Lecture
Games in Automata and Logic

Winter 2026/2027

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Lecture 2 - Games on Finite Graphs

Schedule

	Monday 11-13	Tuesday 15-17
12.10.-18.10.		Lecture 1: Church's Problem
19.10.-25.10.	Exercise 1	Lecture 2: Games on Finite Graphs
26.10.-1.11.	Exercise 2	Lecture 3: Reachability Games
2.11.-8.11.	Exercise 3	Lecture 4: Safety / Büchi Games
9.11.-15.11.	Exercise 4	Lecture 5: Büchi / co-Büchi / generalized Büchi
16.11.-22.11.	No Exercise	No Lecture
23.11.-29.11.	Exercise 5	Lecture 6: Parity Games
30.11.-6.12.	Exercise 6	Lecture 7: Parity Games
7.12.-13.12.	Exercise 7	Lecture 8: S1S
14.12.-20.12.	Exercise 8	Lecture 9:
4.1.-10.1.	Exercise 10	Lecture 11:
11.1.-17.1.	Exercise 11	Lecture 12:
18.1.-24.1.	Exercise 12	Lecture 13:
25.1.-31.1.	Exercise 13 :	Lecture 14:
1.2.-7.2.	Exercise 14	Written Exam ?

Two-Player Games On Finite Graphs

- We define games on finite graphs
- A game consists of a finite arena (game graph)
- Two players: Alice and Eve
- Each player has an objective
- For example, Eve's objective may be to reach a certain vertex in the graph, whereas Alice's objective may be to avoid that vertex forever
- Besides the arena, the game is specified by a winning condition representing Eve's objective
- The winning condition for Alice is the complement of the winning condition for Eve

Arenas

An arena is a tuple $A = (V, V_A, V_E, E)$, where

- V is a finite set of vertices,
- V_A and V_E form a partition of V, that is,
  - $V = V_A \cup V_E$,
  - $V_A \cap V_E = \emptyset$, and
  - $V_A \neq \emptyset$ and $V_E \neq \emptyset$.

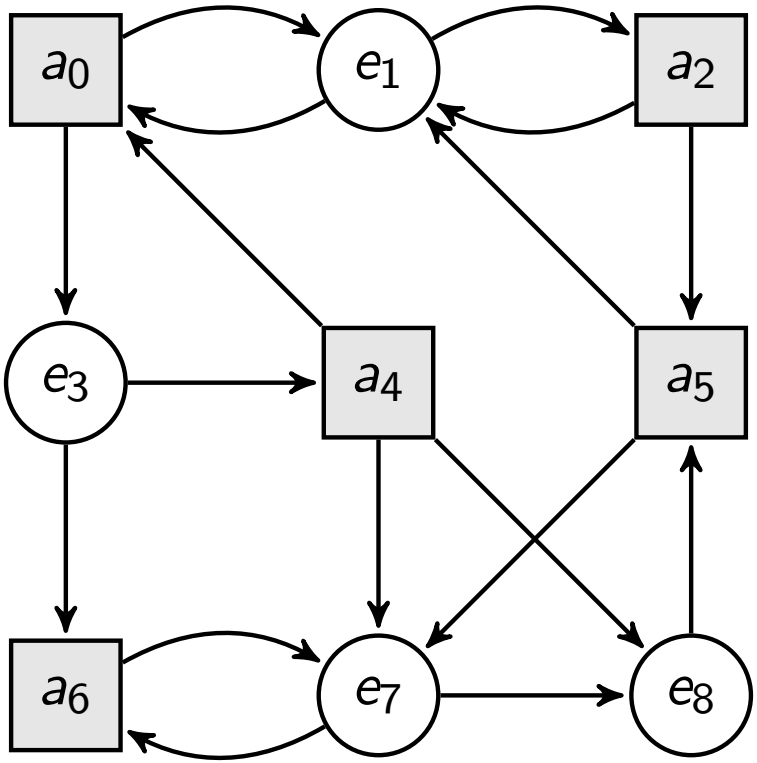
  V_A are Alice's vertices, and V_E are Eve's vertices.
- $E \subseteq (V \times V)$ is a total edge relation, formally: for every $v \in V$ there exists $v' \in V$ such that $(v, v') \in E$.

Given vertices v and v', we say that v' is a successor of v if $(v, v') \in E$.

Note that E does not need to be bipartite.

Example

The following depicts the arena $A = (V, V_A, V_E, E)$, where $V_A = \{a_0, a_2, a_4, a_5, a_6\}$ (rectangular grey vertices), and $V_E = \{e_1, e_3, e_7, e_8\}$ (round vertices).



Plays

Let $A = (V, V_A, V_E, E)$ be an arena.

Informally, a play starts in some initial vertex v.

- If $v \in V_E$, then Eve chooses a vertex v' with $(v, v') \in E$.
- If $v \in V_A$, Alice chooses a vertex v' with $(v, v') \in E$.

Repeating this forever yields an infinite sequence of vertices, the play.

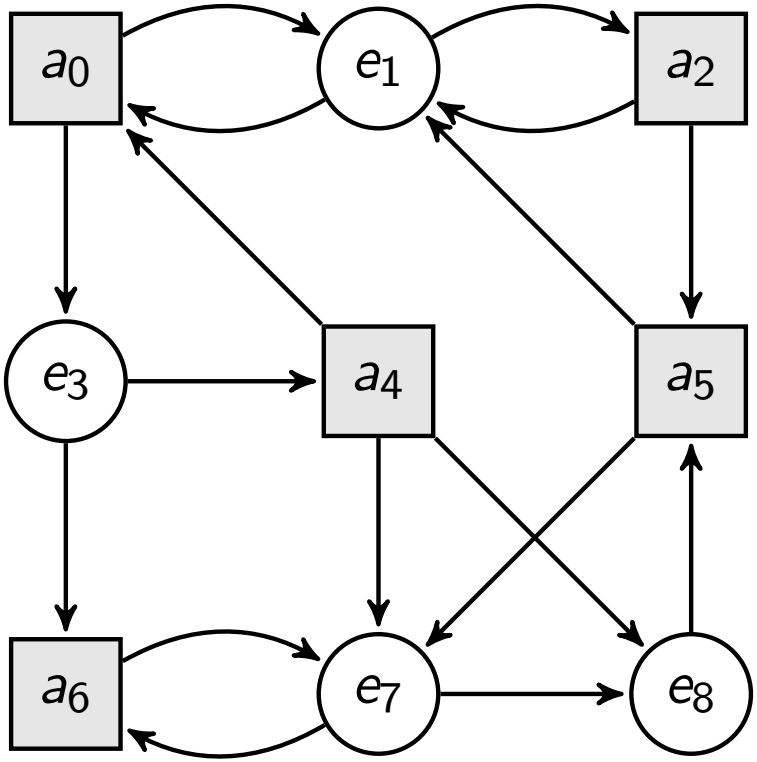
A play on A is an infinite sequence $v_0 v_1 v_2 \ldots$ over V such that $(v_{i-1}, v_i) \in E$ for all $i > 0$.

We use Plays(A) to denote the set of all plays on A.

Given a vertex $v \in V$, a play $v_0 v_1 v_2 \ldots$ starts in v if $v_0 = v$.

Given a set $U \subseteq V$ of vertices, a play $v_0 v_1 v_2 \ldots$ starts in U if there exists $v \in U$ such that $v_0 v_1 v_2 \ldots$ starts in v.

Example



The infinite sequence $e_3 a_4 a_0 e_1 a_0 (e_1 a_2)^\omega$ is a play that starts in e_3.

Objectives and Winning Conditions

An arena $A = (V, V_A, V_E, E)$ specifies **where** the players can move

A winning condition specifies **what** the players try to achieve

Formally, a winning condition is a set $\Omega \subseteq \text{Plays}(A)$ of plays on A (the one's that are according to Eve's objective)

In practice, winning conditions are given by finite representations

A play $\pi \in \text{Plays}(A)$ is winning for Eve if $\pi \in \Omega$. Otherwise, π is winning for Alice.

Reachability Winning Condition

Let $A = (V, V_A, V_E, E)$ be an arena, and let $T \subseteq V$.

If Eve's objective is to eventually reach vertices in T, then the winning condition is Reach(A, T), defined as

$$\{v_0 v_1 v_2 \cdots \in \text{Plays}(A) \mid \text{there exists } i \geq 0 \text{ such that } v_i \in T\}$$

- Eve wants the play to reach a vertex in the target set T at least once
- It does not matter what happens after the first visit to T
- If the play never visits T, then Eve loses (and Alice wins)
- T is a finite representation of this winning condition
- In LTL, the corresponding formula is $\mathsf{F} \bigvee_{v \in T} v$

Pingo

Eve's objective is to eventually reach vertices in T.
What is Alice's objective? Choose one answer.

A Reach T before Eve does.

B Visit T infinitely often.

C Avoid T forever.

D Stay in T once it is reached.

Alice's winning condition is simply the complement of Eve's winning condition (hence C is correct answer).

Hence, not necessary to give the winning condition of Alice

Safety Winning Condition

Let $A = (V, V_A, V_E, E)$ be an arena, and let $T \subseteq V$.

If Eve's objective is to stay in the vertices of T forever, then the winning condition is the safety winning condition Safety(A, T), defined as

$$\{v_0 v_1 v_2 \cdots \in \text{Plays}(A) \mid v_i \in T \text{ for all } i \geq 0\}$$

- T is the set of safe vertices
- Eve wins if the play never leaves T
- If the play visits a vertex outside T, then Eve loses (and Alice wins)
- Again, T is a finite representation of this winning condition
- In LTL, the corresponding formula is $\mathsf{G} \bigvee_{v \in T} v$

Pingo

Eve's objective is to stay in T forever.
What is Alice's objective? Choose one answer.

A Reach T eventually.

B Eventually leave T.

C Stay outside T forever.

D Visit T infinitely often.

Winning Conditions

Reachability and safety are dual winning conditions:

$\text{Safety}(A, T) = \text{Plays}(A) \setminus \text{Reach}(A, V \setminus T)$

Other winning conditions will be introduced later:

- Büchi
- co-Büchi
- Parity
- Streett

Games

A game is a pair (A, Ω), where

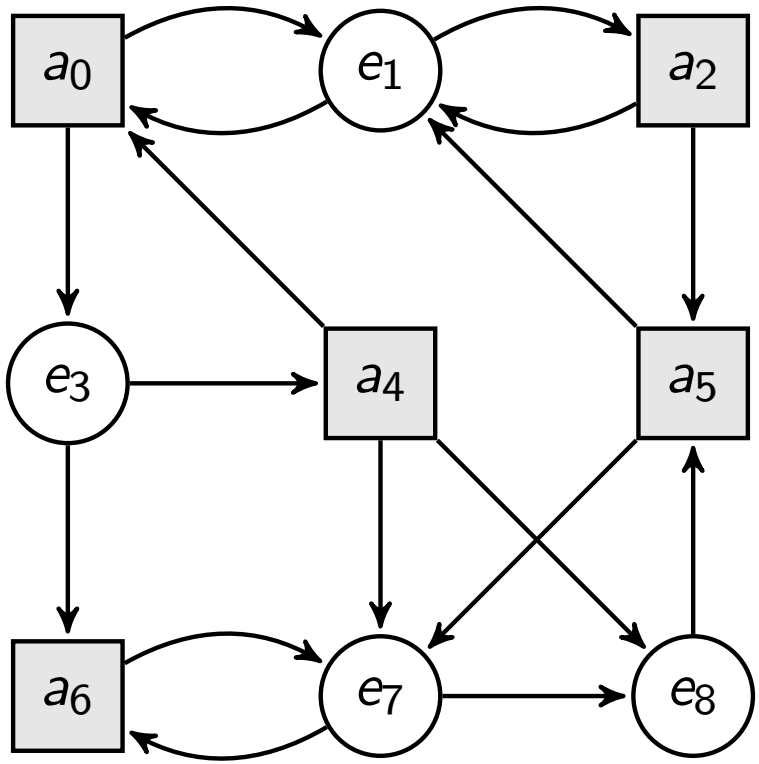
- $A = (V, V_{\mathrm{A}}, V_{\mathrm{E}}, E)$ is an arena, and
- $\Omega \subseteq \mathrm{Plays}(A)$ is a winning condition.

For example, let $A = (V, V_{\mathrm{A}}, V_{\mathrm{E}}, E)$ be an arena and let $T \subseteq V$. Then A and T define

- the reachability game $(A, \mathrm{Reach}(A, T))$
- the safety game $(A, \mathrm{Safety}(A, T))$

Example: Reachability Game

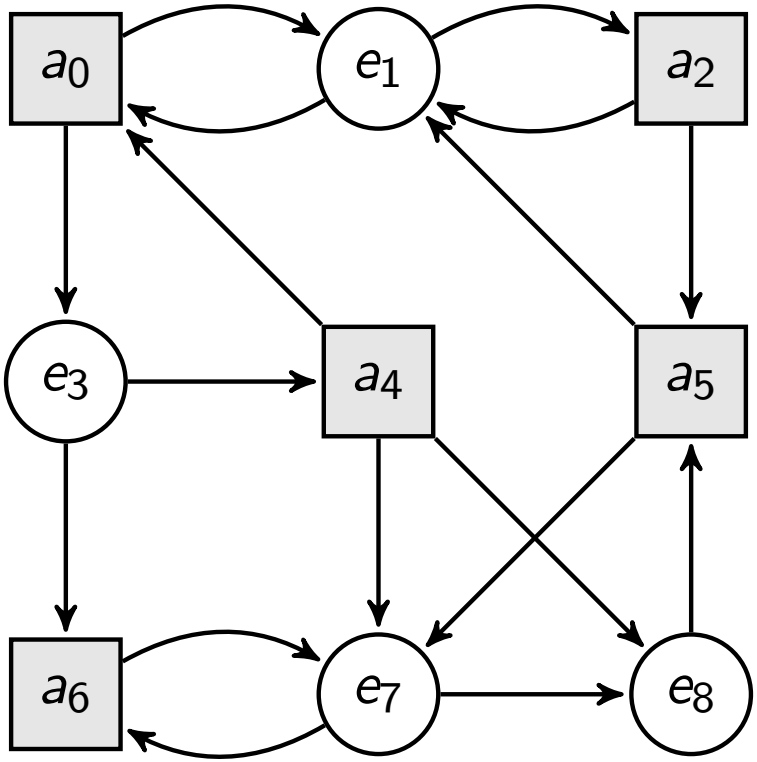
Let $A = (V, V_A, V_E, E)$ be the arena depicted below and let $T = \{a_5\}$.



The play $a_6 e_7 e_8 a_5 (e_1 a_2)^\omega$ is a play in $\mathrm{Reach}(A, T)$.
Hence this play is winning for Eve.

Quiz: Reachability Game

Let $A = (V, V_A, V_E, E)$ be the arena depicted below and let $T = \{a_4\}$.



- Is the play $e_1 a_0 e_3 a_4 (e_7 a_6)^\omega$ winning for Eve?
- What would you do if you were Alice?

Prep for Strategies: Histories

Let $A = (V, V_A, V_E, E)$.

A history on A is a finite nonempty sequence $v_0\ v_1 \dots v_n$ over V such that $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq n$.

We use $\mathrm{Hist}(A)$ to denote the set of all histories on A.

Given a history $\pi = v_0 v_1 \dots v_n$, we use $\mathrm{last}(\pi)$ to denote the last vertex v_n.

We use $\mathrm{Hist}_A(A)$ to denote the set of all histories $\pi \in \mathrm{Hist}(A)$ with $\mathrm{last}(\pi) \in V_A$.

Similarly, we $\mathrm{Hist}_E(A)$ to denote the set of all histories $\pi \in \mathrm{Hist}(A)$ with $\mathrm{last}(\pi) \in V_E$.

(Hence $\mathrm{Hist}_E(A) \cap \mathrm{Hist}_A(A) = \emptyset$ and $\mathrm{Hist}_E(A) \cup \mathrm{Hist}_A(A) = \mathrm{Hist}(A)$.)

Strategies

Informally, a strategy is a function that tells a player which move to make so as to achieve the player's objective, regardless of the opponent's choices.

Let $A = (V, V_{\mathsf{A}}, V_{\mathsf{E}}, E)$.

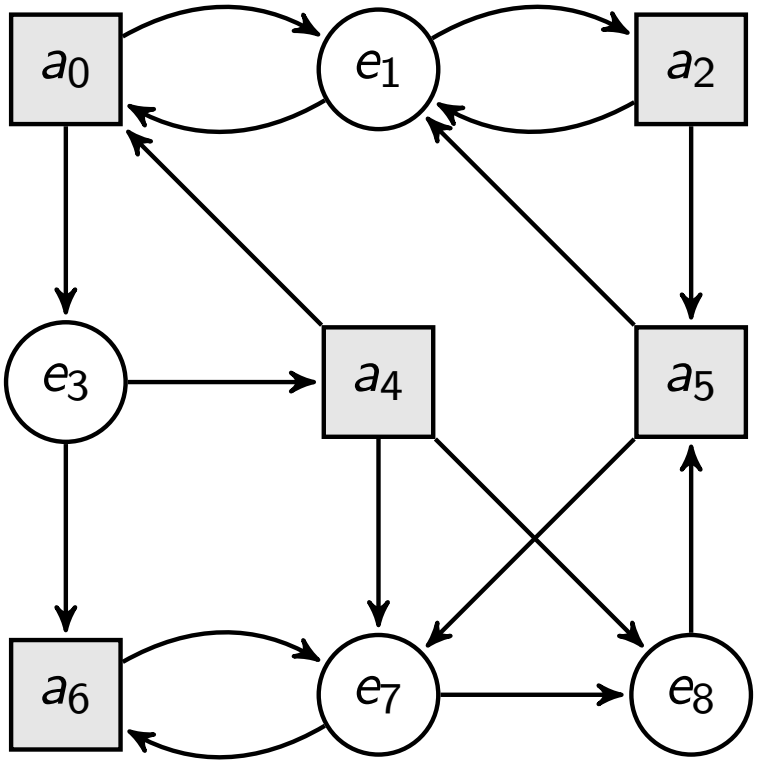
Let $i \in \{\mathsf{A}, \mathsf{E}\}$.

A strategy for player i is a total function $f : \mathrm{Hist}_i(A) \to V$ such that for every $\pi \in \mathrm{Hist}_i(A)$, we have $(\mathrm{last}(\pi), f(\pi)) \in E$.

In general, we are interested in strategies that can be finitely represented.

Example: Strategies

Let $A = (V, V_A, V_E, E)$ be the arena depicted below and let $T = \{a_4\}$.



A strategy f for Alice may define:

- $f(e_1 a_0) = e_3$
- $f(u\, a_0) = e_1$ for all u with $|u| \geq 2$

Memoryless Strategies

Let $A = (V, V_A, V_E, E)$.

A strategy $f : \mathrm{Hist}_i(A) \to V$ is memoryless (positional), if f only depends on the current vertex v, that is, if for every $v \in V_i$, $f(uv) = f(v)$, for all $u \in V^*$.

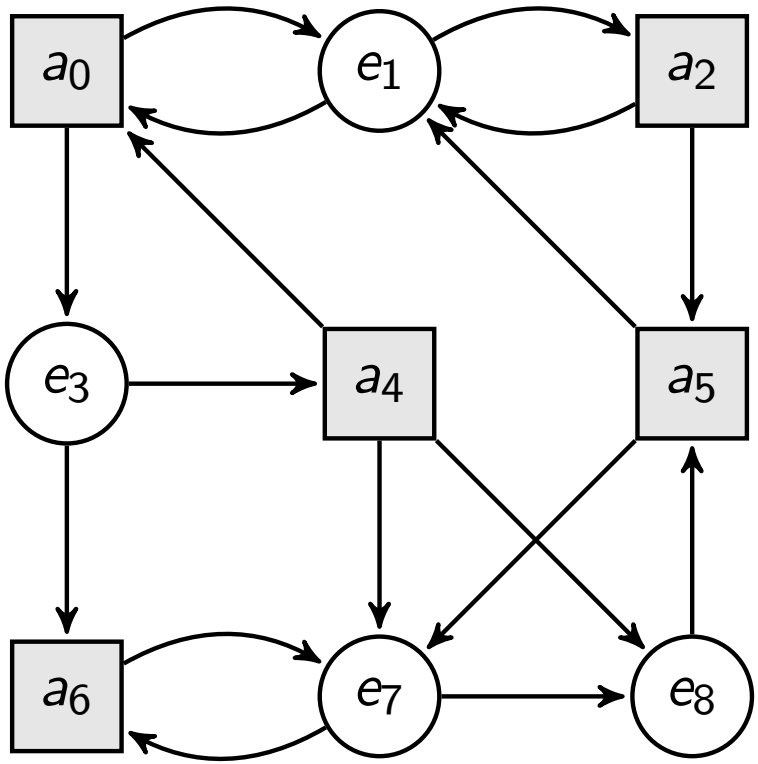
Hence, every memoryless strategy induces a unique function

$$f : V_i \to V$$

satisfying $(v, f(v)) \in E$ for every $v \in V_i$. By abuse of notation, we identify a memoryless strategy with this function.

Example: Memoryless Strategy in Reachability Game

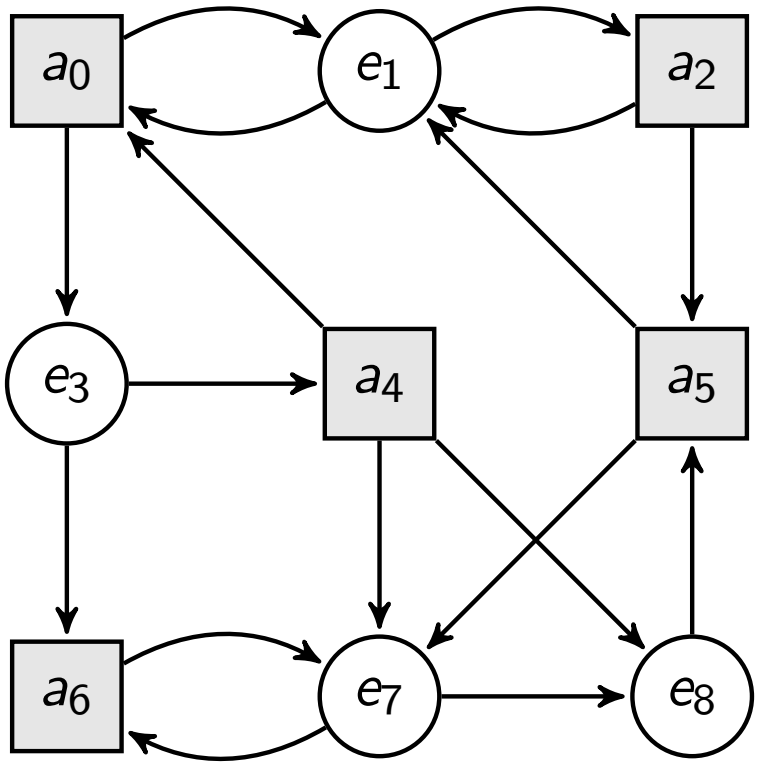
Let us define a memoryless strategy for Eve, if $T = \{a_4\}$:



- $g : \{e_1, e_3, e_7, e_8\} \to V$
- $g(e_1) = a_0$
- $g(e_3) = a_4$
- $g(e_7) = e_8$
- $g(e_8) = a_5$

Example: Memoryless Strategy in Reachability Game

Let us define a memoryless strategy for Alice, if $T = \{a_4\}$:



- $f : \{a_0, a_2, a_4, a_5, a_6\} \to V$
- $f(a_0) = e_1$
- $f(a_2) = a_5$
- $f(a_5) = e_1$
- $f(a_4) = a_0$
- $f(a_6) = e_7$

Strategies and Plays

Let $A = (V, V_A, V_E, E)$.

Let f be a strategy for Alice, and let g be a strategy for Eve.

Let $v \in V$.

We use play(v, f, g) to denote the uniquely determined play $v_0 v_1 v_2 \ldots$ on A that starts in v and proceeds according to f and g, formally:

$$v_i = \begin{cases} f(v_0 v_1 \ldots v_{i-1}) & \text{if } v_{i-1} \in V_A \\ g(v_0 v_1 \ldots v_{i-1}) & \text{if } v_{i-1} \in V_E \end{cases}$$

Let π be a play that starts in v, and let f be a strategy for Alice.

The play π is consistent with f if there exists a strategy g for Eve such that play$(v, f, g) = \pi$.

Analogous definition for Eve.

Example: Play induced by Strategies

We are given Alice's strategy f and Eve's strategy g.

- $f : \{a_0, a_2, a_4, a_5, a_6\} \to V$
- $f(a_0) = e_1$
- $f(a_2) = a_5$
- $f(a_5) = e_1$
- $f(a_4) = a_0$
- $f(a_6) = e_7$

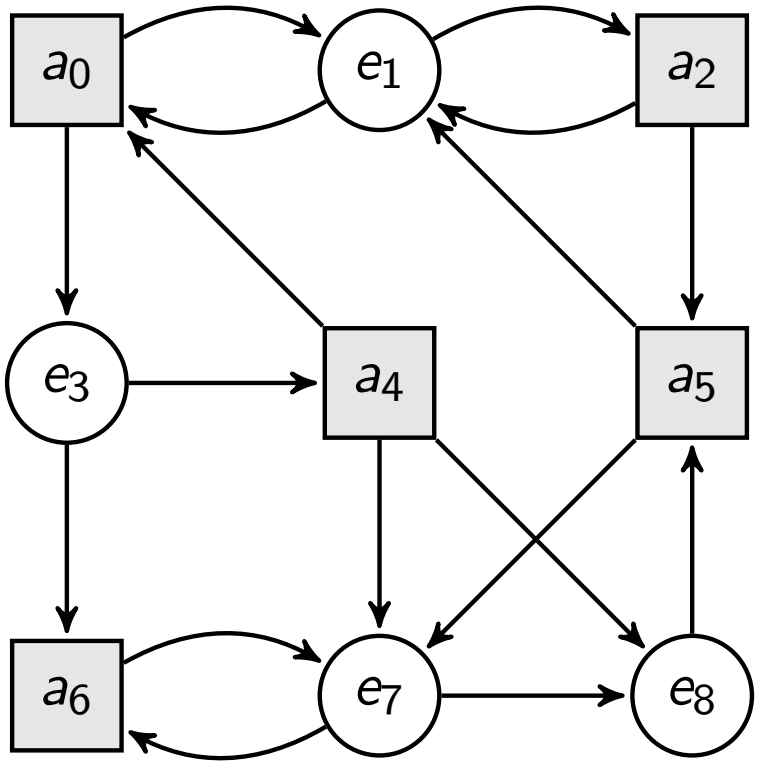
- $g : \{e_1, e_3, e_7, e_8\} \to V$
- $g(e_1) = a_0$
- $g(e_3) = a_4$
- $g(e_7) = e_8$
- $g(e_8) = a_5$

Then $\text{play}(e_1, f, g) = e_1 a_0 e_1 a_0 e_1 \ldots$

Hence $\text{play}(e_1, f, g) \notin \text{Reach}(A, \{a_4\})$

Example: Play consistent with Strategy

We are given a memoryless strategy f for Alice, if $T = \{a_4\}$

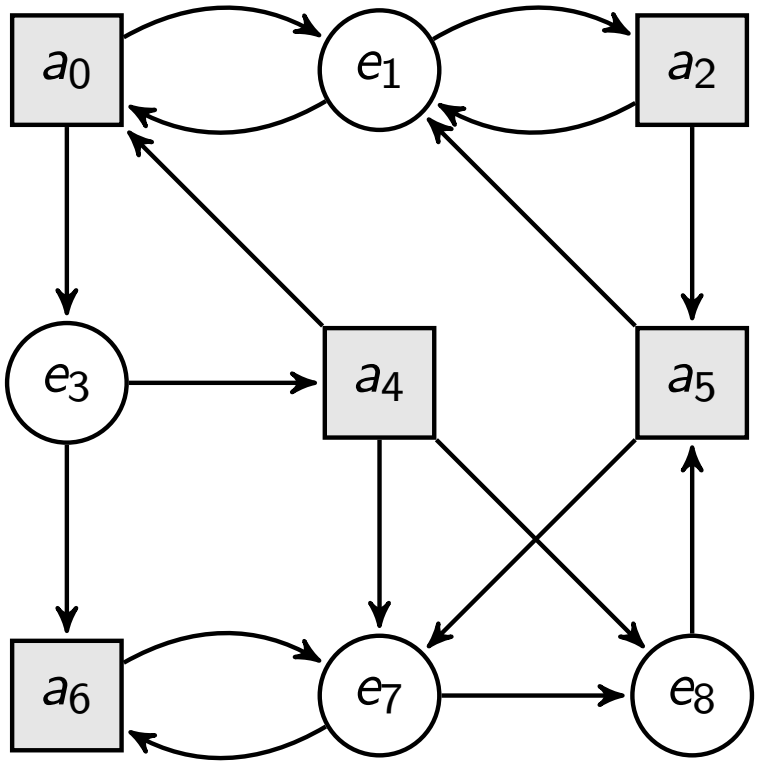


- $f : \{a_0, a_2, a_4, a_5, a_6\} \to V$
- $f(a_0) = e_1$
- $f(a_2) = a_5$
- $f(a_5) = e_1$
- $f(a_4) = a_0$
- $f(a_6) = e_7$

The play $e_7 e_8 (a_5 e_1 a_2)^\omega$ is consistent with f: there exists a strategy g for Eve such that play(e_7, f, g) equals $e_7 e_8 (a_5 e_1 a_2)^\omega$.

Pingo: Play consistent with Strategy

We are given a memoryless strategy f for Alice, if $T = \{a_4\}$

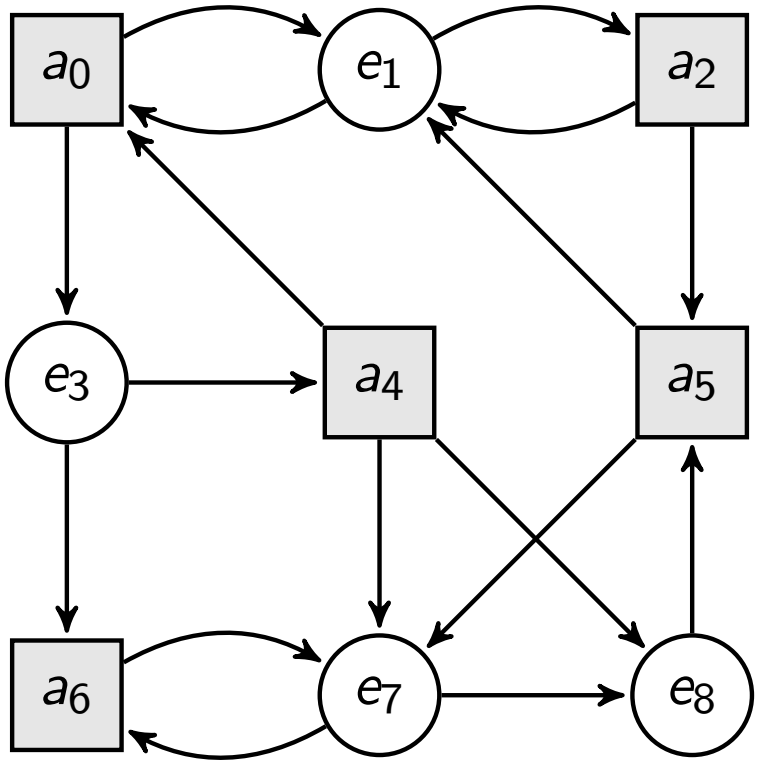


- $f : \{a_0, a_2, a_4, a_5, a_6\} \to V$
- $f(a_0) = e_1$
- $f(a_2) = a_5$
- $f(a_5) = e_1$
- $f(a_4) = a_0$
- $f(a_6) = e_7$

Is the play $e_7 a_6 e_7 e_8 (a_5 e_1 a_2)^\omega$ consistent with f? A: Yes B: No

PINGO: Reachability Game

Let $T = \{a_4\}$. We are given the following mapping g.



- $g : \{e_1, e_3, e_7, e_8\} \to V$
- $g(e_1) = a_2$
- $g(e_3) = a_6$
- $g(e_7) = a_6$
- $g(e_8) = a_5$

Is g a memoryless strategy for Eve? A: Yes B: No

Winning Strategies

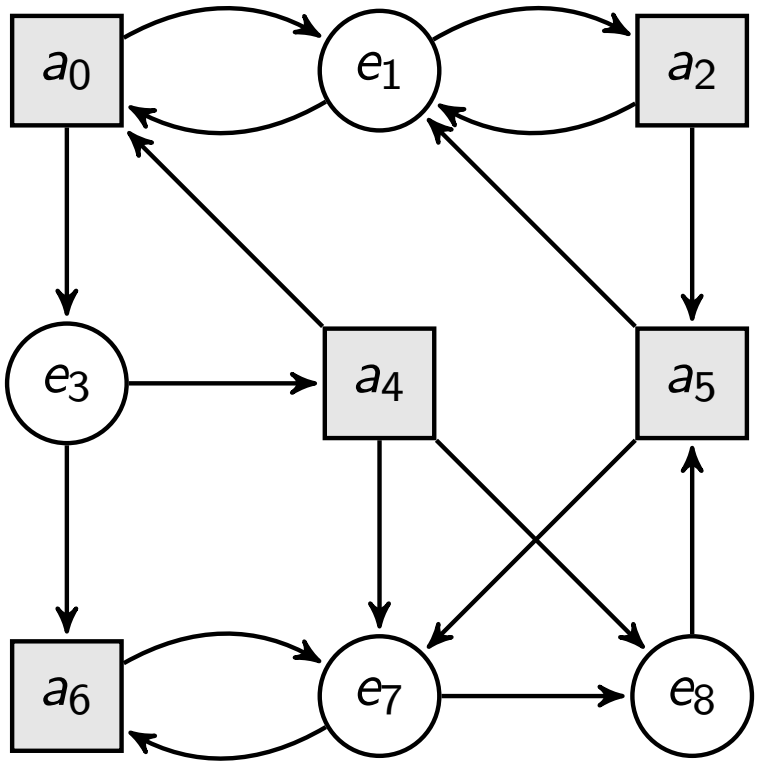
A strategy g for Eve is winning for Eve from vertex v if play(v, f, g) is winning for Eve, for **every** strategy f for Alice.

Eve wins from v if there **exists** a strategy g for Eve that is winning for Eve from v.

We then also say: Eve has a winning strategy from v.

Example: Reachability Game and Winning Strategies

Let $T = \{a_4\}$. Eve's strategy g defined below is obviously a winning strategy from e_3.

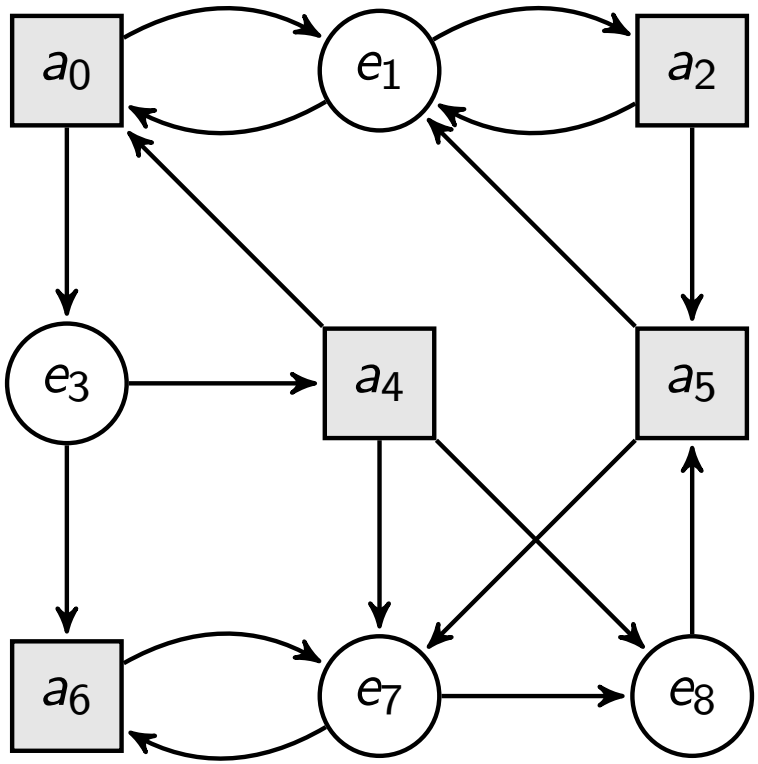


- $g : \{e_1, e_3, e_7, e_8\} \to V$
- $g(e_1) = a_2$
- $g(e_3) = a_4$
- $g(e_7) = a_6$
- $g(e_8) = a_5$

Is g a winning strategy for Eve from e_7? And from a_2?

Example: Reachability Game and Winning Strategies

Let $T = \{a_4\}$. Eve's strategy g defined below is obviously not a winning strategy from e_3.

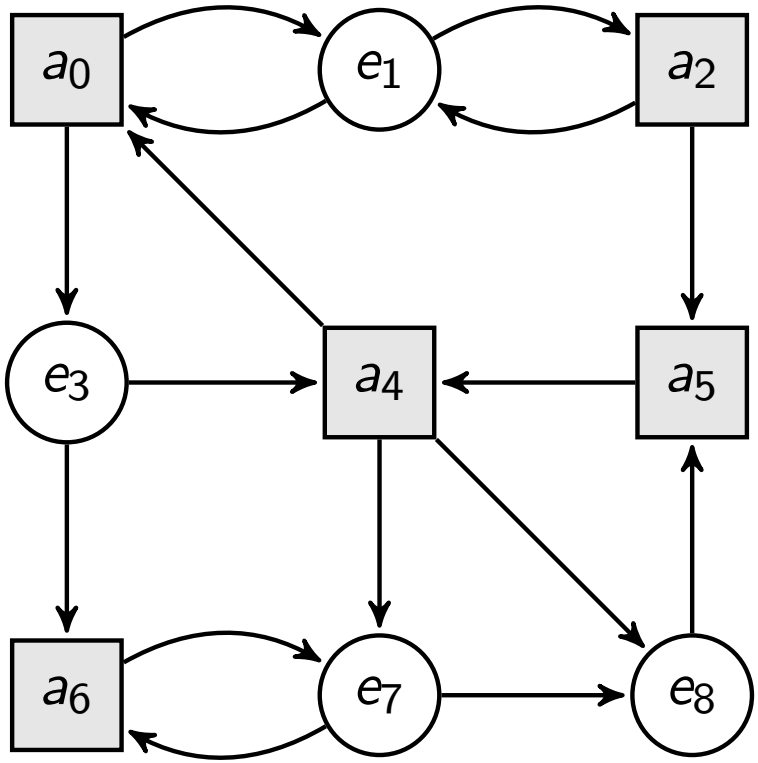


- $g : \{e_1, e_3, e_7, e_8\} \to V$
- $g(e_1) = a_2$
- $g(e_3) = a_6$
- $g(e_7) = a_6$
- $g(e_8) = a_5$

Wait...why is this so obvious?

Example: Reachability Game and Winning Strategies

Let $T = \{a_4\}$. Let us change the arena and Eve's strategy g a bit.

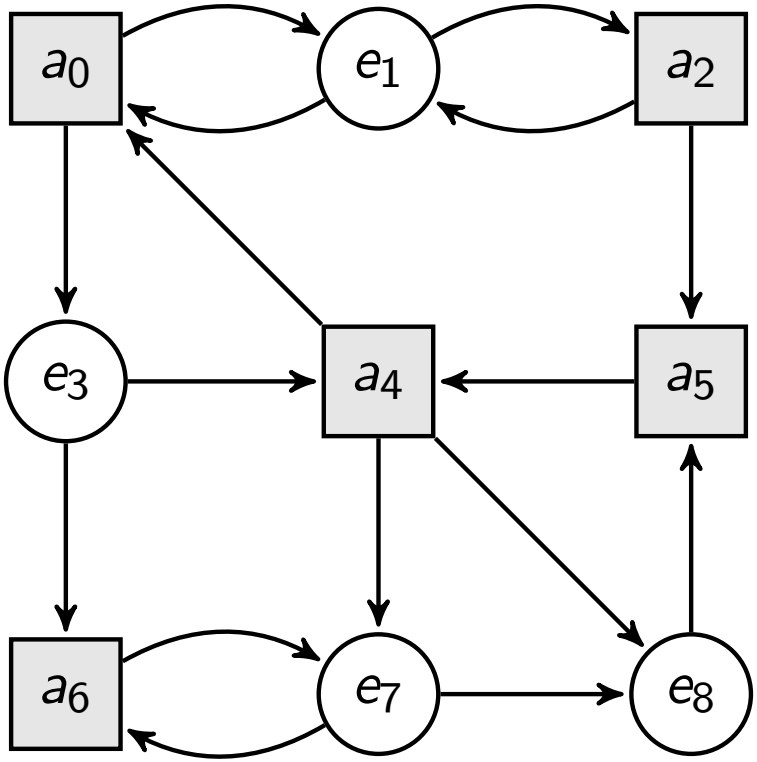


- $g : \{e_1, e_3, e_7, e_8\} \to V$
- $g(e_1) = a_2$
- $g(e_3) = a_6$
- $g(e_7) = e_8$
- $g(e_8) = a_5$

Is g a winning strategy from e_3 for Eve? And from e_1?

Example: Reachability Game

Let $T = \{a_4\}$. Is there a winning strategy for Eve from e_1? ("Does Eve win from e_1?")



Summary / What to Remember

Remember:

- Arena: defines what moves are possible
- Winning condition: defines which infinite plays are winning for Eve
- Strategy: defines how does a player choose moves