Lecture
Games in Automata and Logic

Winter 2026/2027

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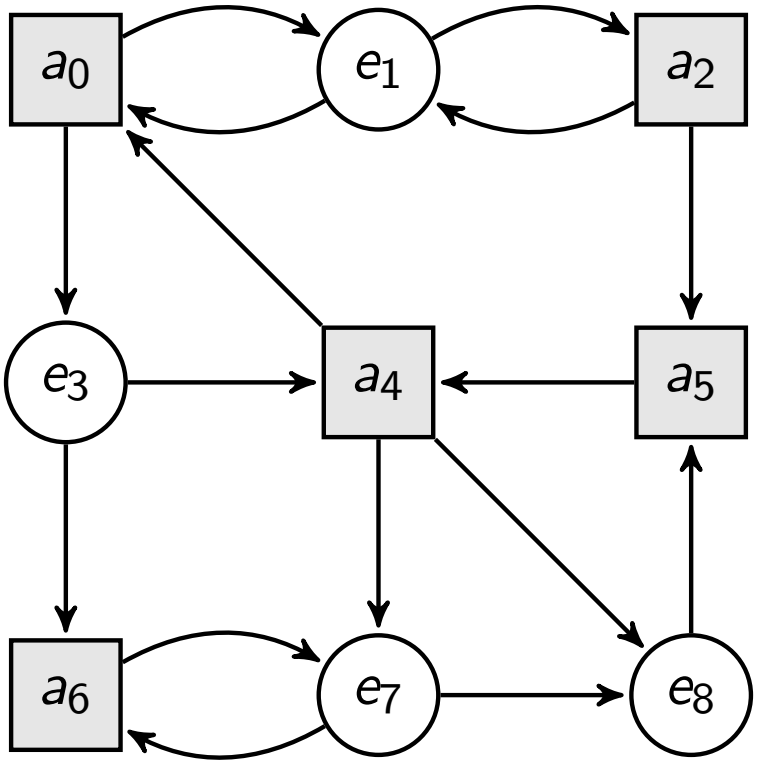
Lecture 3 - Reachabiity Games

Schedule

	Monday 11-13	Tuesday 15-17
12.10.-18.10.		Lecture 1: Church's Problem
19.10.-25.10.	Exercise 1	Lecture 2: Games on Finite Graphs
26.10.-1.11.	Exercise 2	Lecture 3: Reachability Games
2.11.-8.11.	Exercise 3	Lecture 4: Safety / Büchi Games
9.11.-15.11.	Exercise 4	Lecture 5: Büchi / co-Büchi / generalized Büchi
16.11.-22.11.	No Exercise	No Lecture
23.11.-29.11.	Exercise 5	Lecture 6: Parity Games
30.11.-6.12.	Exercise 6	Lecture 7: Parity Games
7.12.-13.12.	Exercise 7	Lecture 8: S1S
14.12.-20.12.	Exercise 8	Lecture 9:
4.1.-10.1.	Exercise 10	Lecture 11:
11.1.-17.1.	Exercise 11	Lecture 12:
18.1.-24.1.	Exercise 12	Lecture 13:
25.1.-31.1.	Exercise 13 :	Lecture 14:
1.2.-7.2.	Exercise 14	Written Exam ?

Example: Reachability Game

Let $T = \{a_4\}$.
Is there a winning strategy for Eve from e_1?



Winning Regions

Let $G = (A, \Omega)$ be a game, where

- $A = (V, V_{\mathsf{A}}, V_{\mathsf{E}}, E)$ is an arena, and
- $\Omega \subseteq \mathrm{Plays}(A)$ is a winning condition.

The winning region of Eve, denoted by $\mathrm{W}_{\mathsf{E}}(G)$, is the set of vertices v such that Eve wins from v.

That is, the winning region of Eve consists of all vertices from which Eve can force the objective, regardless of how Alice plays.

Analogously, we define the winning region of Alice, denoted by $\mathrm{W}_{\mathsf{A}}(G)$,

Winning Regions

The proof of the following is immediate from the definition of winning regions.

Lemma 1

Let $G = (A, \Omega)$ be a game. Then $\mathsf{W}_\mathsf{E}(G) \cap \mathsf{W}_\mathsf{A}(G) = \emptyset$.

Proof.

Towards contradiction, suppose there exists some $v \in \mathsf{W}_\mathsf{E}(G) \cap \mathsf{W}_\mathsf{A}(G)$. By definition, Alice wins from v. That is, Alice has a winning strategy f from v, that is, the play $\mathrm{play}(v, f, g)$ is winning for Alice for every strategy g for Eve $(\star)$. Similarly, Eve wins from v, that is, Eve has a winning strategy g' from v. Consider the game $\pi = \mathrm{play}(v, f, g')$. Then π is winning for Alice by $(\star)$, that is $\pi \notin \Omega$, and π is winning for Eve, that is $\pi \in \Omega$, contradiction. □

Determined Games

A game $G = (A, \Omega)$ is determined if $\mathrm{W_E}(G) \cup \mathrm{W_A}(G) = V$.

That is, for every vertex v, either $v \in \mathrm{W_E}(G)$ or $v \in \mathrm{W_A}(G)$.

Whether a game is determined is **not** immediate and requires a proof, depending on the class of games.

A game $G = (A, \Omega)$ is memoryless determined if it is determined and player i has a memoryless strategy from every vertex $v \in \mathrm{W}_i(G)$ for all $i \in \{\mathrm{A}, \mathrm{E}\}$.

Game Problems

Game Decision Problem

Input: A game $G = (A, \Omega)$ and a vertex v in A.

Question: Does Eve win from v?

Game Computation Problem

Input: A game $G = (A, \Omega)$.

Output: The winning regions $\mathrm{W_E}(G)$ and $\mathrm{W_A}(G)$.

Game Synthesis Problem

Input: A game $G = (A, \Omega)$.

Output: A winning strategy for Eve from her winning region $\mathrm{W_E}(G)$.

Reachability Games: Outline

- We study the computation problem for reachability games, that is,

Reachability Game Computation Problem

Input: A finite arena $A = (V, V_A, V_E, E)$ and $T \subseteq V$.

Output: The winning regions $W_E(G)$ and $W_A(G)$ for the reachability game $G = (A, \text{Reach}(A, T))$.

- We develop an algorithm and prove its correctness and computational complexity
- We prove that reachability games are memoryless determined

Computing $W_E(G)$ - Prep 1

Let $U \subseteq V$.

$$\begin{aligned} \text{Force}_E(U) \quad := \quad & \{v \in V_E \mid (v,u) \in E \text{ for some } u \in U\} \cup \\ & \{v \in V_A \mid (v,u) \in E \text{ implies } u \in U\} \end{aligned}$$

Note the usage of existential and universal quantification: for a vertex v to be in $\text{Force}_E(U)$,

- there only needs to exist (at least) one successor that is in U if $v \in V_E$
- all successors must be in U if $v \in V_A$

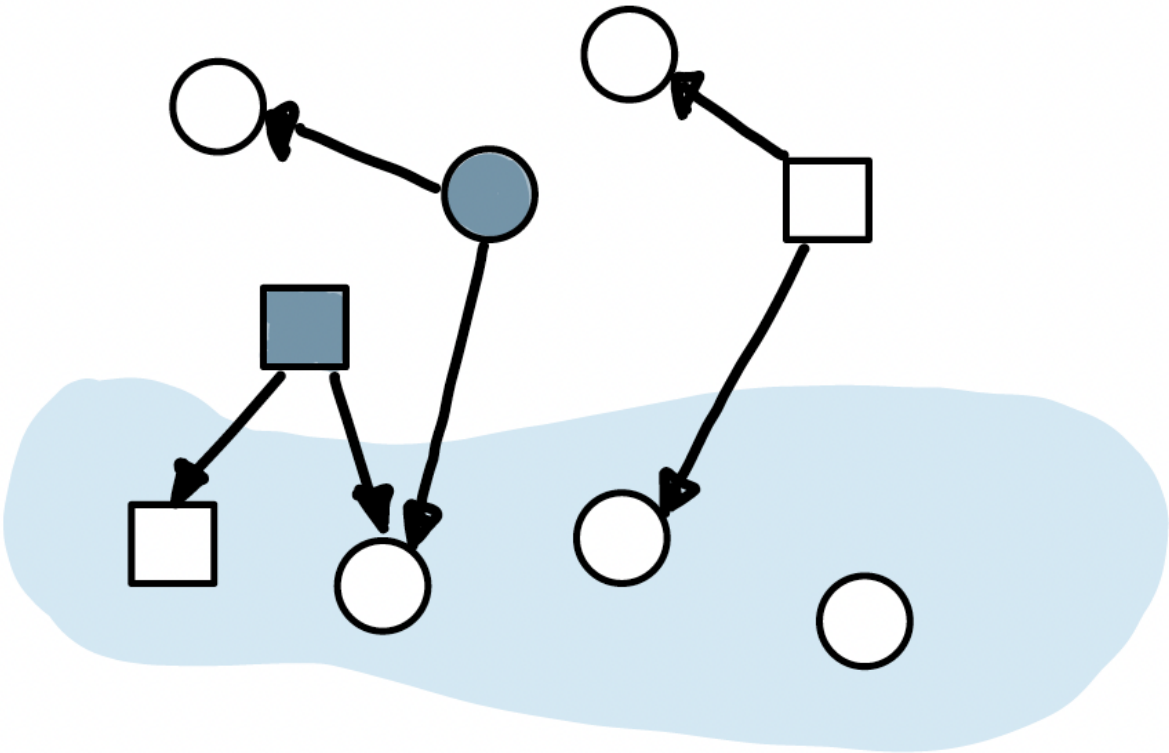
Hence, for every $v \in \text{Force}_E(U)$ ...

- ... if $v \in V_E$, then Eve can pick successor u in U
- ... if $v \in V_A$, Alice cannot but pick successor u in U.

Illustration $\mathrm{Force}_{\mathsf{E}}(U)$

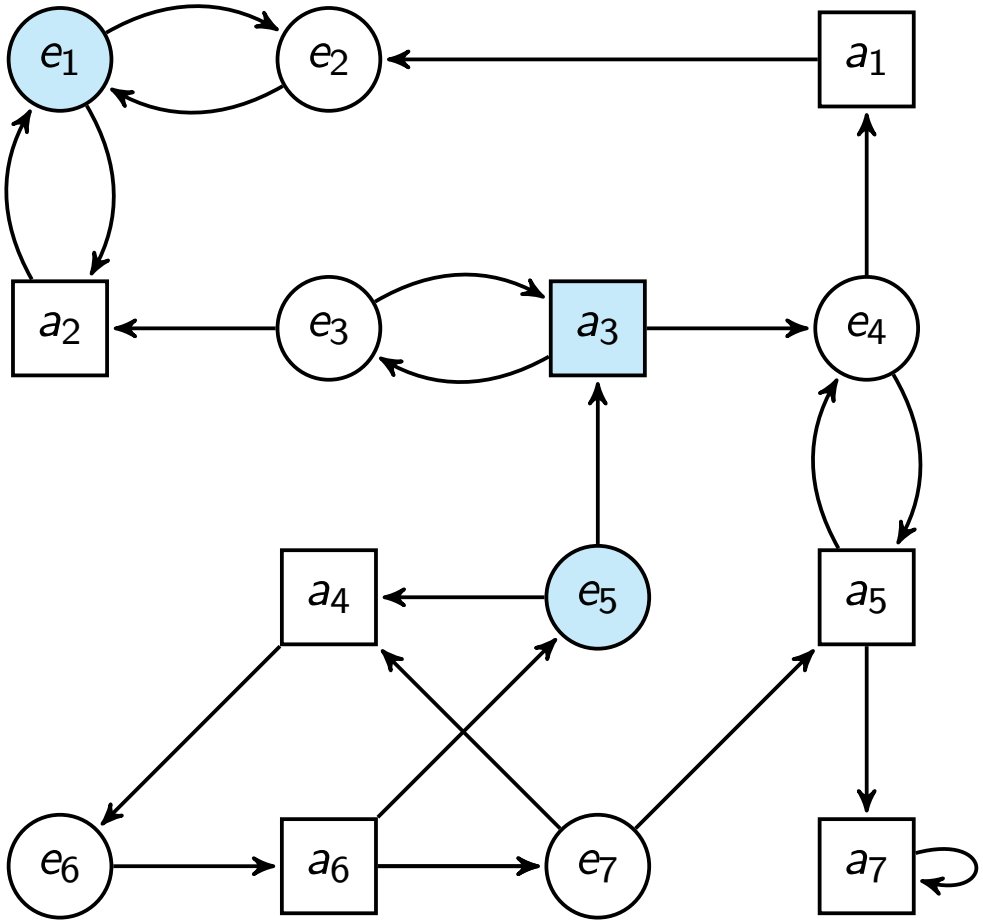
Let $U \subseteq V$.

$$\begin{aligned} \mathrm{Force}_{\mathsf{E}}(U) \quad := \quad & \{v \in V_{\mathsf{E}} \mid (v,u) \in E \text{ for some } u \in U\} \cup \\ & \{v \in V_{\mathsf{A}} \mid (v,u) \in E \text{ implies } u \in U\} \end{aligned}$$



Example $\mathrm{Force}_E(U)$

$\mathrm{Force}_E(\{e_1, e_5, a_3\}) = \{a_2, e_2, e_5, e_3\}$.



Monotonicity of Force_E

Let $U \subseteq V$.

$$\begin{aligned} \text{Force}_E(U) \quad := \quad & \{v \in V_E \mid (v,u) \in E \text{ for some } u \in U\} \cup \\ & \{v \in V_A \mid (v,u) \in E \text{ implies } u \in U\} \end{aligned}$$

Lemma 2 (Monotonicity of Force_E)

For all $S, T \subseteq V$, if $S \subseteq T$, then $\text{Force}_E(S) \subseteq \text{Force}_E(T)$.

Proof.

Let $v \in \text{Force}_E(S)$. We distinguish two cases.

- If $v \in V_E$, then there exists $u \in S$ with $(v,u) \in E$. By $S \subseteq T$, there exists $u \in T$ with $(v,u) \in E$. Hence $v \in \text{Force}_E(T)$.
- If $v \in V_A$, then $u \in S$ whenever $(v,u) \in E$. By $S \subseteq T$, $u \in T$ whenever $(v,u) \in E$. Hence $v \in \text{Force}_E(T)$.

Computing W_E - Prep 2: Attractors for Eve

Let $T \subseteq V$ be a set of target vertices.

$$\begin{aligned} \mathrm{Attr}_E^0(T) &:= T \\ \mathrm{Attr}_E^{i+1}(T) &:= \mathrm{Attr}_E^i(T) \cup \mathrm{Force}_E(\mathrm{Attr}_E^i(T)) \end{aligned}$$

Define

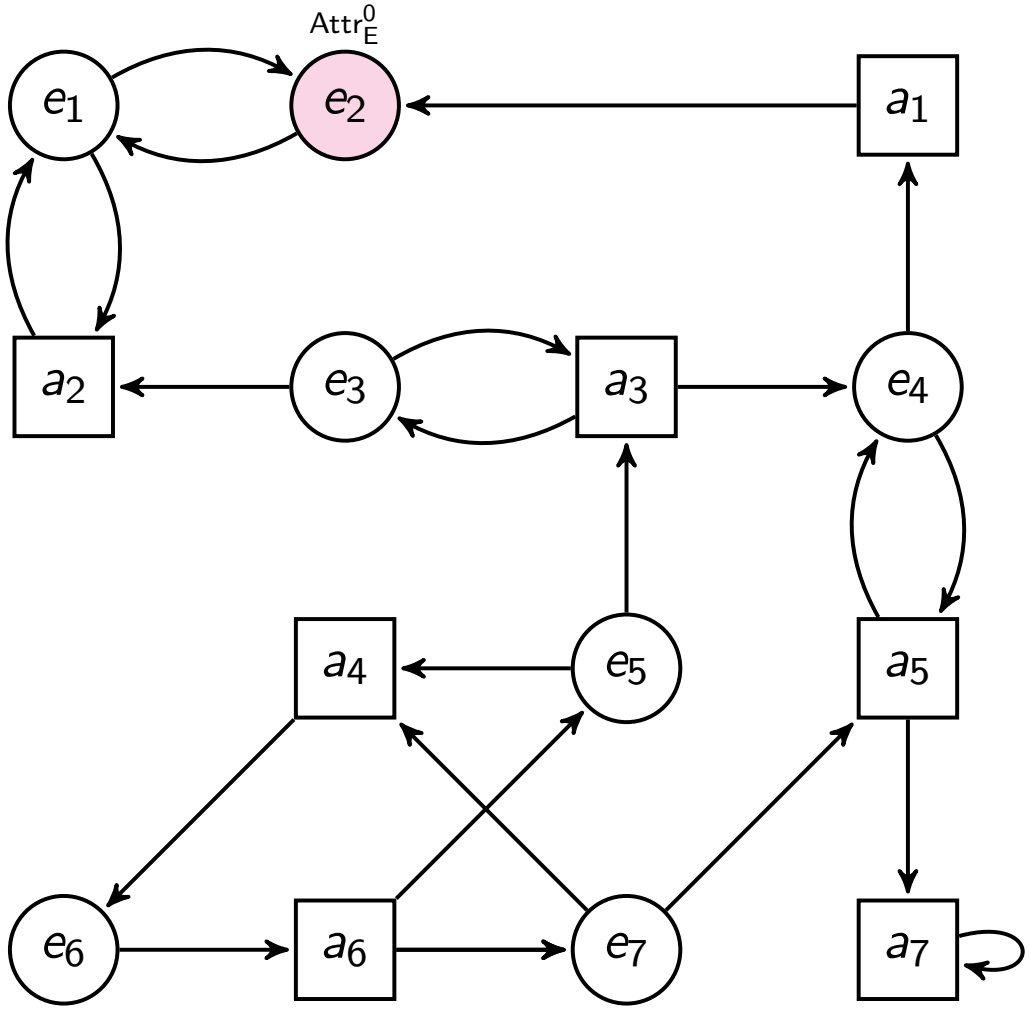
$$\mathrm{Attr}_E(T) := \bigcup_{i \geq 0} \mathrm{Attr}_E^i(T)$$

The attractor $\mathrm{Attr}_E(T)$ of Eve is the set of vertices from which Eve can force the play to be attracted towards T:

- Start with T
- Add every vertex in V_E from which Eve can move into the current set
- Add every vertex in V_A from which Alice must move to the current set
- Repeat until no more vertices can be added

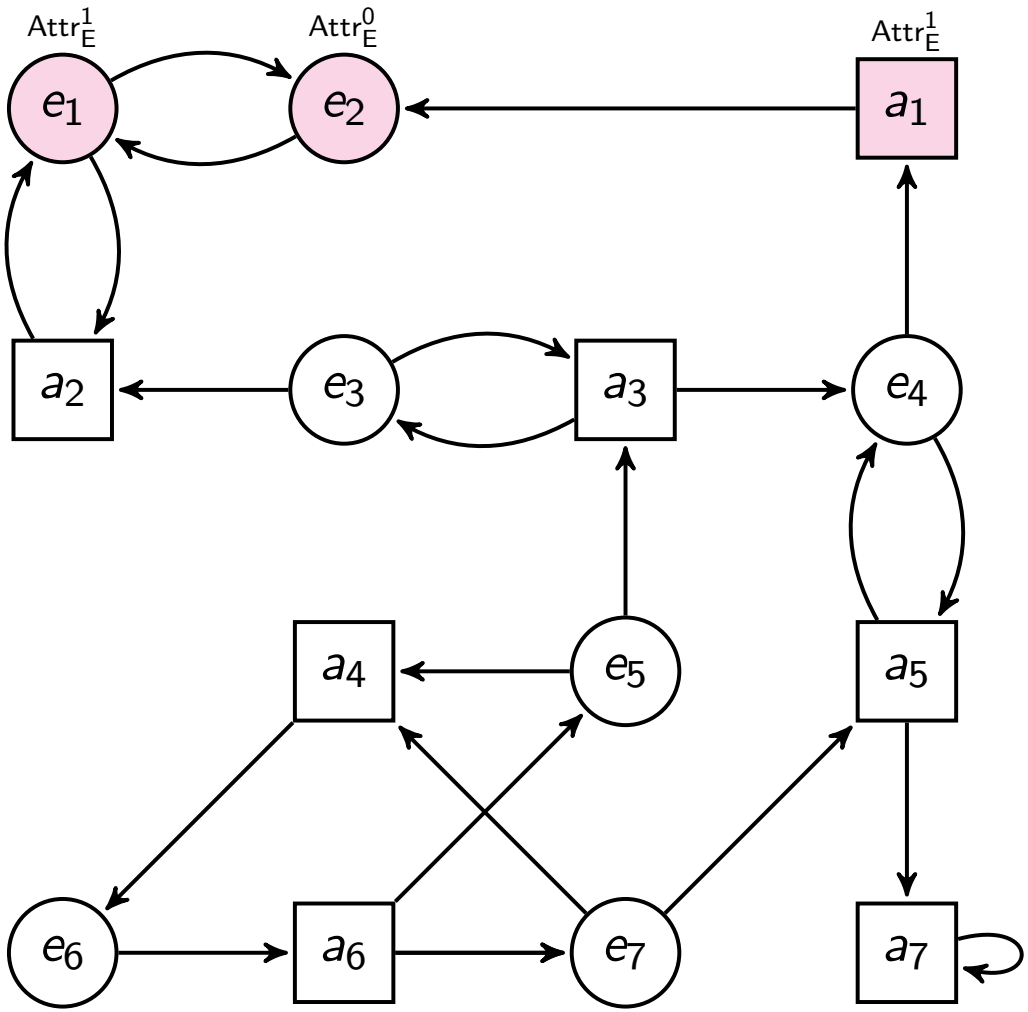
Reachability Game Computation Problem: Attractors

Consider the reachability game $(A, \mathrm{Reach}(A, T))$, where A is depicted below, and $T = \{e_2\}$.



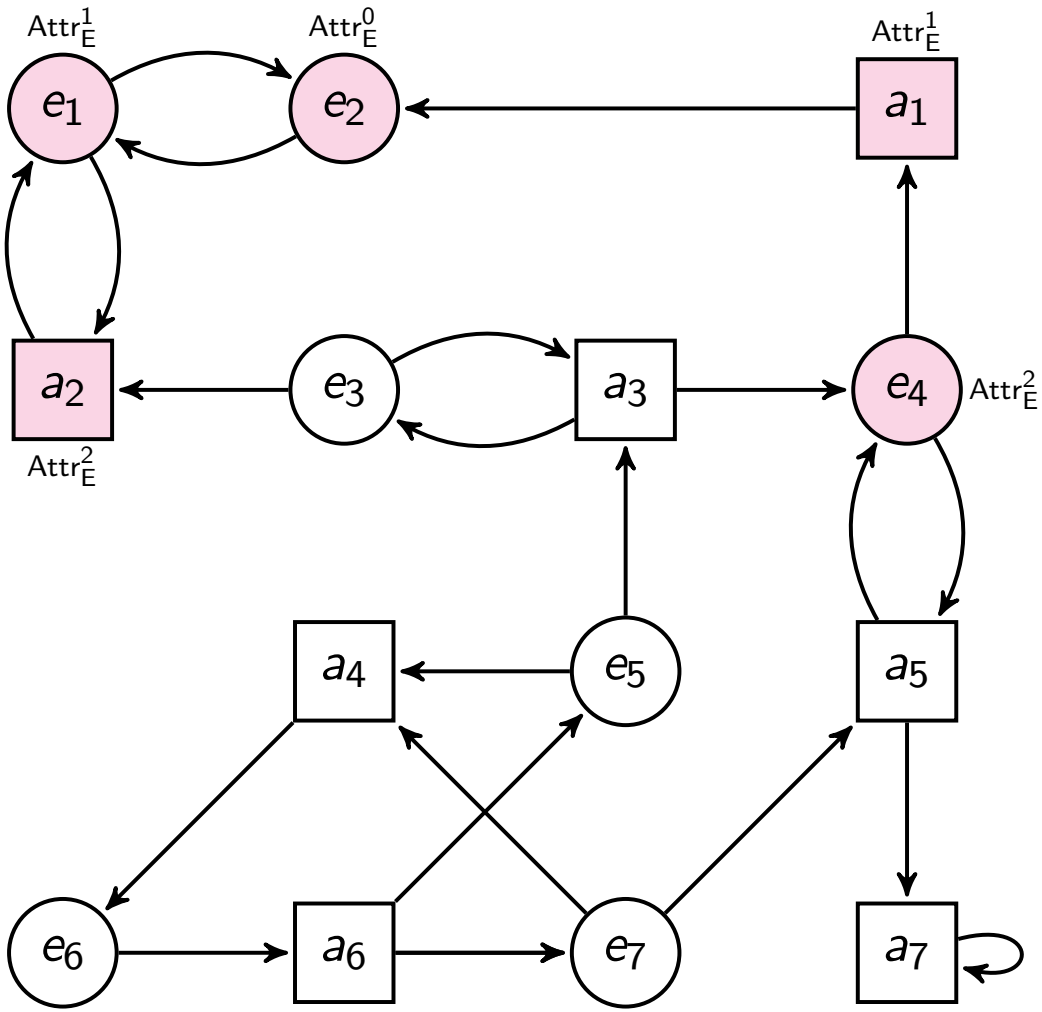
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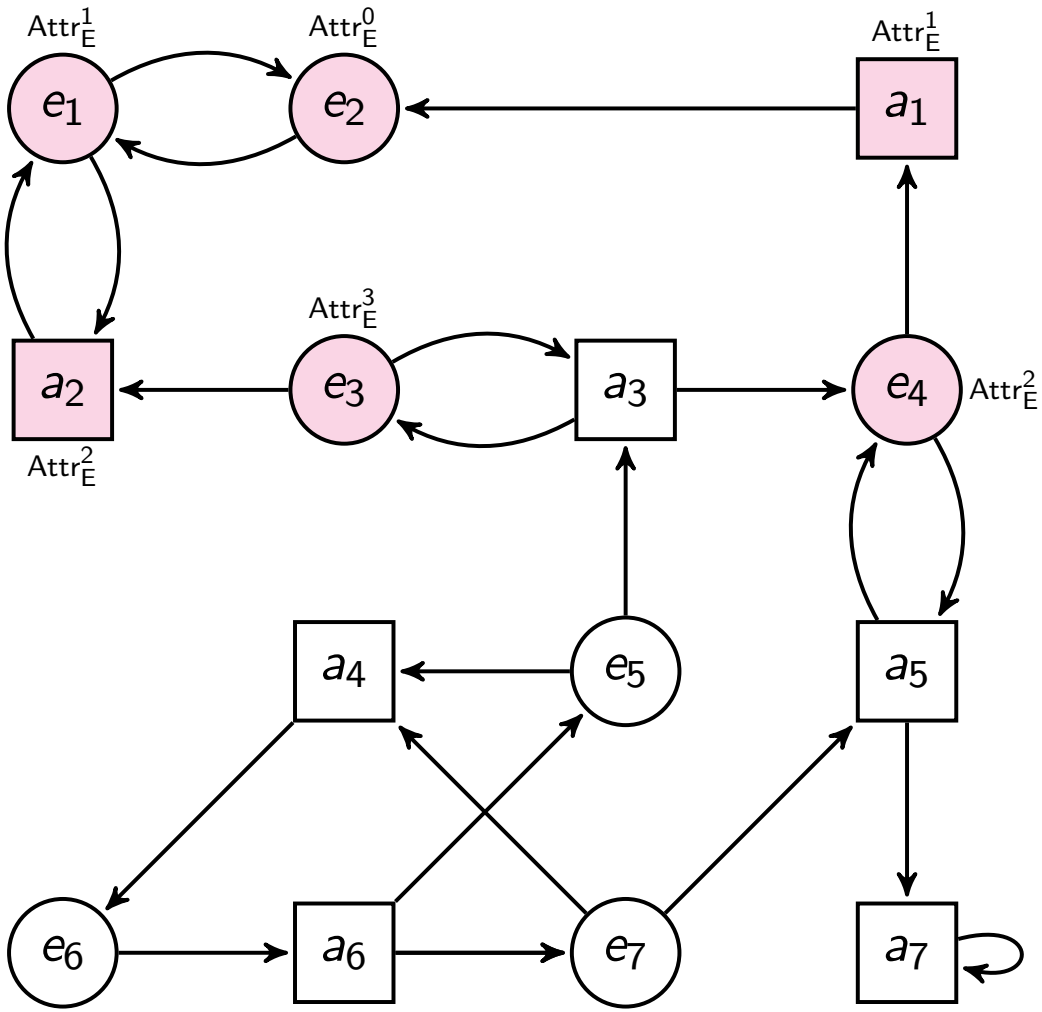
Reachability Game Computation Problem: Attractors

Consider the reachability game $(A, \text{Reach}(A, T))$, where A is depicted below, and $T = \{e_2\}$.



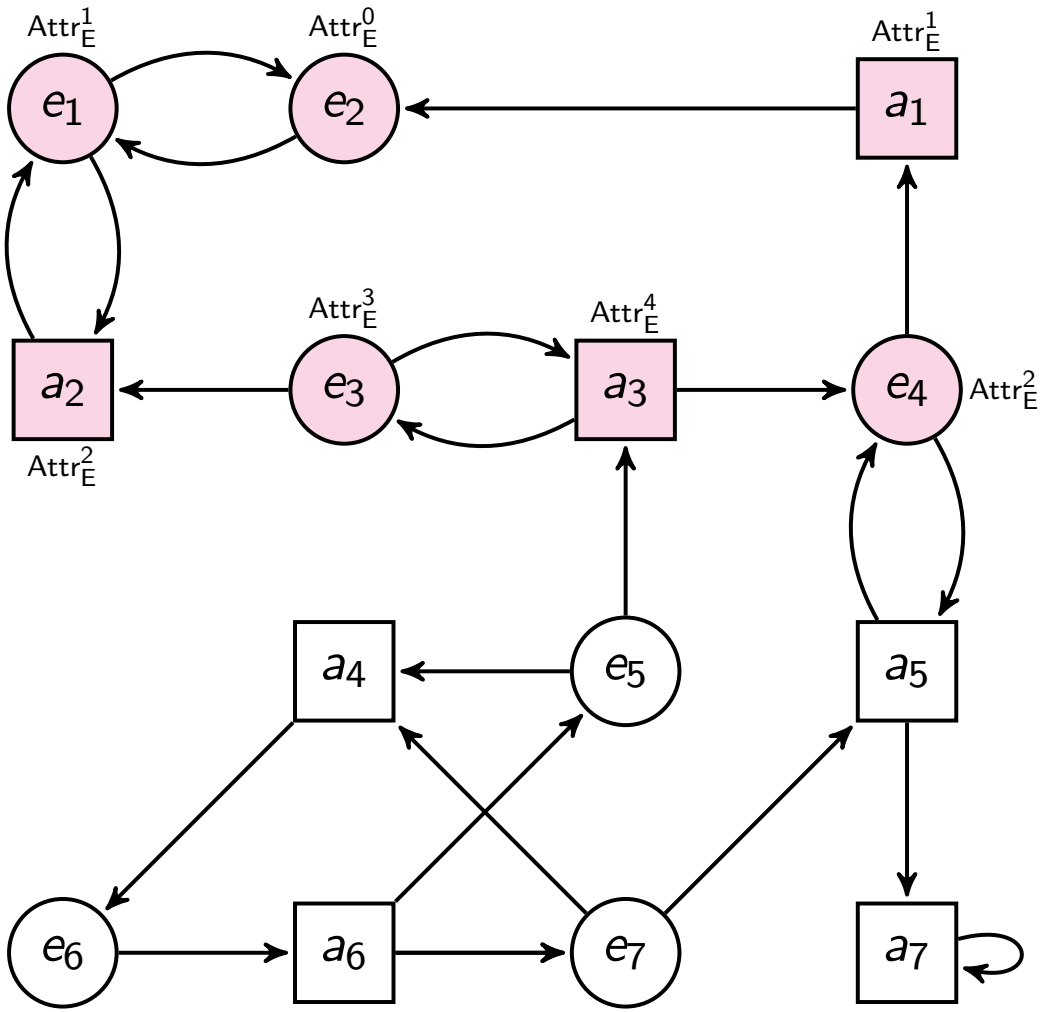
Reachability Game Computation Problem: Attractors

Consider the reachability game $(A, \text{Reach}(A, T))$, where A is depicted below, and $T = \{e_2\}$.



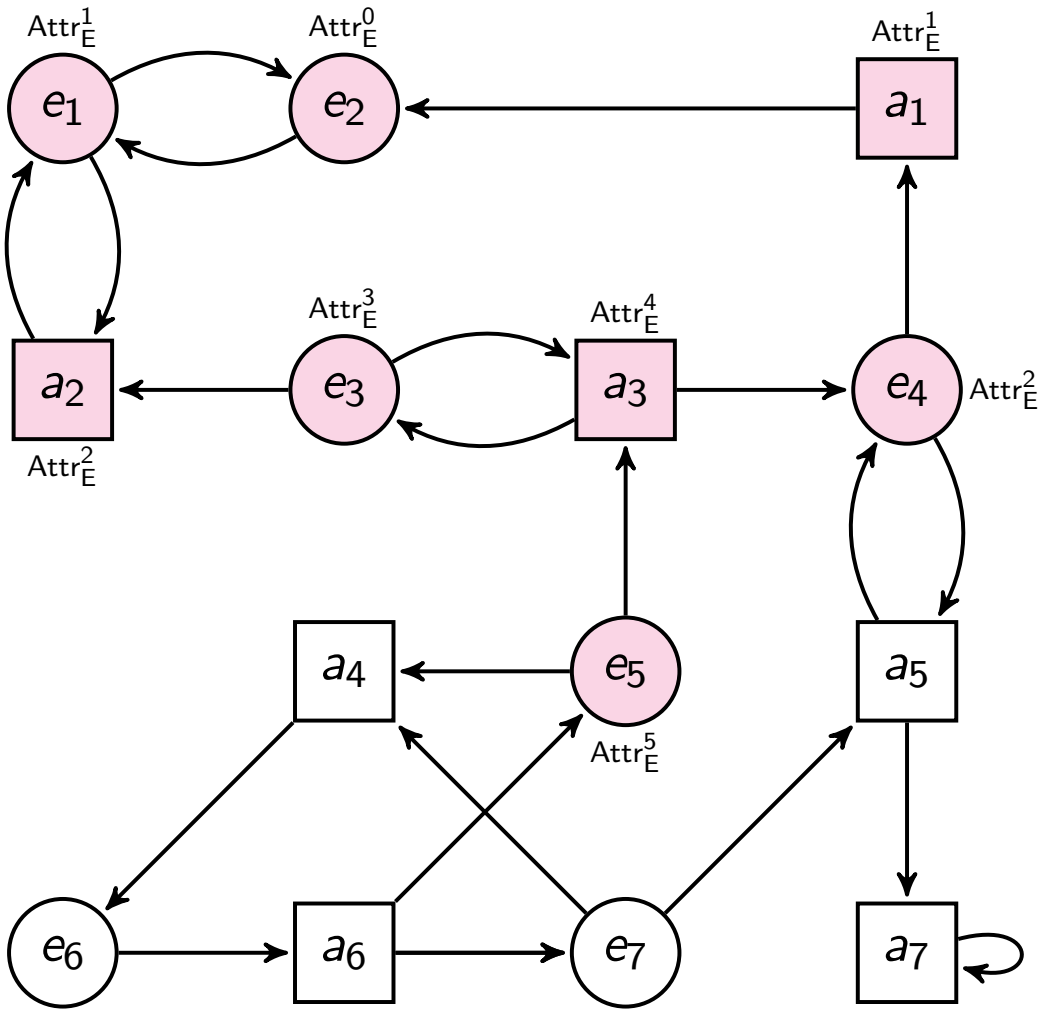
Reachability Game Computation Problem: Attractors

Consider the reachability game $(A, \text{Reach}(A, T))$, where A is depicted below, and $T = \{e_2\}$.



Reachability Game Computation Problem: Attractors

Consider the reachability game $(A, \text{Reach}(A, T))$, where A is depicted below, and $T = \{e_2\}$.



Computing W_E

Lemma 3

$\mathrm{Attr}_E(T) = W_E(G)$

Proof.

- For $\mathrm{Attr}_E(T) \subseteq W_E(G)$ (soundness), we prove that for every vertex $v \in \mathrm{Attr}_E(T)$ there exists a winning strategy for Eve from v. We even prove something stronger: there exists a strategy g for Eve such that g is winning for Eve for every $v \in \mathrm{Attr}_E(T)$, and g is even memoryless (Lemma 4).
- For $W_E(G) \subseteq \mathrm{Attr}_E(T)$ (completeness), we prove the contraposition: for every vertex $v \notin \mathrm{Attr}_E(T)$, there exists no winning strategy for Eve from v. That is, for every strategy g for Eve from v, there exists some strategy f for Alice such that $\mathrm{play}(v, f, g) \notin \mathrm{Reach}(A, T)$. We even prove something stronger: there exists a strategy f for A such that f is winning for Alice for every $v \notin \mathrm{Attr}_E(T)$, and f is even memoryless (Lemma 5).



A Memoryless Strategy Winning for Eve

Lemma 4

There exists a memoryless strategy g for Eve such that g is winning from v for every $v \in \mathrm{Attr}_{\mathsf{E}}(T)$.

Proof.

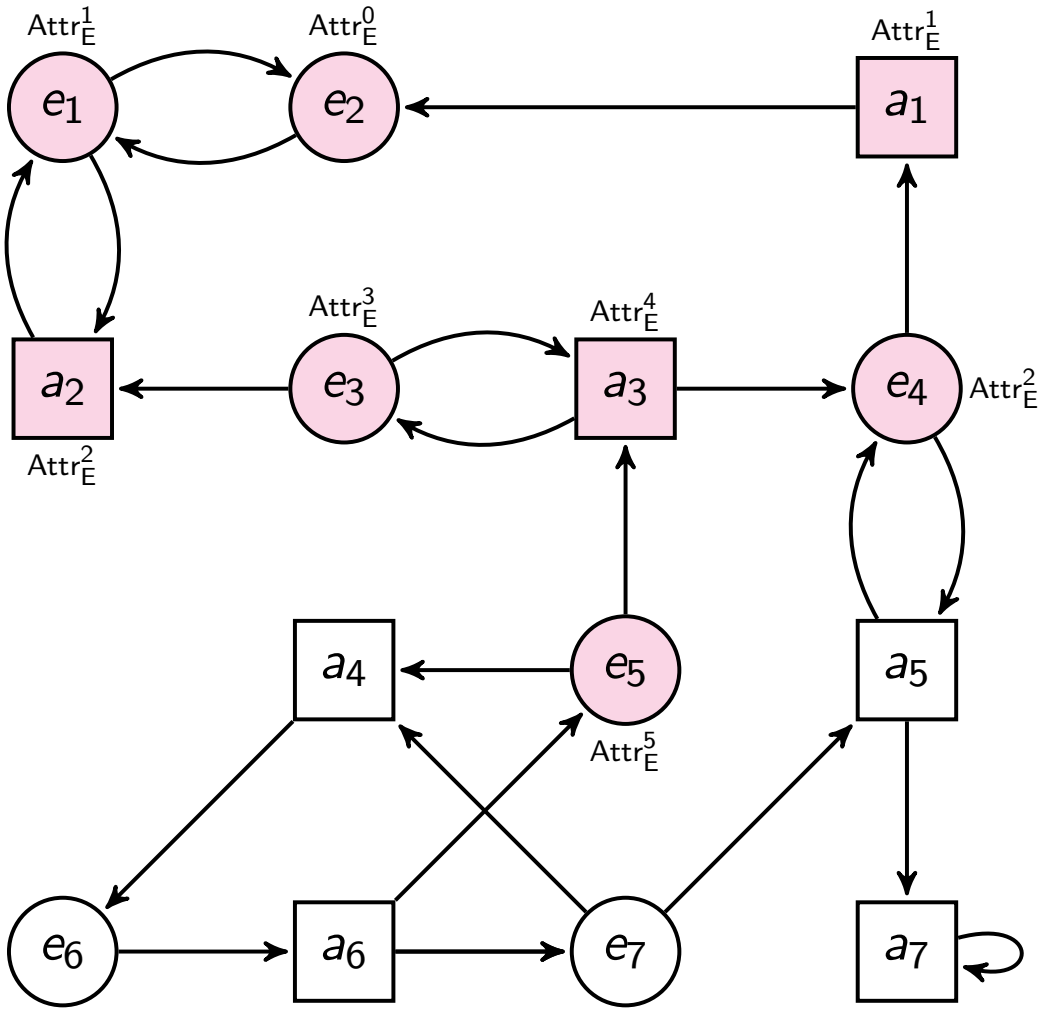
For all $v \in \mathrm{Attr}_{\mathsf{E}}(T)$ define $\mathrm{level}(v) = \min\{i \mid v \in \mathrm{Attr}^i_{\mathsf{E}}(T)\}$. Hence $\mathrm{level}(v) = 0$ implies $v \in T$, and $\mathrm{level}(v) = i > 0$ implies $v \in \mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}^{i-1}_{\mathsf{E}}(T)) \setminus \mathrm{Attr}^{i-1}_{\mathsf{E}}(T)$.

Define $g : V_{\mathsf{E}} \to V$ as follows:

- if $v \in \mathrm{Attr}_{\mathsf{E}}(T)$ and $\mathrm{level}(v) = i > 0$, then set $g(v) = u$ for some $u \in \mathrm{Attr}^{i-1}_{\mathsf{E}}(T)$ (must exist because $v \in V_{\mathsf{E}} \cap \mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}^{i-1}_{\mathsf{E}}(T))$)
- otherwise, set $g(v) = u$ for some u with $(v, u) \in E$ (must exist as E is total)

Reachability Game Computation Problem: Attractors

Let A be the arena depicted below and let $T = \{e_2\}$. Consider the reachability game $(A, \text{Reach}(A, T))$.



- $g : V_E \to V$
- $g(e_1) = e_2$
- $g(e_2) = e_1$
- $g(e_3) = a_2$
- $g(e_4) = a_1$
- $g(e_5) = a_3$
- $g(e_6) = a_6$
- $g(e_7) = a_4$

A Memoryless Strategy Winning for Eve

Lemma 4

There exists a memoryless strategy g for Eve such that g is winning from v for every $v \in \mathrm{Attr}_{\mathsf{E}}(T)$.

Proof.

For all $v \in \mathrm{Attr}_{\mathsf{E}}(T)$ define $\mathrm{level}(v) = \min\{i \mid v \in \mathrm{Attr}^i_{\mathsf{E}}(T)\}$. Hence $\mathrm{level}(v) = 0$ implies $v \in T$, and $\mathrm{level}(v) = i > 0$ implies $v \in \mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}^{i-1}_{\mathsf{E}}(T)) \setminus \mathrm{Attr}^{i-1}_{\mathsf{E}}(T)$.

Define $g : V_{\mathsf{E}} \to V$ as follows:

- if $v \in \mathrm{Attr}_{\mathsf{E}}(T)$ and $\mathrm{level}(v) = i > 0$, then set $g(v) = u$ for some $u \in \mathrm{Attr}^{i-1}_{\mathsf{E}}(T)$ (must exist because $v \in V_{\mathsf{E}} \cap \mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}^{i-1}_{\mathsf{E}}(T))$)
- otherwise, set $g(v) = u$ for some u with $(v, u) \in E$ (must exist as E is total)

Let $v \in \mathrm{Attr}_{\mathsf{E}}(T)$. Let us prove that g is winning for Eve from v.

A Memoryless Strategy Winning for Eve (ctd.)

Proof ctd.

Let $\pi = v_0 v_1 v_2 \dots$ be a play from v that is consistent with g.

We prove: for all $i \geq 0$, if $v_i \in \mathrm{Attr}_{\mathsf{E}}(T)$ and $\mathrm{level}(v_i) > 0$, then $v_{i+1} \in \mathrm{Attr}_{\mathsf{E}}(T)$ and $\mathrm{level}(v_i) > \mathrm{level}(v_{i+1})$.
So let $v_i \in \mathrm{Attr}_{\mathsf{E}}(T)$ and $\mathrm{level}(v_i) = j$ with $j > 0$. Case distinction:

- Suppose $v_i \in V_{\mathsf{E}}$. Since π is consistent with g, we have $v_{i+1} \in \mathrm{Attr}_{\mathsf{E}}^{j-1}(T)$ by definition of g.
- Suppose $v_i \in V_{\mathsf{A}}$. By $v_i \in \mathrm{Force}_{\mathsf{E}}(\mathrm{Attr}_{\mathsf{E}}^{j-1}(T))$, we have $v_{i+1} \in \mathrm{Attr}_{\mathsf{E}}^{j-1}(T)$.

Hence in both cases $\mathrm{level}(v_i) > \mathrm{level}(v_{i-1})$.

Since $v \in \mathrm{Attr}_{\mathsf{E}}(T)$ and $\mathrm{level}(v) = j$ for some $j \in \mathbb{N}$, using the above claim we can infer that there exists some $i \geq 0$ such that $\mathrm{level}(v_i) = 0$ and hence $v_i \in T$. Hence $\pi \in \mathrm{Reach}(A, T)$. Hence g is winning for Eve from v. □

A Memoryless Strategy Winning for Alice

Lemma 5

There exists a memoryless strategy f for Alice such that f is winning from v for every $v \notin \mathrm{Attr}_{\mathsf{E}}(T)$.

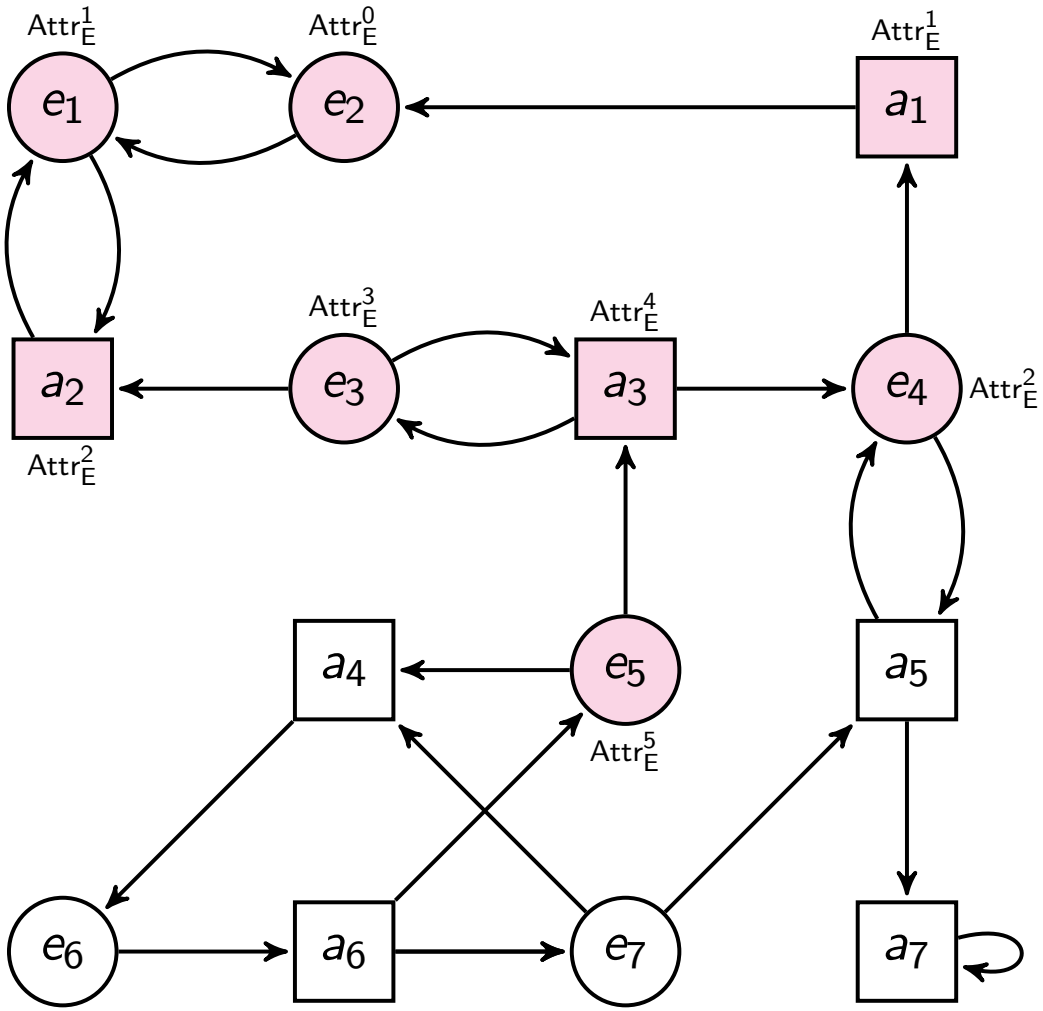
Proof.

Define $f : V_{\mathsf{A}} \to V$ as follows:

- If $v \notin \mathrm{Attr}_{\mathsf{E}}(T)$, then set $f(v) = u$ for some u with $(v, u) \in E$ and $u \notin \mathrm{Attr}_{\mathsf{E}}(T)$ (must exist by definition of $\mathrm{Attr}_{\mathsf{E}}(T)$)
- Otherwise, set $f(v) = u$ for some u with $(v, u) \in E$ (must exist as E is total)

Reachability Game Computation Problem: Attractors

Let A be the arena depicted below and let $T = \{e_2\}$. Consider the reachability game $(A, \text{Reach}(A, T))$.



- $f : V_A \to V$
- $f(a_1) = e_2$
- $f(a_2) = e_1$
- $f(a_3) = e_3$
- $f(a_4) = e_6$
- $f(a_5) = a_7$
- $f(a_6) = e_7$
- $f(a_7) = a_7$

A Memoryless Strategy Winning for Alice

Lemma 5

There exists a memoryless strategy f for Alice such that f is winning from v for every $v \notin \mathrm{Attr}_{\mathsf{E}}(T)$.

Proof.

Define $f : V_{\mathsf{A}} \to V$ as follows:

- If $v \notin \mathrm{Attr}_{\mathsf{E}}(T)$, then set $f(v) = u$ for some u with $(v, u) \in E$ and $u \notin \mathrm{Attr}_{\mathsf{E}}(T)$ (must exist by definition of $\mathrm{Attr}_{\mathsf{E}}(T)$)
- Otherwise, set $f(v) = u$ for some u with $(v, u) \in E$ (must exist as E is total)

Let $v \notin \mathrm{Attr}_{\mathsf{E}}(T)$. Let us prove that f is winning for Alice from v. Let $\pi = v_0 v_1 v_2 \ldots$ be a play from v that is consistent with f.

We prove: for all $i \geq 0$, if $v_i \notin \mathrm{Attr}_{\mathsf{E}}(T)$, then $v_{i+1} \notin \mathrm{Attr}_{\mathsf{E}}(T)$.

A Memoryless Strategy Winning for Alice ctd.

Proof ctd.

Case distinction:

- Suppose $v_i \in V_A$. Since π is consistent with f, we have $v_{i+1} \notin \mathrm{Attr}_E(T)$ by definition of f.
- Suppose $v_i \in V_E$. By $v_i \notin \mathrm{Attr}_E(T)$, there does not exist any $u \in \mathrm{Attr}_E(T)$ with $(v_i, u) \in E$. Hence $v_{i+1} \notin \mathrm{Attr}_E(T)$.

Since $v \notin \mathrm{Attr}_E(T)$, using the above claim we can infer that $v_i \notin \mathrm{Attr}_E(T)$ for all $i \geq 0$. In particular, $v_i \notin T$ for all $i \geq 0$. Hence $\pi \notin \mathrm{Reach}(A, T)$. Hence f is winning for Alice from v. □

Reachability Games

Theorem 6

Let $A = (V, V_A, V_E, E)$ and let $T \subseteq V$.

1. The reachability game $G = (A, \text{Reach}(A, T))$ is memoryless determined.
2. The winning regions $W_E(G)$ and $W_A(G)$ and the corresponding memoryless strategies can be computed in polynomial time.

Proof.

By Lemmas 4 and 5, we get $W_E(G) = \text{Attr}_E(T)$ and $W_A(G) = V \setminus \text{Attr}_E(T)$, hence $W_E(G) \cup W_A(G) = V$. We can compute $\text{Attr}_E(T)$ in time linear in $|V| + |E|$ (see Exercise). □