Universität Leipzig
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Exercises

Games in Automata and Logic

Solutions for Exercise Sheet 1 (for 19.10.2026)

Exercise 1.1

Explain the meaning of the following statements over natural numbers. Prove or disprove whether the statements are true.

(a) $\forall x \exists y\ x < y$ For every number x there exists a number y that is greater than x. This is true. Proof: Let $x \in \mathbb{N}$. Define $y = x + 1$. Then $x < y$.

(b) $\exists y \forall x\ x < y$ There exists some number y such that all numbers x are smaller than y. This is not true; towards contradiction, suppose such a number y exists. Take $x = y + 1$. But $x < y$ does not hold.

(c) $\forall x \forall z \exists y\ x + y > z$ For all numbers x and z there exists number y such that $x + y > z$. This is true. Proof: Let $x, z \in \mathbb{N}$. Define $y = z + 1$. Clearly $x + (z + 1) > z$.

(d) $\exists x \forall y \exists z\ x + y = z$ There exists x such that for all y there exists a number z such that $x + y = z$. This is true. Proof: define $x = 0$. Let $y \in \mathbb{N}$. Define $z = y$. Clearly $0 + y = z$ holds.

Exercise 1.2

We play a game with two players, Alice and Eve. The game consists of one round and the rules are:

- Alice picks a natural number x.
- Eve picks a natural number y.

Eve wins the game if, and only if, $x < y$.

(a) Does Eve have a winning strategy? Yes. She can always simply pick the successor of x.

(b) Does Alice have a winning strategy? No. For every x she picks, Eve can pick the successor so that Eve wins.

(c) Write two quantified formulas in first-order logic over the natural numbers that express:

- Eve has a winning strategy. $\forall x \exists y\ x < y$.
- Alice has a winning strategy. $\exists x \forall y\ y \leq x$.

Exercise 1.3

We play the two-player-game from Lecture 1. The rules of the game are:

- Alice picks a bit in $\{0,1\}$,
- Eve responds with a bit in $\{0,1\}$,
- this goes on forever.

A play is an infinite sequence $(a_1,e_1)(a_2,e_2)\ldots$ over $\{0,1\}^2$. For each of the following winning conditions for Eve, decide whether Eve has a winning strategy. If yes, give a strategy and decide whether it is memoryless and/or finite-memory. If no, explain why not.

(a) There exists $n \geq 1$ such that $e_i = 1$ for all $i \geq n$. Yes, Eve has a winning strategy: he can always pick 1, completely independent from Alice's choice. This is a memoryless strategy.

(b) For all $n \geq 1$ there exists $i \geq n$ such that $e_i = a_i$. Yes, Eve has a winning strategy, she can always pick the same bit as Alice. This is a memoryless strategy.

(c) For all $n \geq 1$ there exists $i \geq n$ such that $e_i = a_i$, but infinitely often $e_i \neq a_i$. Yes, Eve has a winning strategy: if i is even, she picks $e_i = a_i$, if i is odd, then she picks $e_i = 1 - a_i$. This is not memoryless, because she needs some memory from the history, namely the parity of the round number i. For example, $f(h)(a) \neq f(h')(a)$ if $|h|$ is even and $|h'|$ is odd. But note that the strategy is finite-memory: for checking the parity of i, she only needs to remember one bit in a parity counter: for the first round this counter is 1, for the second round it is 0, for the third round it is 1 etc. In particular, the strategy does not need the precise round number i.

(d) For all $i \geq 1$, if i is odd, then $e_i = 0$, and if i is even, then

$$e_i = \begin{cases} 1 & \text{if the number of 1's in } a_1,\ldots,a_i \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$

Yes, Eve has a winning strategy: if i is odd, she picks 0, if i is odd, she additionally checks the parity of the number of 1's in $a_1,\ldots,a_i$. If it is even, she picks 1, otherwise she picks 0. This strategy is not memoryless, but finite memory (two parity bits).

(e) For all $i \geq 1$, $e_i = 1$ if the number of 1's in $a_1,\ldots,a_i$ is equal to the number of 0's in $a_1,\ldots,a_i$; otherwise $e_i = 0$. Yes, Eve has a winning strategy, but it is not even finite-memory. Eve should implement an integer counter that keeps the difference between the number of 1's in $a_1,\ldots,a_i$ and the number of 0's in $a_1,\ldots,a_i$. However, the value of this counter is unbounded.

(f) If Alice plays infinitely many 1's, then Eve must play only finitely many 1's. If Alice plays only finitely many 1's, then Eve plays infinitely many 1's. Eve has no winning strategy. Suppose towards contradiction that Eve has a winning strategy. Consider the following strategy for Alice:

$$a_1 = 0, \quad a_{i+1} = e_i \quad \text{for all } i \geq 1.$$

Let $(a_1, e_1)(a_2, e_2)$... be the play resulting from these two strategies. By definition of Alice's strategy, we have

$$a_1 a_2 a_3 \cdots = 0 e_1 e_2 e_3 \ldots$$

Hence we have

$$a_i = 1 \text{ infinitely often} \iff e_i = 1 \text{ infinitely often}$$

But Eve's winning strategy requires

$$a_i = 1 \text{ infinitely often} \iff e_i = 1 \text{ only finitely often}$$

Hence the play is not winning for Eve, contradicting the assumption that her strategy is winning.