Lecture
Games in Automata and Logic

Winter 2026/2027

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Lecture 4 - Büchi Games

Schedule

	Monday 11-13	Tuesday 15-17
12.10.-18.10.		Lecture 1: Church's Problem
19.10.-25.10.	Exercise 1	Lecture 2: Games on Finite Graphs
26.10.-1.11.	Exercise 2	Lecture 3: Reachability Games
2.11.-8.11.	Exercise 3	Lecture 4: Safety / Büchi Games
9.11.-15.11.	Exercise 4	Lecture 5: Büchi / co-Büchi / generalized Büchi
16.11.-22.11.	No Exercise	No Lecture
23.11.-29.11.	Exercise 5	Lecture 6: Parity Games
30.11.-6.12.	Exercise 6	Lecture 7: Parity Games
7.12.-13.12.	Exercise 7	Lecture 8: S1S
14.12.-20.12.	Exercise 8	Lecture 9:
4.1.-10.1.	Exercise 10	Lecture 11:
11.1.-17.1.	Exercise 11	Lecture 12:
18.1.-24.1.	Exercise 12	Lecture 13:
25.1.-31.1.	Exercise 13 :	Lecture 14:
1.2.-7.2.	Exercise 14	Written Exam ?

Safety Games

Theorem 7

Let $A = (V, V_A, V_E, E)$ and let $T \subseteq V$.

1. The safety game $G = (A, \text{Safety}(A, T))$ is memoryless determined.
2. The winning regions $W_E(G)$ and $W_A(G)$ and the corresponding memoryless strategies can be computed in polynomial time.

Proof.

Let $A' = (V, V'_A, V'_E, E)$, where $V'_A = V_E$ and $V'_E = V_A$, and let $T' = V \setminus T$. Consider the reachability game $G' = (A', \text{Reach}(A', T'))$. Let $W_A(G')$ and $W_E(G')$ be the winning regions of G' (as by Theorem 6), and let f' and g' be the corresponding memoryless strategies.

Let us prove that $W_A(G') \subseteq W_E(G)$. Define $g : V_E \to V$ by $g = f'$. Let $v \in W_A(G')$, let f be an arbitrary strategy for Alice, and let $\pi \in \text{play}(v, f, g)$. By Lemma 5, $\pi \notin \text{Reach}(A', V \setminus T)$, that is, $\pi \in \text{Safety}(A, T)$.

Proof ctd.

Proof.

Let us prove that $W_E(G') \subseteq W_A(G)$. Define $f : V_A \to V$ by $f = g'$. Let $v \in W_E(G')$, let g be an arbitrary strategy for Eve, and let $\pi \in \text{play}(v, f, g)$. By Lemma 4, $\pi \in \text{Reach}(A', V \setminus T)$, that is, $\pi \notin \text{Safety}(A, T)$.

So set $W_E(G) = W_A(G')$ and $W_A(G) = W_E(G')$. By Theorem 6, G is memoryless determined and the two winning regions can be computed in polynomial time.



Recap: $\mathrm{Force}_{\mathsf{E}}(U)$ and $\mathrm{Attr}_{\mathsf{E}}(T)$

Let $U \subseteq V$.

$$\begin{aligned} \mathrm{Force}_{\mathsf{E}}(U) \;:=\; & \{v \in V_{\mathsf{E}} \mid (v,u) \in E \text{ for some } u \in U\} \cup \\ & \{v \in V_{\mathsf{A}} \mid (v,u) \in E \text{ implies } u \in U\} \end{aligned}$$

Let $T \subseteq V$ be a set of target vertices.

$$\begin{aligned} \mathrm{Attr}^0_{\mathsf{E}}(T) &:= T \\ \mathrm{Attr}^{i+1}_{\mathsf{E}}(T) &:= \mathrm{Attr}^i_{\mathsf{E}}(T) \cup \mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}^i_{\mathsf{E}}(T)) \end{aligned}$$

Define

$$\mathrm{Attr}_{\mathsf{E}}(T) := \bigcup_{i \geq 0} \mathrm{Attr}^i_{\mathsf{E}}(T)$$

Analogous definitions for $\mathrm{Force}_{\mathsf{A}}(U)$ and $\mathrm{Attr}_{\mathsf{A}}(T)$

Monotonicity of the Attractor

Lemma 8

For all $S, T \subseteq V$, if $S \subseteq T$ then $\mathrm{Attr}_{\mathrm{E}}(S) \subseteq \mathrm{Attr}_{\mathrm{E}}(T)$.

Proof.

We prove the following claim:

For all $S, T \subseteq V$, if $S \subseteq T$ then $\mathrm{Attr}^i_{\mathrm{E}}(S) \subseteq \mathrm{Attr}^i_{\mathrm{E}}(T)$ for all $i \geq 0$.

Proof by induction on i. For the induction base, we have $\mathrm{Attr}^0_{\mathrm{E}}(S) = S \subseteq T = \mathrm{Attr}^0_{\mathrm{E}}(T)$. So suppose the claim holds for i. We prove it also holds for $i+1$.

$$\begin{aligned} \mathrm{Attr}^{i+1}_{\mathrm{E}}(S) &= \mathrm{Attr}^i_{\mathrm{E}}(S) \cup \mathrm{Force}_{\mathrm{E}}(\mathrm{Attr}^i_{\mathrm{E}}(S)) \\ &\subseteq \mathrm{Attr}^i_{\mathrm{E}}(T) \cup \mathrm{Force}_{\mathrm{E}}(\mathrm{Attr}^i_{\mathrm{E}})(T) \quad \text{(by ind hyp and Lemma} \\ &= \mathrm{Attr}^{i+1}_{\mathrm{E}}(T) \end{aligned}$$

Closure of Attractor under Force

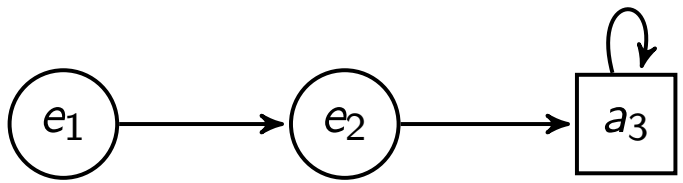
Lemma 9

For all $T \subseteq V$, $\mathrm{Force}_E(\mathrm{Attr}_E(T)) \subseteq \mathrm{Attr}_E(T)$.

Proof.

Let $v \in \mathrm{Force}_E(\mathrm{Attr}_E(T))$. By definition, $v \in \mathrm{Force}_E(\mathrm{Attr}_E^i(T))$ for some $i \geq 0$, hence $v \in \mathrm{Attr}_E^{i+1}(T)$, hence $v \in \mathrm{Attr}_E(T)$. □

However, note that $\mathrm{Attr}_E(T) \subseteq \mathrm{Force}_E(\mathrm{Attr}_E(T))$ does not hold in general: consider the following arena, together with $T = \{e_2\}$.



Then $\mathrm{Attr}_E(T) = \{e_1, e_2\}$, but $\mathrm{Force}_E(\mathrm{Attr}_E(T)) = \{e_1\}$.

Closure of Attractor under Attractor

Lemma 10

For all $T \subseteq V$, $\mathrm{Attr}_{\mathsf{E}}(T) = \mathrm{Attr}_{\mathsf{E}}(\mathrm{Attr}_{\mathsf{E}}(T))$.

Proof.

Exercise 4.3



Pingo

Which of the following is equivalent to

$$\mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}_{\mathsf{E}}(T))$$

A $\mathrm{Force}_{\mathsf{A}}(V \setminus \mathrm{Attr}_{\mathsf{E}}(T))$

B $V \setminus (\mathrm{Force}_{\mathsf{A}}(\mathrm{Attr}_{\mathsf{E}}(T)))$

C $V \setminus (\mathrm{Force}_{\mathsf{A}}(V \setminus \mathrm{Attr}_{\mathsf{E}}(T)))$

(Multiple answers may be possible.)

Pingo Solution

We have $\text{Force}_E(\text{Attr}_E(T)) \neq \text{Force}_A(V \setminus \text{Attr}_E(T))$:

- if $v \in V_A \cap \text{Force}_E(\text{Attr}_E(T))$, then every successor of v is in $\text{Attr}_E(T)$
- if $v \in V_A \cap \text{Force}_A(V \setminus \text{Attr}_E(T))$, then there exists some successor of v that is not in $\text{Attr}_E(T)$

Pingo Solution II

We have $\text{Force}_\text{E}(\text{Attr}_\text{E}(T)) \neq V \setminus (\text{Force}_\text{A}(\text{Attr}_\text{E}(T)))$:

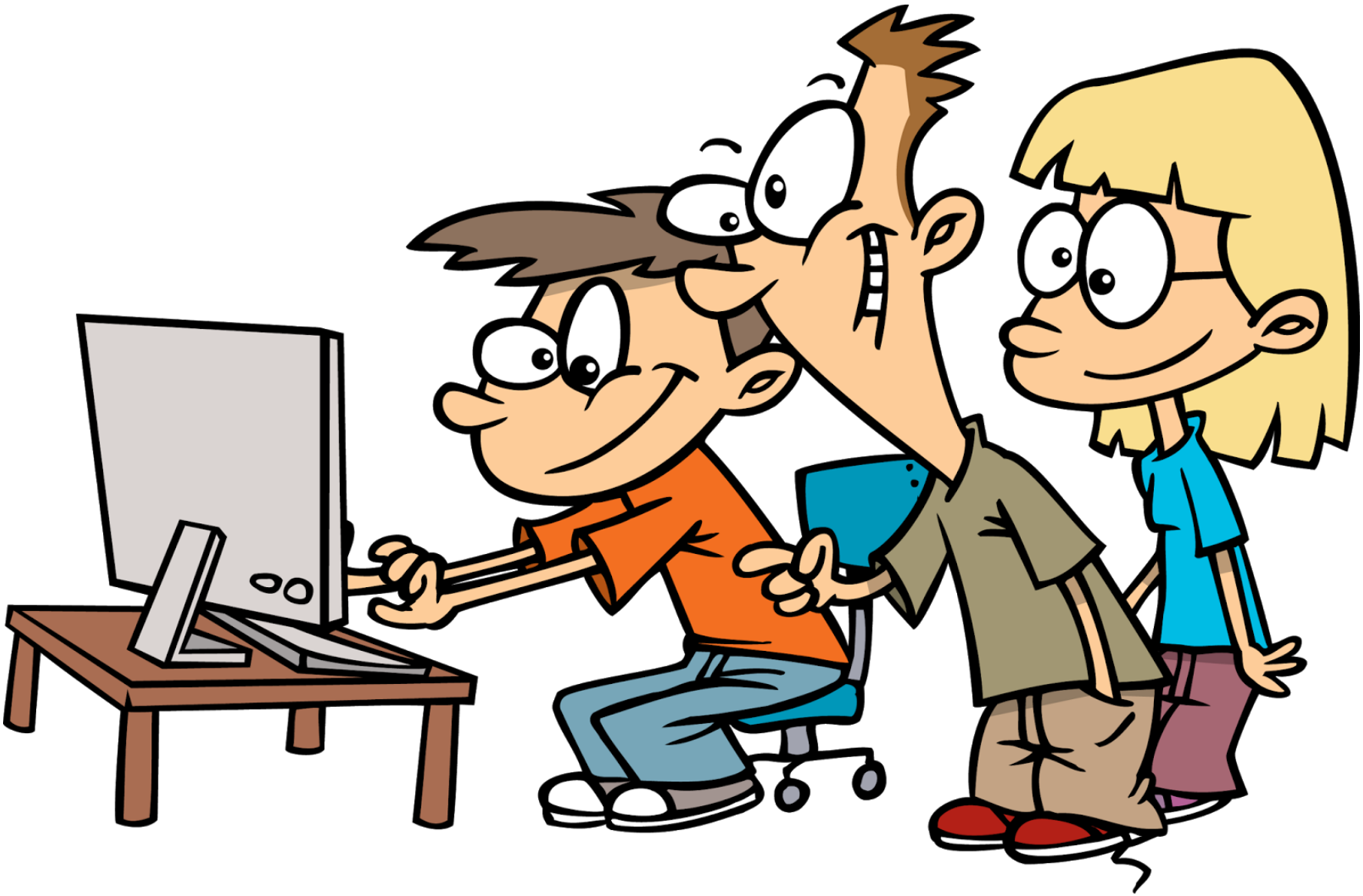
- if $v \in V_\text{A} \cap \text{Force}_\text{E}(\text{Attr}_\text{E}(T))$, then every successor of v is in $\text{Attr}_\text{E}(T)$
- if $v \in V_\text{A} \cap V \setminus (\text{Force}_\text{A}(\text{Attr}_\text{E}(T)))$, then no successor of v is in $\text{Attr}_\text{E}(T)$.

Pingo Solution III

We have $\text{Force}_E(\text{Attr}_E(T)) = V \setminus (\text{Force}_A(V \setminus \text{Attr}_E(T)))$:

- if $v \in V_A \cap \text{Force}_E(\text{Attr}_E(T))$, then every successor of v is in $\text{Attr}_E(T)$
- if $v \in V_A \cap V \setminus (\text{Force}_A(\text{Attr}_E(T)))$, then then there exists no successor that is $V \setminus \text{Attr}_E(T))$, that is, all successors are in $\text{Attr}_E(T)$.
- if $v \in V_E \cap \text{Force}_E(\text{Attr}_E(T))$, then there exists some successor of v is in $\text{Attr}_E(T)$
- if $v \in V_E \cap V \setminus (\text{Force}_A(\text{Attr}_E(T)))$, then there exists some successor of v that is not in $V \setminus \text{Attr}_E(T))$, that is, a successor that is in $\text{Attr}_E(T)$.

Groupwork



Towards Büchi

Let $A = (V, V_A, V_E, E)$ be an arena and let $T \subseteq V$. Let $G = (A, \text{Reach}^2(A, T))$, where $\text{Reach}^2(A, T)$ is defined by

$$\{v_0 v_1 v_2 \cdots \in \text{Plays}(A) \mid \text{there exist } i < j \text{ such that } v_i \in T \text{ and } v_j \in T\}$$

that is, Eve's objective is to visit T at least two times.

Determine the winning region $W_E(G)$!

Solution To Groupwork

- Two perspectives
- Eve's perspective:
  - Eve can enforce reaching T when in $\mathsf{Attr}_\mathsf{E}(T)$
  - Once she has reached T, she wants to enforce visiting $\mathsf{Attr}_\mathsf{E}(T)$ in one step:

    $$T \cap \mathsf{Force}_\mathsf{E}(\mathsf{Attr}_\mathsf{E}(T))$$

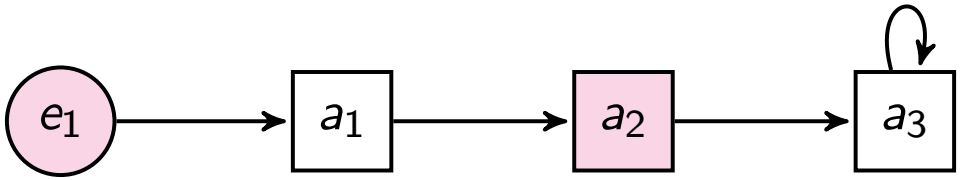
    Quiz: why not $T \cap \mathsf{AttrE}(\mathsf{Attr}_\mathsf{E}(T))$?
  - From which vertices can she enforce to visit this set?

    $$\mathsf{Attr}_\mathsf{E}(T \cap \mathsf{Force}_\mathsf{E}(\mathsf{Attr}_\mathsf{E}(T)))$$

Hence $\mathsf{W}_\mathsf{E}(G) = \mathsf{Attr}_\mathsf{E}(T \cap \mathsf{Force}_\mathsf{E}(\mathsf{Attr}_\mathsf{E}(T)))$.

Solution To Groupwork – Example

Let $A = (V, V_A, V_E, E)$ be the arena depicted below, and let $T = \{e_1, a_2\}$.



- Eve can enforce reaching T when in $\mathrm{Attr}_E(T)$

$$\mathrm{Attr}_E(T) = \{e_1, a_1, a_2\}$$

- Once she has reached T, she wants to enforce visiting $\mathrm{Attr}_E(T)$ in one step:

$$T \cap \mathrm{Force}_E(\mathrm{Attr}_E(T)) = \{e_1\}$$

- From which vertices can she enforce to visit this set?

$$\mathrm{Attr}_E(T \cap \mathrm{Force}_E(\mathrm{Attr}_E(T))) = \{e_1\}$$

Solution To Groupwork II

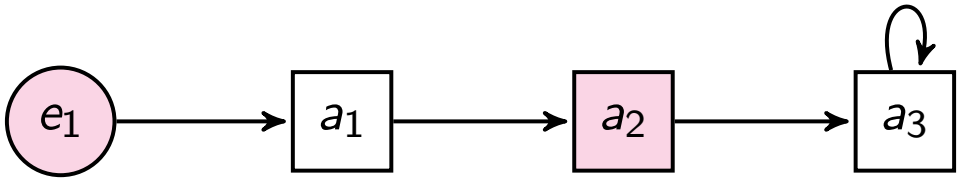
- Two perspectives: Eve or Alice
- Alice's perspective:
  - Eve can enforce reaching T when in $\mathrm{Attr}_{\mathsf{E}}(T)$
  - Alice can avoid visiting T when in $V \setminus \mathrm{Attr}_{\mathsf{E}}(T)$
  - Alice can force the play to visit $V \setminus \mathrm{Attr}_{\mathsf{E}}(T)$ in one step from $\mathrm{Force}_{\mathsf{A}}(V \setminus \mathrm{Attr}_{\mathsf{E}}(T))$
  - Eve can enforce a second visit to T only from those in T that are not in $\mathrm{Force}_{\mathsf{A}}(V \setminus \mathrm{Attr}_{\mathsf{E}}(T))$, hence

$$T \setminus \mathrm{Force}_{\mathsf{A}}(V \setminus \mathrm{Attr}_{\mathsf{E}}(T))$$

Hence $\mathrm{W}_{\mathsf{E}}(G) = \mathrm{Attr}_{\mathsf{E}}(T \setminus \mathrm{Force}_{\mathsf{A}}(V \setminus \mathrm{Attr}_{\mathsf{E}}(T)))$.

Solution To Groupwork II – Example

Let $A = (V, V_A, V_E, E)$ be the arena depicted below, and let $T = \{e_1, a_2\}$.



- Which vertices allow Alice to prevent any future visit to T

$$V \setminus \mathrm{Attr}_E(T) = \{a_3\}$$

- From which vertices can Alice immediately force the play into this set?

$$\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T)) = \{a_2, a_3\}$$

- Hence, from which vertices in T can Eve force a return to T?

$$T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))) = \{e_1\}$$

- And from where can Eve force the first visit to one of these vertices?

$$\mathrm{Attr}_E(T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T)))) = \{e_1\}$$

Questions for Understanding

In the exercise you will prove that the game is memoryless determined and

$$W_E(G) = \mathrm{Attr}_E(T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))))$$

Why do we use

$$\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))$$

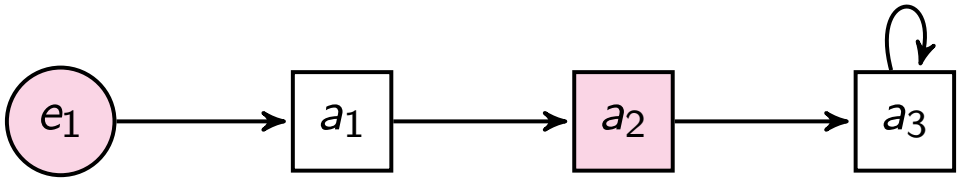
and not

$$\mathrm{Attr}_A(V \setminus \mathrm{Attr}_E(T))?$$

- Force_A...one step vs. Attr_A...many steps
- Counterexample from the slide before, where $\mathrm{Attr}_A(V \setminus \mathrm{Attr}_E(T)) = V$ so that $W_E(G) = \emptyset$
- If Alice needs more than one step to reach the set from which she can avoid $\mathrm{Attr}_E(T)$ forever, Eve may visit T a second time along the way

Solution To Groupwork II – Example

Let $A = (V, V_A, V_E, E)$ be the arena depicted below, and let $T = \{e_1, a_2\}$.



- Which vertices allow Alice to prevent any future visit to T

$$V \setminus \mathrm{Attr}_E(T) = \{a_3\}$$

- From which vertices can Alice immediately force the play into this set?

$$\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T)) = \{a_2, a_3\}$$

- Hence, from which vertices in T can Eve force a return to T?

$$T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))) = \{e_1\}$$

- And from where can Eve force the first visit to one of these vertices?

$$\mathrm{Attr}_E(T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T)))) = \{e_1\}$$

Questions for Understanding II

In the exercise you will prove that the game is memoryless determined and

$$W_E(G) = \mathrm{Attr}_E(T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))))$$

Why do we remove from T and not from $\mathrm{Attr}_E(T)$?

- We want to consider vertices immediately after a visit to T, not in $\mathrm{Attr}_E(T)$
- Counterexample from the slide before, where $\mathrm{Attr}_E(\mathrm{Attr}_E(T) \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T)))) = \emptyset$ so that $W_E(G) = \emptyset$

Questions for Understanding III

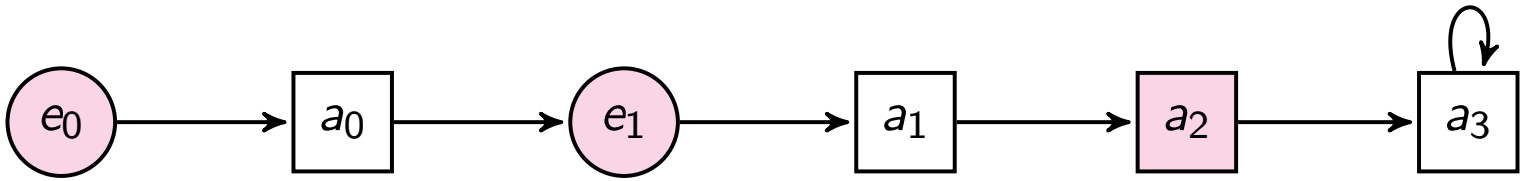
In the exercise you will prove that the game is memoryless determined and

$$W_E(G) = \mathrm{Attr}_E(T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))))$$

How do we need to change the computation of $W_E(G)$ if we change the winning objective of Eve to "visit T at least three times"?

At least three times...

Let $A = (V, V_A, V_E, E)$ be the arena depicted below, and let $T = \{e_0, e_1, a_2\}$.



- $V \setminus \mathrm{Attr}_E(T) = \{a_3\}$ (the vertices from which Alice can avoid T)
- $\mathrm{Force}_A(\{a_3\}) = \{a_2, a_3\}$ (the vertices from which she can force to visit avoiding vertices in one step)
- $T \setminus \{a_2, a_3\} = \{e_0, e_1\}$ (the vertices from which Eve can force to return to T for a second time)
- $V \setminus \mathrm{Attr}_E(\{e_0, e_1\}) = \{a_1, a_2, a_3\}$ (the vertices from which Alice can avoid $\{e_0, e_1\}$)
- $\mathrm{Force}_A(\{a_1, a_2, a_3\}) = \{e_1, a_1, a_2, a_3\}$, and
- $\mathrm{Attr}_E(\{e_0, e_1\} \setminus \{e_1, a_1, a_2, a_3\}) = \{e_0\}$ (the vertices from which Eve can force to return to T for a third time)

Questions for Understanding IV

In the exercise you will prove that the game is memoryless determined and

$$W_E(G) = \mathrm{Attr}_E(T \setminus (\mathrm{Force}_A(V \setminus \mathrm{Attr}_E(T))))$$

How do we need to change the computation of $W_E(G)$ if we change the winning objective of Eve to "visit T at least k times", for $k \in \mathbb{N}$?

How do we need to change the computation of $W_E(G)$ if we change the winning objective of Eve to "visit T infinitely often"?

Büchi Winning Condition

Let $A = (V, V_A, V_E, E)$ be an arena and let $T \subseteq V$.

Given a play π of the form $v_0 v_1 v_2 \ldots$ on A, define

$$\inf(\pi) = \{v \in V \mid v = v_i \text{ for infinitely many } i \geq 0\}$$

to be the set of vertices that occur infinitely often in π.

If Eve's objective is to infinitely visit vertices in T, then the winning condition is Büchi(A, T), defined as

$$\{\pi \in \text{Plays}(A) \mid \inf(\pi) \cap T \neq \emptyset\}$$

- Eve wants the play to recurrently visit vertices in T
- If from a certain point on, the play does not visit vertices in T anymore, she loses (and Alice wins)
- In LTL, the corresponding formula is $\text{GF} \bigvee_{v \in T} v$

Pingo

In a Büchi game, Eve wants to visit T infinitely often.
Which statements are sufficient to guarantee that a given infinite play is winning for Eve?

A The play reaches T at least once.

B For every $n \in \mathbb{N}$, the play visits T at least n times.

C The play remains in T for infinitely many consecutive steps.

D The play reaches T after every move of Eve.

(Multiple answers may be possible.)

Pingo

In a Büchi game, Eve wants to visit T infinitely often.
Which of the following is equivalent to Eve winning the play?

A The play eventually reaches T and stays there forever.

B For every position N in the play, there is a later position $N' > N$ such that the vertex at N' belongs to T.

C There exists $N \in \mathbb{N}$ such that $v_N \in T$ and $v_{N+1} \in T$.

D The play visits every vertex in T infinitely often.

(Multiple answers may be possible.)

Pingo

Eve's objective is to visit vertices in T infinitely often.
What is Alice's objective? Choose one answer.

A Reach T before Eve does.

B Visit T infinitely often.

C Visit T only finitely many times.

D Stay in T once it is reached.

- Alice's winning condition is the complement of Eve's winning condition (hence C is the correct answer)
- Given $A = (V, V_A, V_E, E)$ and $T \subseteq V$, let us define

$$\text{coBüchi}(A, T) = \{\pi \in \text{Plays}(A) \mid \inf(\pi) \cap T = \emptyset\}$$