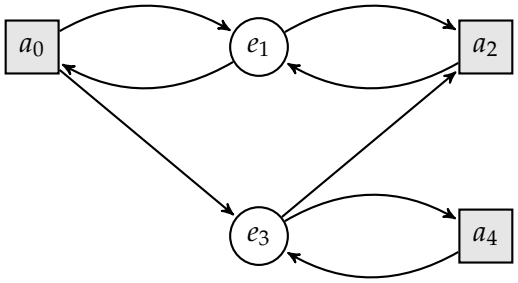
Exercises

Games in Automata and Logic

Solutions for Exercise Sheet 2 (for 26.10.2026)

Exercise 2.1 (Reachability and Safety)

Consider the arena $A = (V, V_A, V_E, E)$ depicted below, where $V_A = \{a_0, a_2, a_4\}$ and $V_E = \{e_1, e_3\}$. Let $T = \{a_0, e_3, a_4\}$.



(a) Give a play from a_0 that is in Safety(A, T). $a_0\,(e_3\,a_4)^\omega$

(b) Give a play from a_0 that is in Reach(A, T) but not in Safety(A, T). $a_0\,e_3\,(a_2\,e_1)^\omega$

(c) Can you give a play from a_0 that is in Safety(A, T) but not in Reach(A, T)? Prove your answer. No this is not possible. A play $v_1 v_2 \dots$ is in Safety(A, T) if $v_i \in T$ for all $i \geq 1$. Hence there also exist $i \geq 1$ such that $v_i \in T$. Hence the play is in Reach(A, T).

(d) Give a play from a_0 that is in Reach$(A, V \setminus T)$. Is this play in Safety(A, T)? Prove your answer. We have $V \setminus T = \{e_1, a_2\}$. A play from a_0 that is in Reach$(A, \{e_1, a_2\})$ is, e.g., $(a_0\,e_1)^\omega$. This play is obviously not in Safety(A, T), because already the second vertex is not in T. Since being in Reach$(A, V \setminus T)$ actually requires to leave T, no play in Reach$(A, V \setminus T)$ can at the same time be in Safety(A, T).

Exercise 2.2

Prove that for every arena $A = (V, V_A, V_E, E)$ and every target set $T \subseteq V$, we have

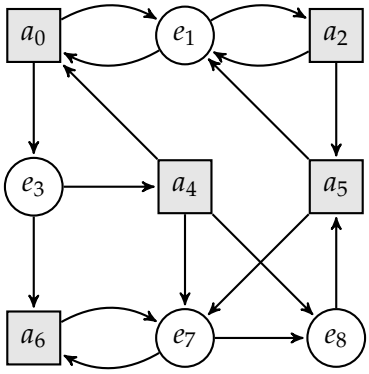
$$\text{Safety}(A, T) = \text{Plays}(A) \setminus \text{Reach}(A, V \setminus T).$$

SOLUTION: Let $\pi = v_1 v_2 \ldots$ be a play in $\text{Plays}(A)$.

$$\begin{aligned}
& \pi \in \text{Safety}(A, T) \\
\iff\ & v_i \in T \text{ for all } i \geq 1 \\
\iff\ & \text{there exists no } i \geq 1 \text{ such that } v_i \in V \setminus T \\
\iff\ & \pi \notin \text{Reach}(A, V \setminus T) \\
\iff\ & \pi \in \text{Plays}(A) \setminus \text{Reach}(A, V \setminus T)
\end{aligned}$$

Exercise 2.3

Consider the following arena from Lecture 2.



Let $T = \{a_4\}$. We play the reachability game $(A, \text{Reach}(A, T))$. We define the following strategy $g : \{e_1, e_3, e_7, e_8\} \to V$ for Eve by

$$g(e_1) = a_2, \quad g(e_3) = a_4, \quad g(e_7) = a_6, \quad g(e_8) = a_5.$$

(a) Is g a memoryless strategy? Yes, because a strategy is memoryless if it only depends on the current vertex.

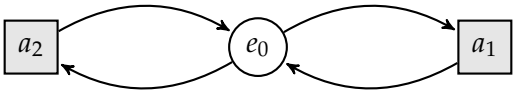
(b) Is g a winning strategy for Eve from e_3? Yes, strategy g is winning for Eve from e_3 if $\text{play}(e_3, f, g)$ is in $\text{Reach}(A, T)$ for *every* Alice-strategy f. This is indeed the case here, because from e_3 it takes exactly one move by Eve to reach a_4 (and this is consistent with her strategy g).

(c) Is g a winning strategy for Eve from e_7? No. Consider any Alice-strategy f that maps a_6 to e_7. Then the play $\text{play}(e_7, f, g)$ is of the form $(e_7\, a_6)^\omega$ and hence not in $\text{Reach}(A, T)$.

(d) Is g a winning strategy for Eve from a_0? No. Consider any Alice-strategy f that maps a_0 and a_2 to e_1. Then the play $\text{play}(a_0, f, g)$ is of the form $a_0\,(e_1\, a_2)^\omega$ and hence not in $\text{Reach}(A, T)$.

Exercise 2.4

Consider the following arena.



Consider the following strategy g for Eve:

- the first time Eve visits e_0, she moves to a_1;
- the second time Eve visits e_0, she moves to a_2;
- the third time Eve visits e_0, she moves to a_1;
- and so on, alternating between a_1 and a_2.

(a) Is g a strategy for Eve? A strategy for Eve is a function $g : \mathrm{Hist}_E(A) \to V$ such that for every history $\pi \in \mathrm{Hist}_E(A)$ we have $(\mathrm{last}(\pi), g(\pi)) \in E$. So let us construct such a function for the strategy described above: $g(\pi \cdot e_0) \mapsto a_1$ if the number of occurrences of e_0 in π is even, and $g(\pi \cdot e_0) \mapsto a_2$ otherwise.

(b) Is g memoryless? Explain your answer. No, because for that, f must only depend on the current vertex e_0. But this is not the case, see next task.

(c) Give two histories π and π' such that $\mathrm{last}(\pi) = \mathrm{last}(\pi') = e_0$, but $g(\pi) \neq g(\pi')$. $\pi = e_0$ with $g(\pi) = a_1$ and $\pi' = e_0 a_1 e_0$ with $g(\pi') = a_2$.

Exercise 2.5

Recall the infinite game with two players Alice and Eve from Lecture 1, where

- Alice picks a bit in $\{0, 1\}$,
- Eve responds with a bit in $\{0, 1\}$, and
- this goes on forever.

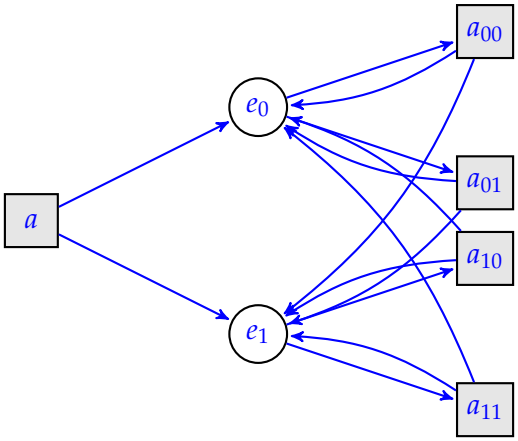
A play is therefore an infinite sequence $(a_1, e_1)(a_2, e_2)(a_3, e_3) \cdots$. Eve wins if, and only if, $e_i = 1 - a_i$ for all $i \geq 1$.

Transform this game into an equivalent game played on a finite arena whose winning condition is a reachability or safety condition. Your construction should:

- Specify the vertices of the new arena and indicate which vertices belong to Alice and which belong to Eve.
- Specify all possible edges and explain how they correspond to the choices made in the original game.
- Give the reachability or safety set.
- Prove that the new game is equivalent to the original game, i.e. that a play is winning for Eve in the original game if and only if the corresponding play in the constructed arena satisfies the new winning condition.

- Briefly explain why your construction preserves the existence of winning strategies for both players.

SOLUTION: We depict the arena $A = (V, V_A, V_E, E)$ with $V_A = \{a, a_{00}, a_{10}, a_{01}, a_{11}\}$ and $V_E = \{e_0, e_1\}$ below. Alice's choice of the first bit is modelled by moving to either e_0 or e_1. Eve's choice of the next bit is modelled by moving to the corresponding a_{ij} vertex; e.g. if Alice's choice is 1 followed by Eve's choice 0, then Alice moves to e_1, and Alice moves to a_{10}. For modelling the winning condition, we can use the safety winning condition with $T = \{a, e_1, e_0, a_{01}, a_{10}\}$ (where the important vertices actually are a_{01} and a_{10}).



Let $\pi = (a_1, e_1)(a_2, e_2)\ldots$ be a play in the original game. A play $\pi' = v_1 v_2 v_3 \ldots$ on A *corresponds* to π iff $v_1 = a$, and

$$v_i = \begin{cases} e_0 & \text{if } a_{i/2} = 0 \\ e_1 & \text{if } a_{i/2} = 1 \end{cases}$$

for all even i, and

$$v_i = \begin{cases} a_{00} & \text{if } a_{(i-1)/2} = 0 \text{ and } e_{(i-1)/2} = 0 \\ a_{01} & \text{if } a_{(i-1)/2} = 0 \text{ and } e_{(i-1)/2} = 1 \\ a_{10} & \text{if } a_{(i-1)/2} = 1 \text{ and } e_{(i-1)/2} = 0 \\ a_{11} & \text{if } a_{(i-1)/2} = 1 \text{ and } e_{(i-1)/2} = 1 \end{cases}$$

for all odd $i \geq 3$. We prove that for π we have $e_i = 1 - a_i$ for all $i \geq 1$ if, and only if, π' is in Safety(A, T).

Suppose $e_i = 1 - a_i$ for all $i \geq 1$. By definition, $v_1 = a \in T$, and $v_i \in \{e_0, e_1\} \subseteq T$ for all even i, and $v_i \in \{e_{01}, e_{10}\} \subseteq T$ for all odd $i \geq 3$. Hence π' is in Safety(A, T).

Suppose $e_i \neq 1 - a_i$ for some $i \geq 1$, that is, $e_i = 0$ and $a_i = 0$, or $e_i = 1$ and $a_i = 1$. Then $v_{2i+1} = e_{00}$ or $v_{2i+1} = e_{10}$, hence Hence π' is not in Safety(A, T).