Formalisms for Data Languages

Part I: Register Automata and Freeze LTL

Karin Quaas

Universität Leipzig

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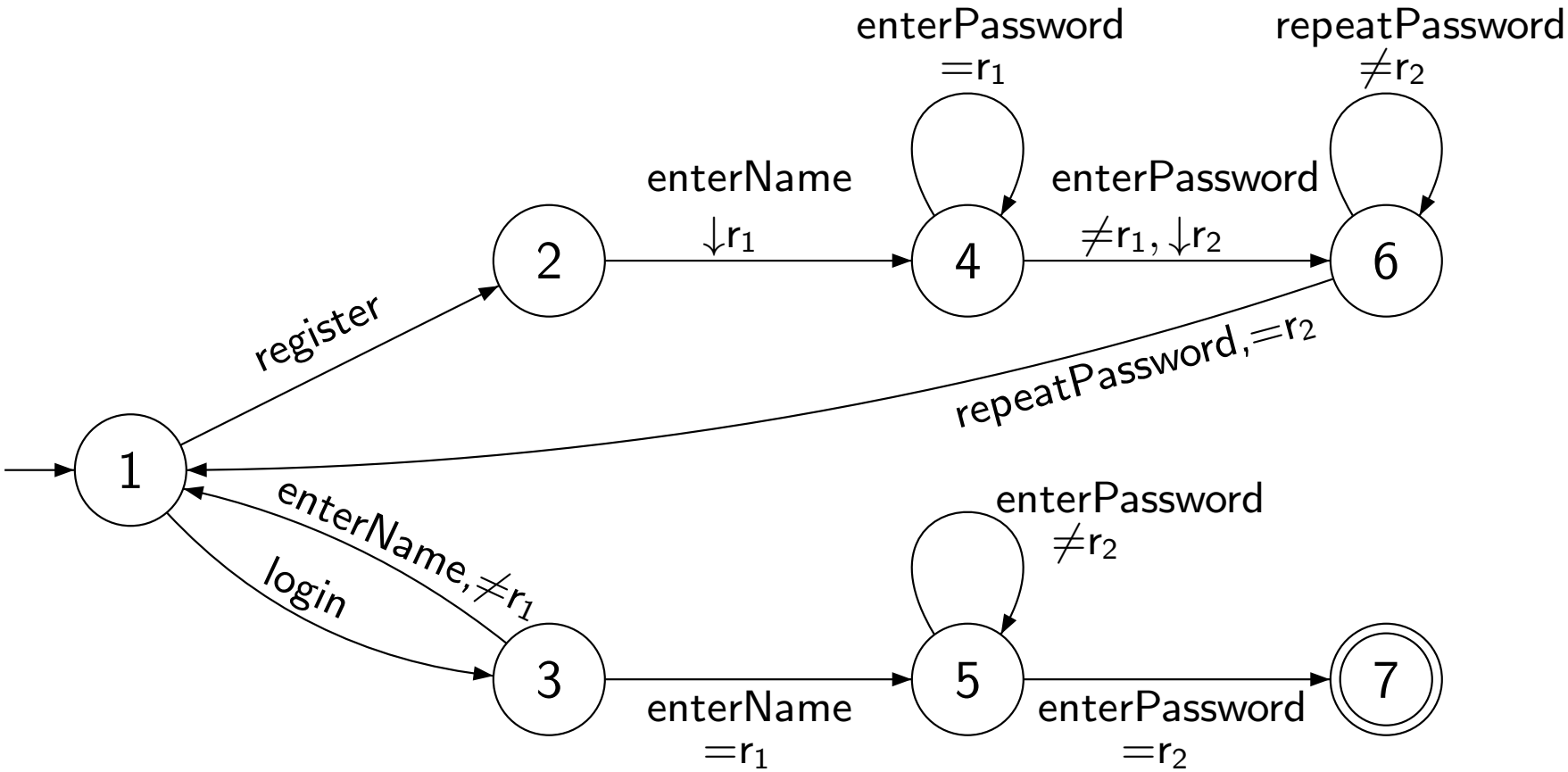
Data Languages

Σ...a finite alphabet
D...an infinite data domain, e.g., $\mathbb{N}$, $\mathbb{R}_{\geq 0}$, `ASCII strings`

A **data word** is a finite sequence $(a_1, d_1)(a_2, d_2)\ldots(a_n, d_n)$ in $(\Sigma \times D)^*$.

A **data language** is a subset of $(\Sigma \times D)^*$.

Register Automata



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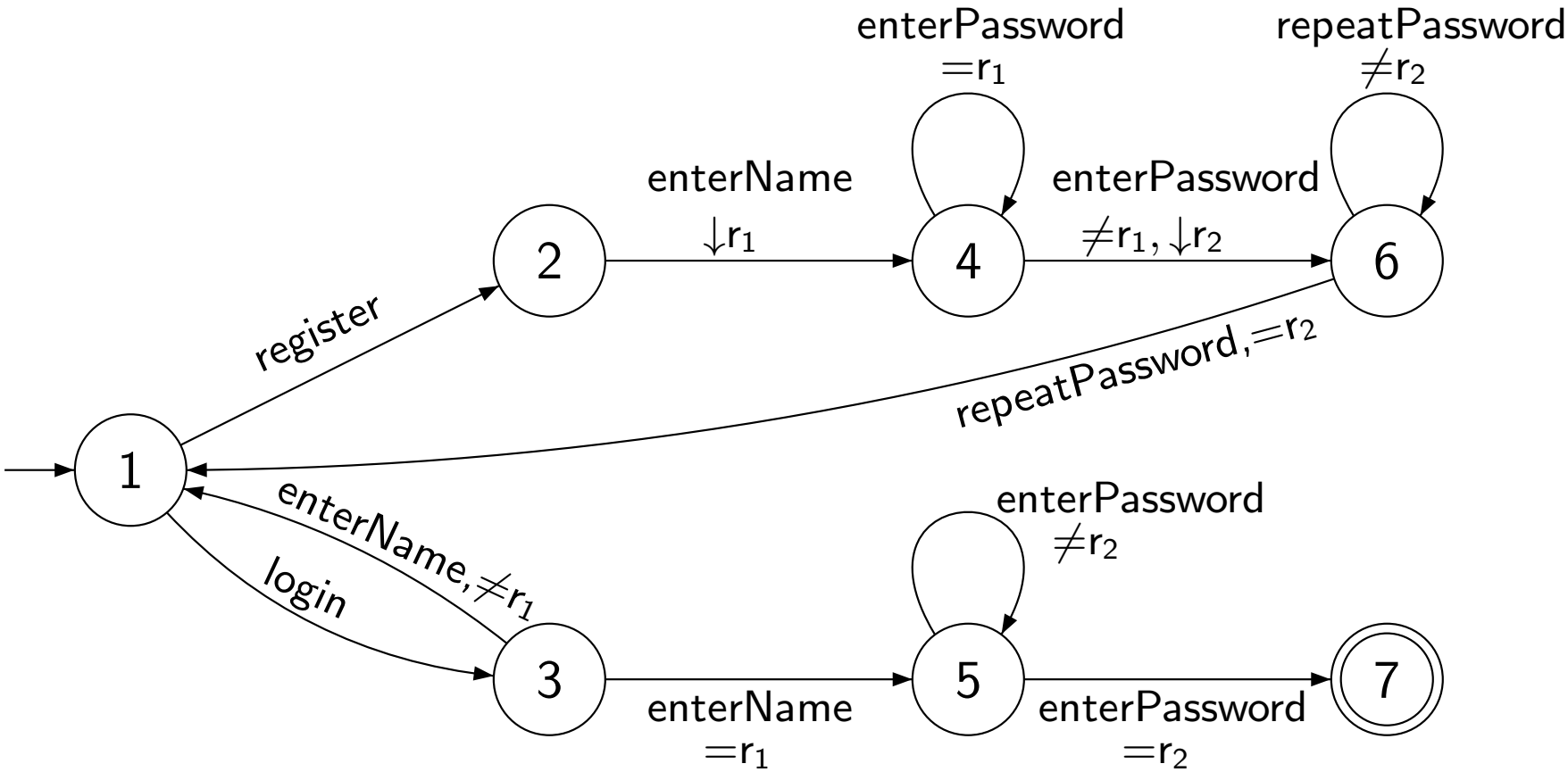

$$(1, \bot, \bot) \xrightarrow[\text{xyz}]{\text{register}} (2, \bot, \bot)$$

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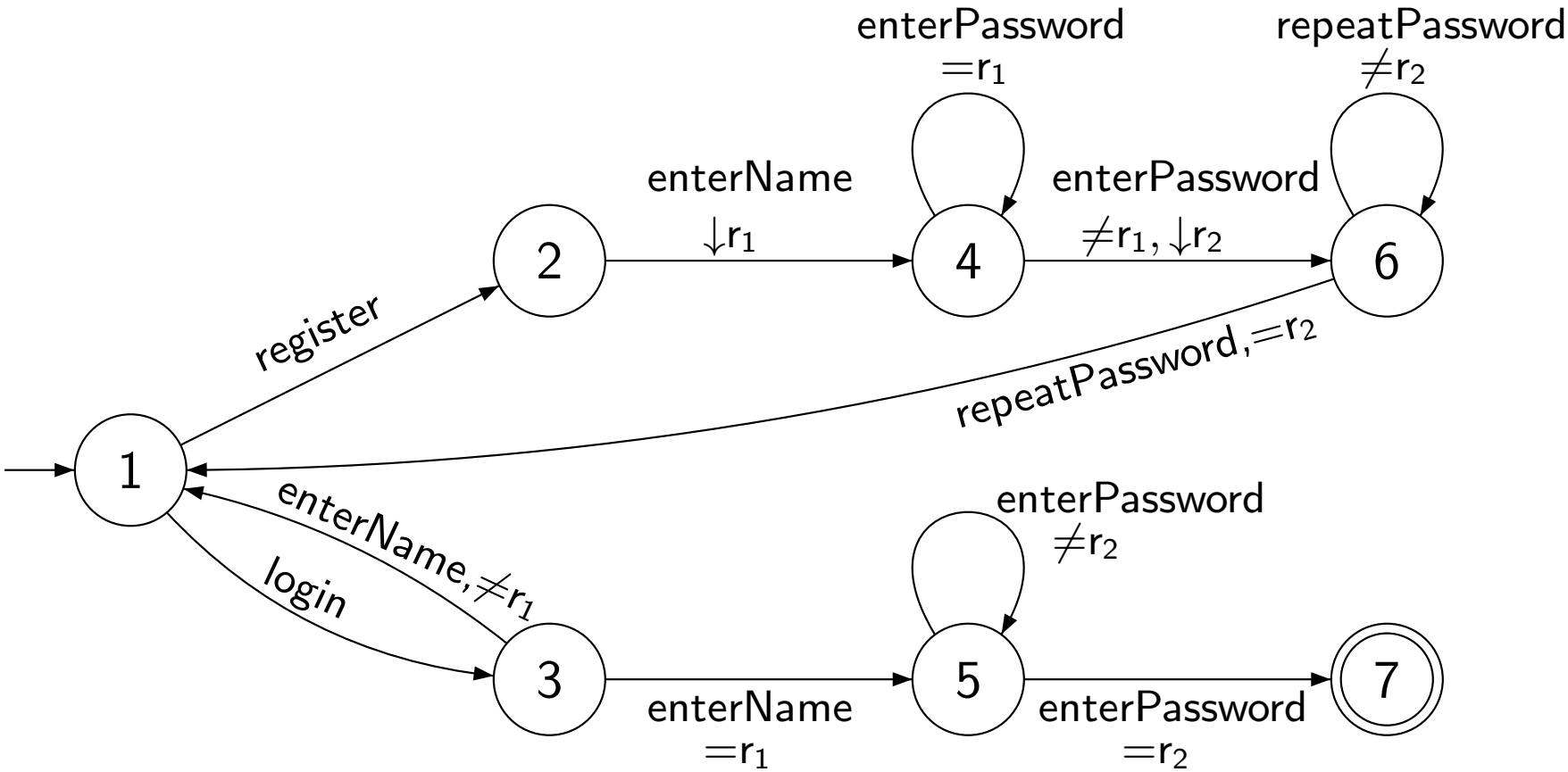

$$(1, \bot, \bot) \xrightarrow[\texttt{xyz}]{\text{register}} (2, \bot, \bot) \xrightarrow[\texttt{karin}]{\text{enterName}} (4, \texttt{karin}, \bot)$$

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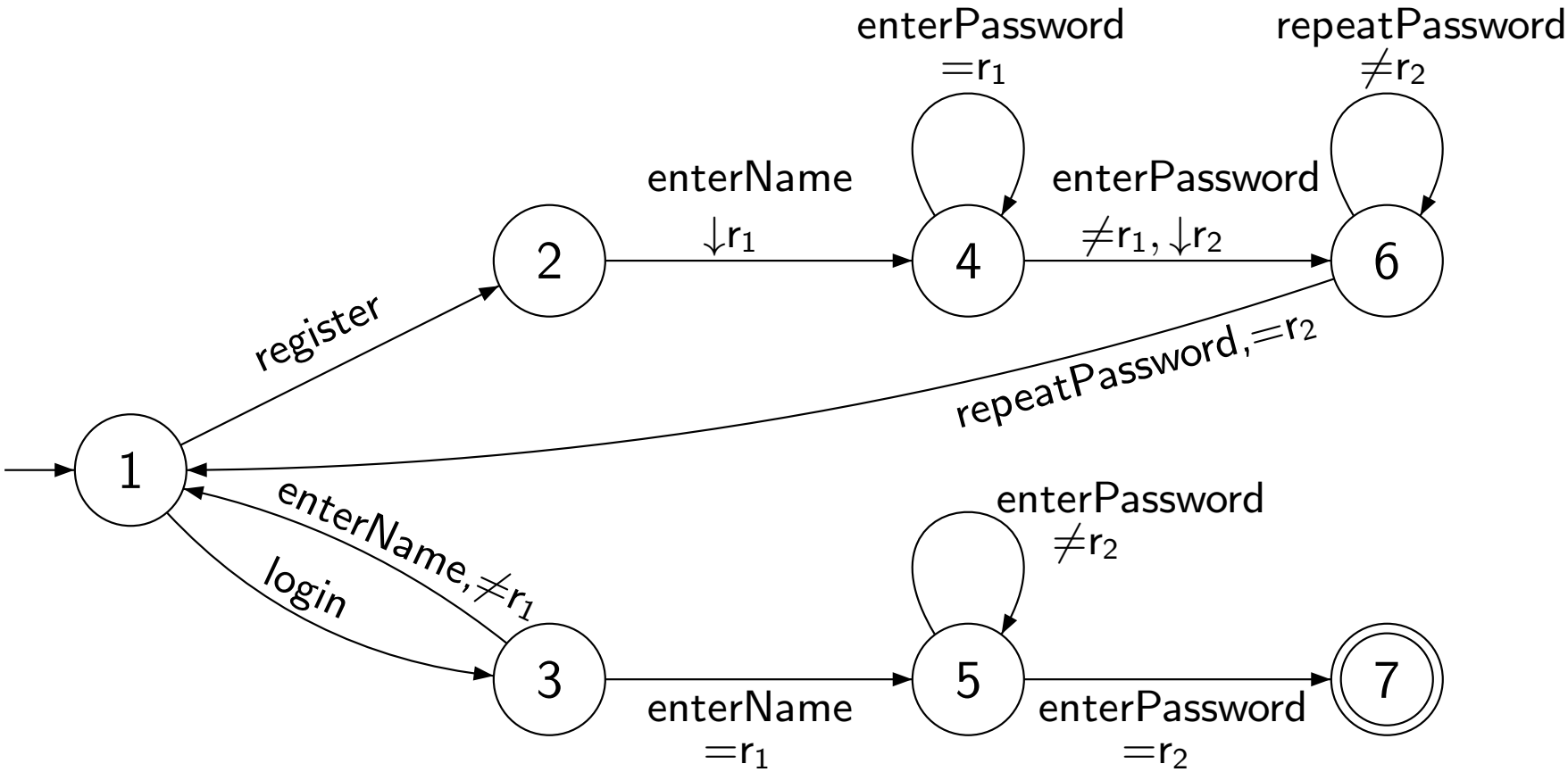

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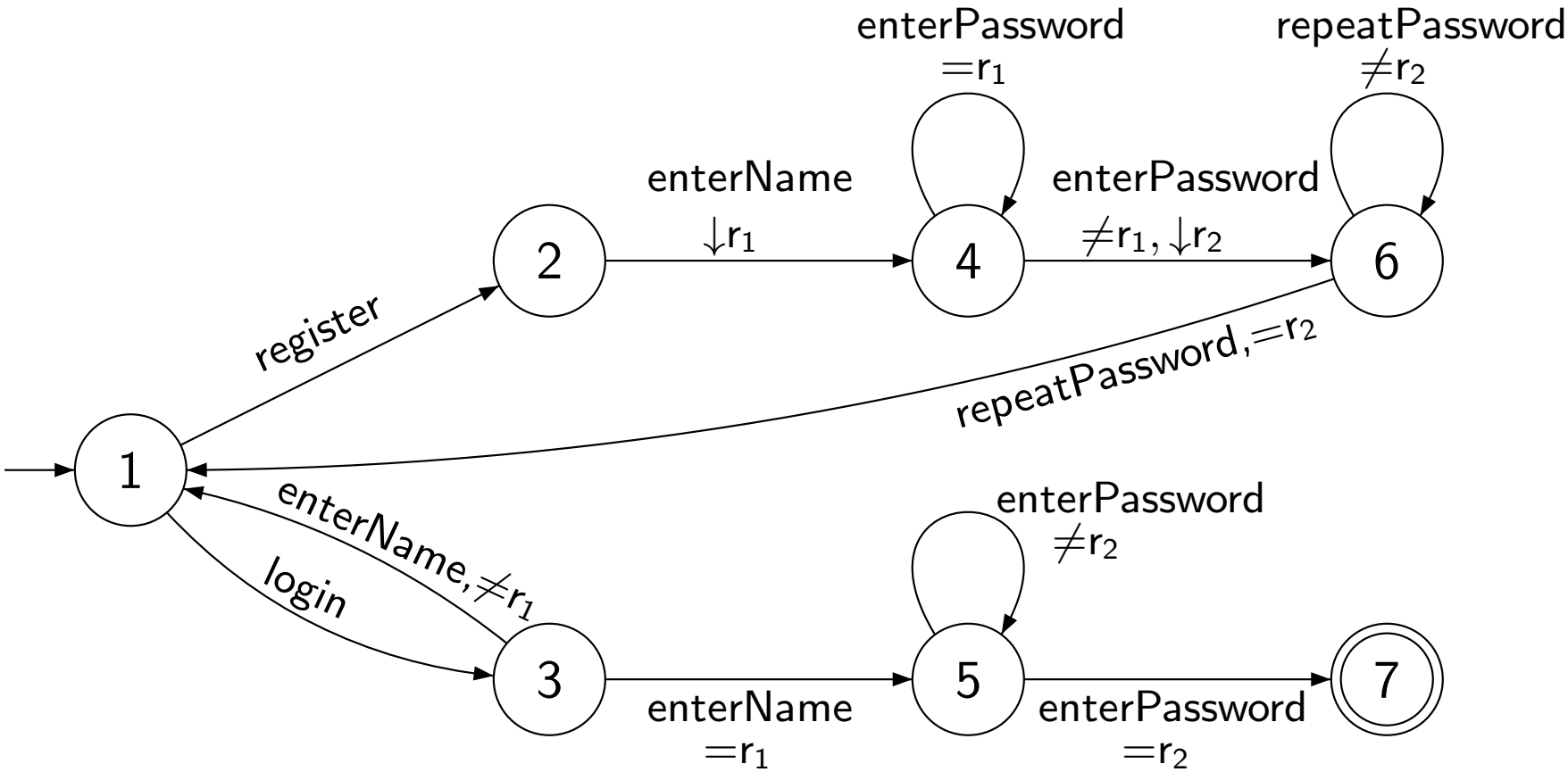

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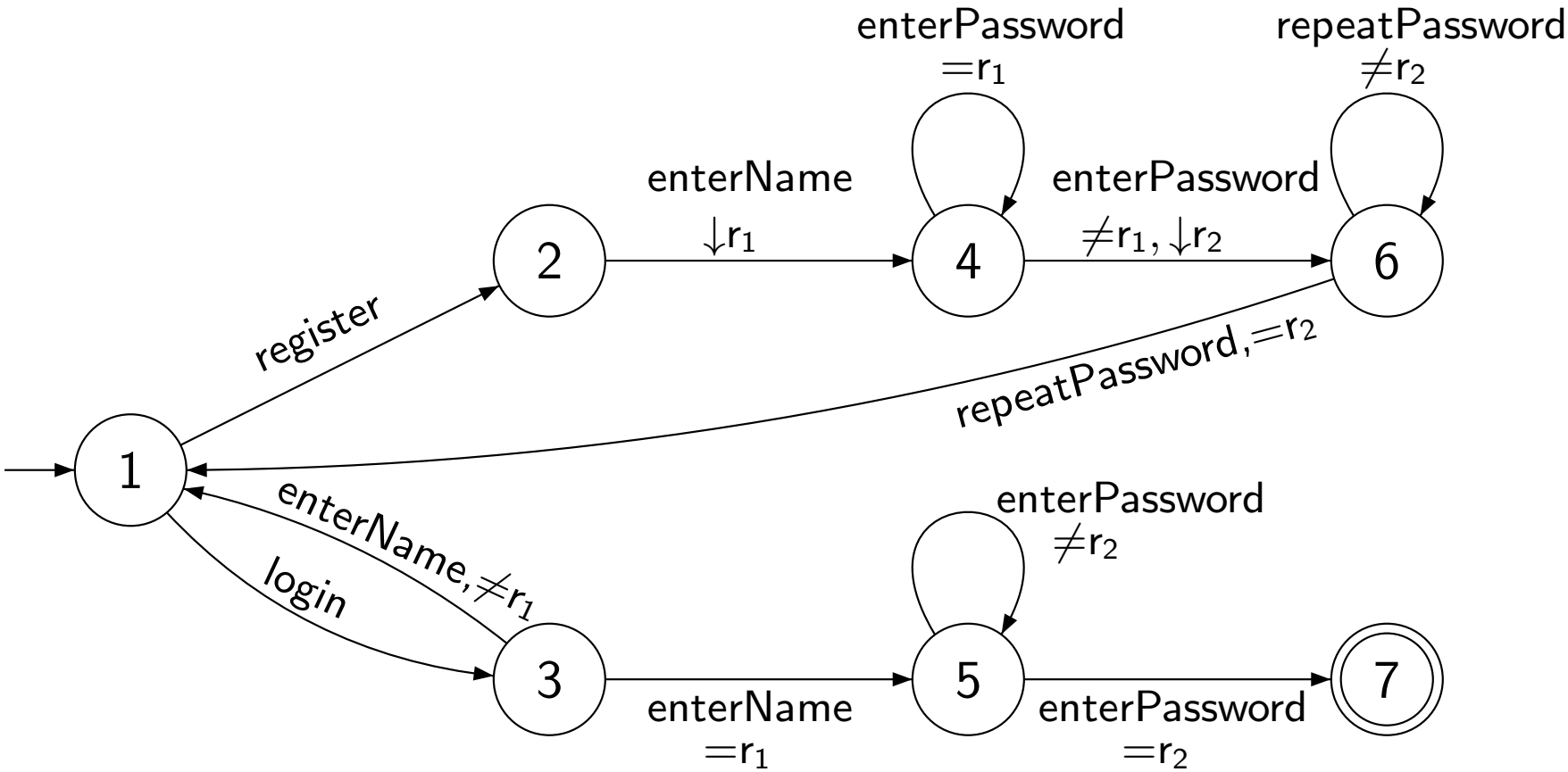

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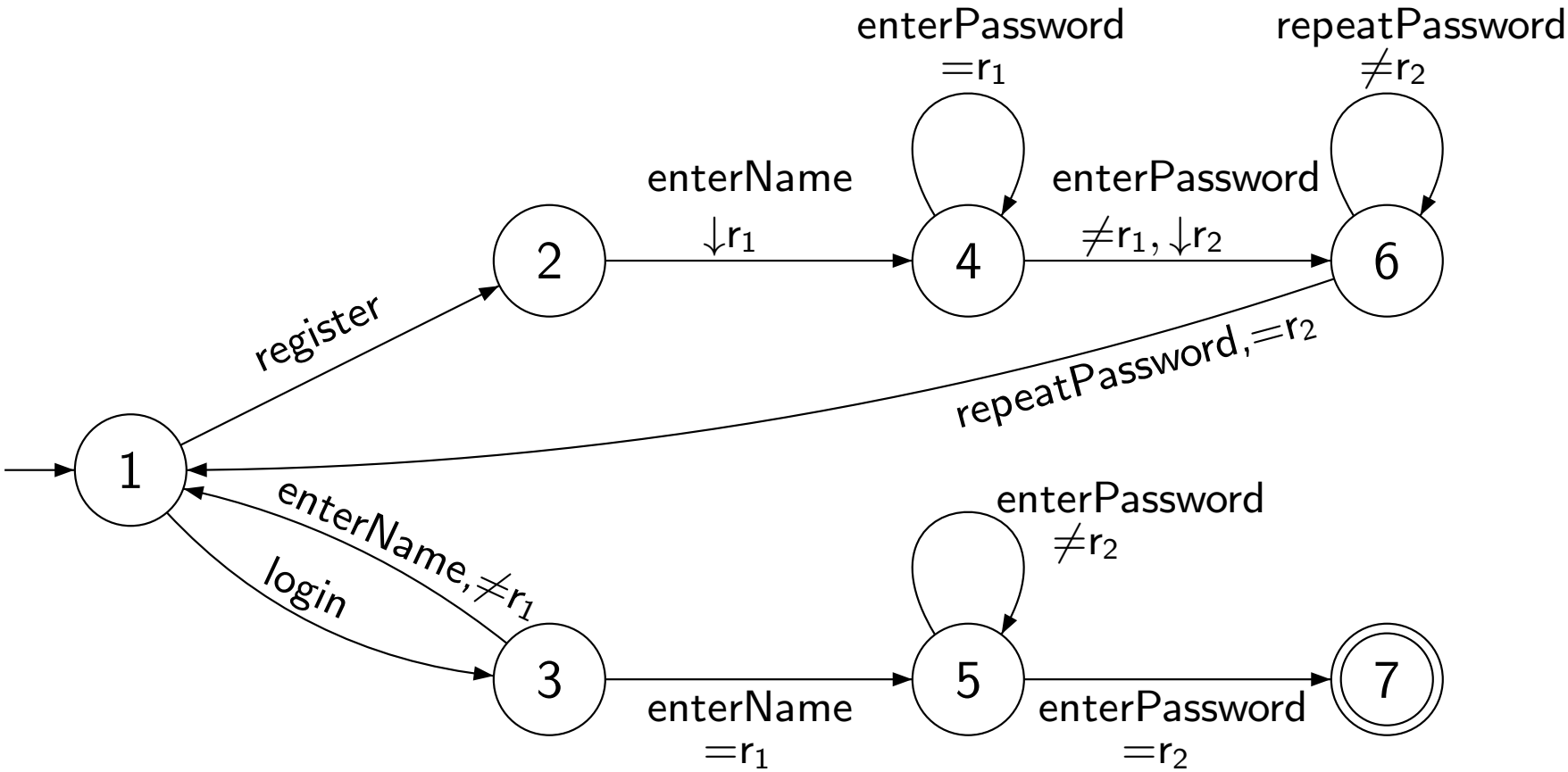

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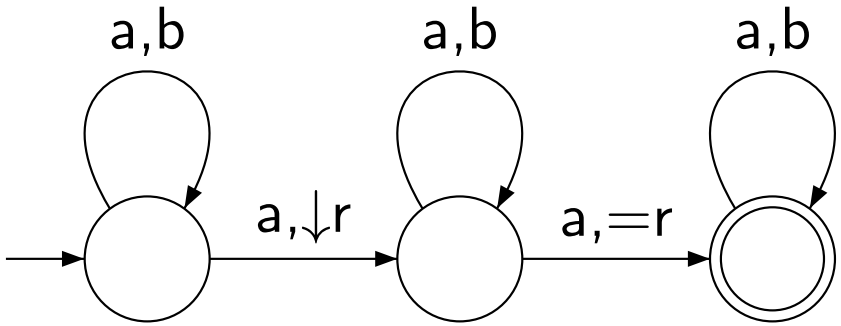
Register Automata

Closure Properties

Recognizable data languages are closed under **union**, **intersection**, but not closed under **complement**:



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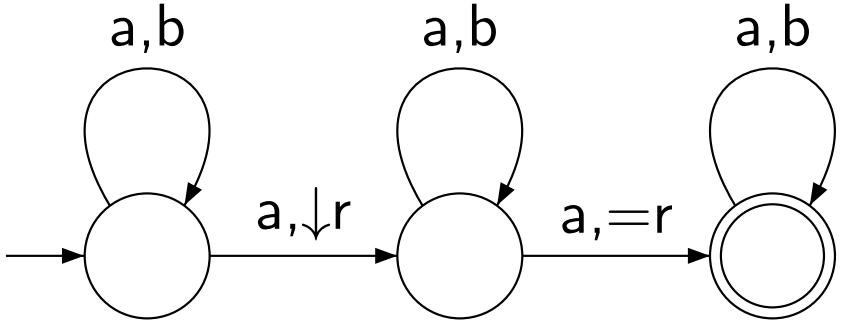

$$L = \{(a_1, d_1) \dots (a_n, d_n) \in (\Sigma \times \mathrm{D})^+ \mid \exists i < j . a_i = a_j = a, d_i = d_j\}$$

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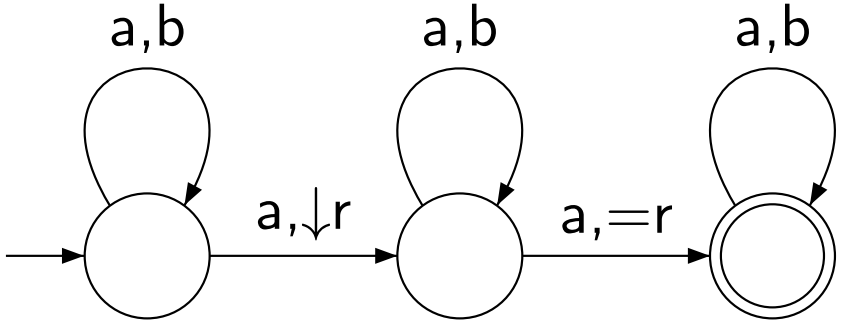

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$$(\Sigma \times \mathrm{D})^* \backslash L = \{w \in (\Sigma \times \mathrm{D})^* \mid \forall i < j . a_i = a_j = a \Rightarrow d_i \neq d_j\}$$

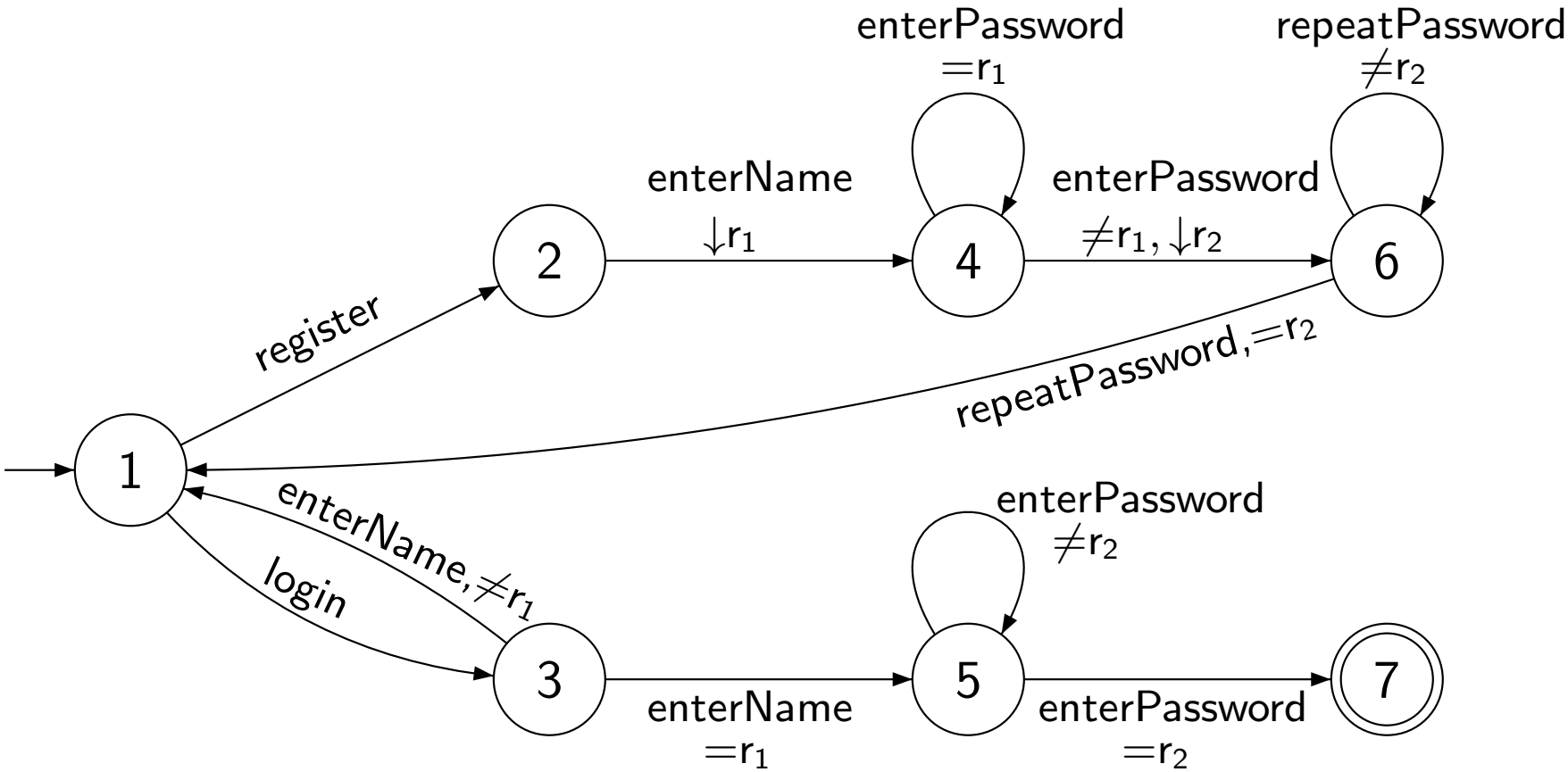
and cannot be recognized by a register automaton

The Non-Emptiness Problem

Instance: A register automaton $\mathcal{A}$.

Question: Does $\mathcal{A}$ accept a data word?

Register Automata



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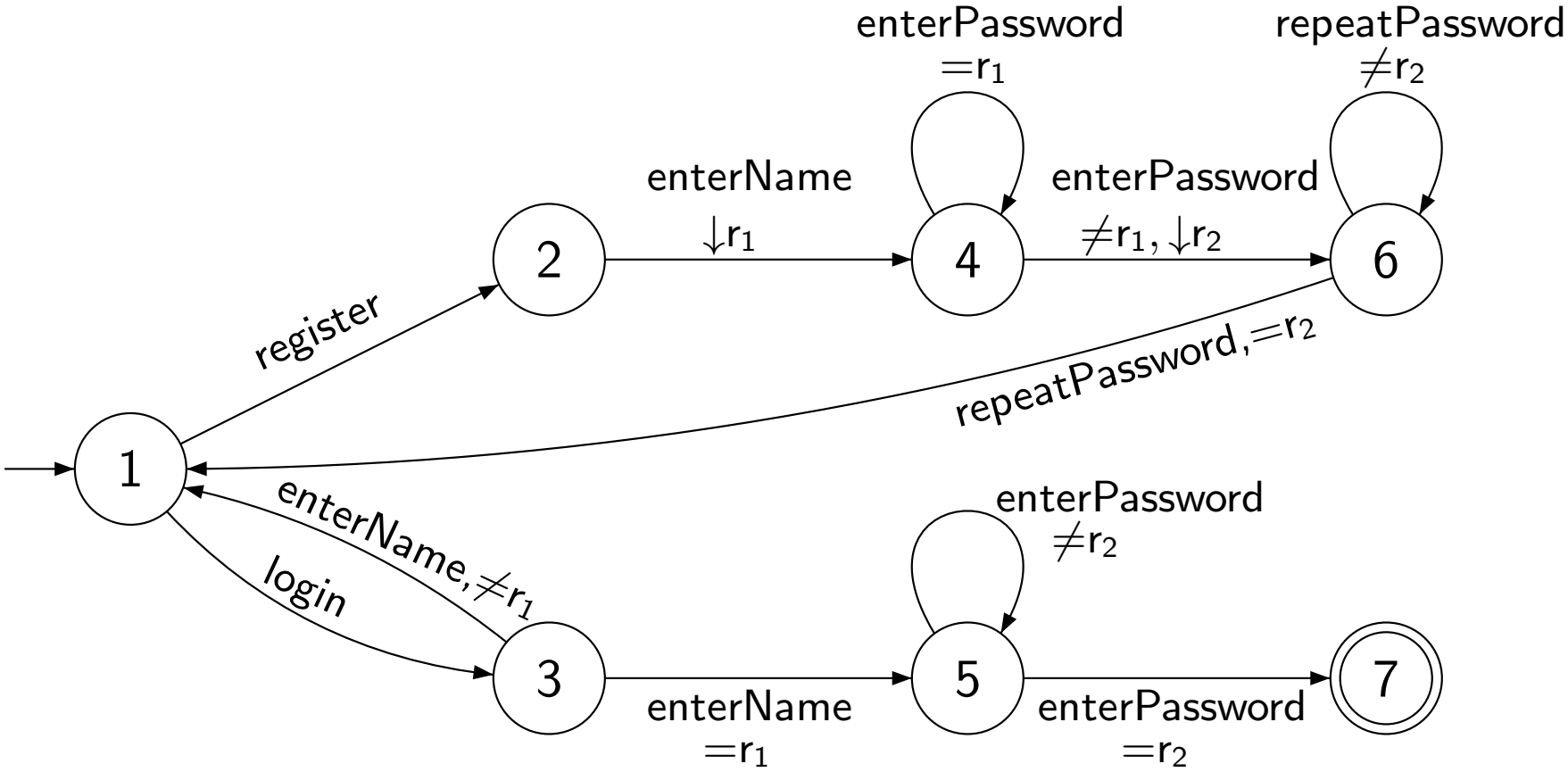
There is a successful run in $\mathcal{A}$, if, and only if, there is a successful run in $\mathcal{S}_{\mathcal{A}}$.

Register Automata

Decision Problems:The Non-Emptiness Problem



$$(1,\perp,\perp) \xrightarrow[\texttt{xyz}]{\text{register}} (2,\perp,\perp) \xrightarrow[\texttt{karin}]{\text{enterName}} (4,\texttt{karin},\perp) \xrightarrow[\texttt{karin}]{\text{enterPassword}} (4,\texttt{karin},\perp)$$
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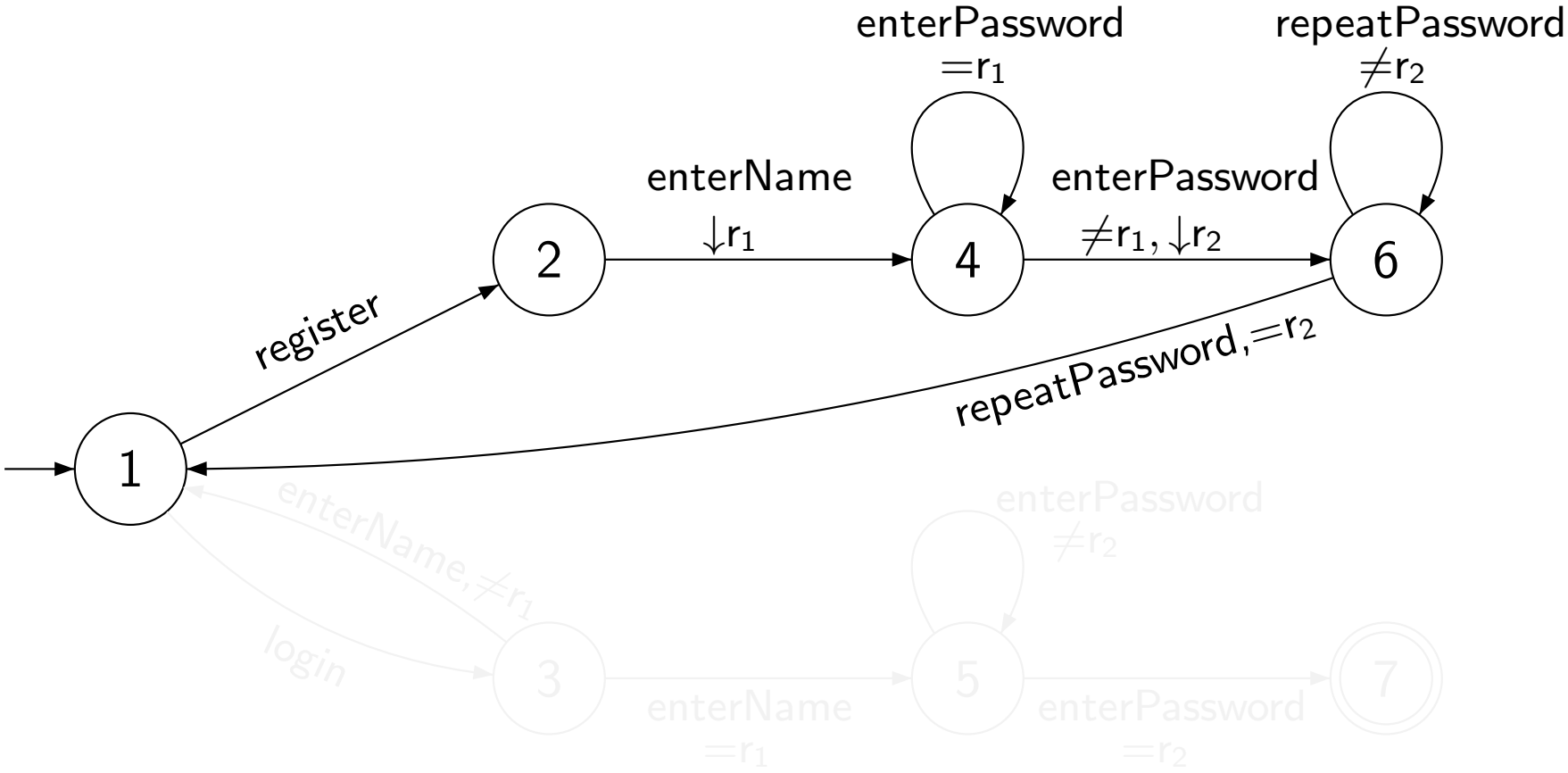

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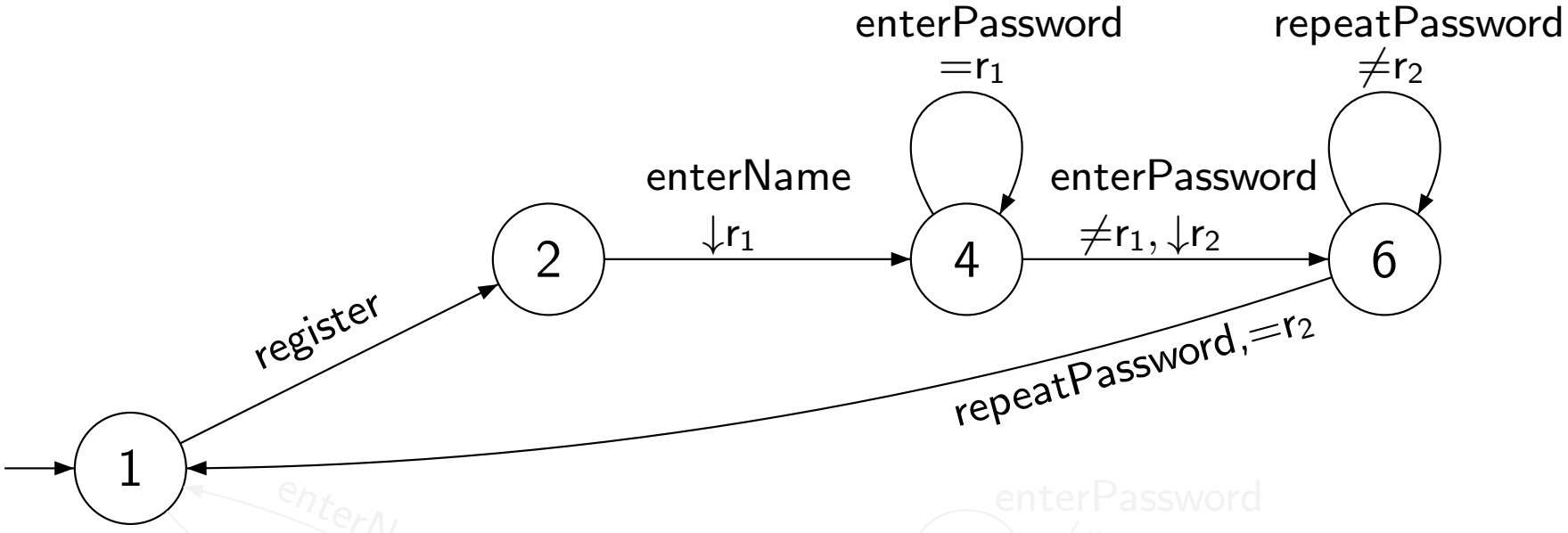

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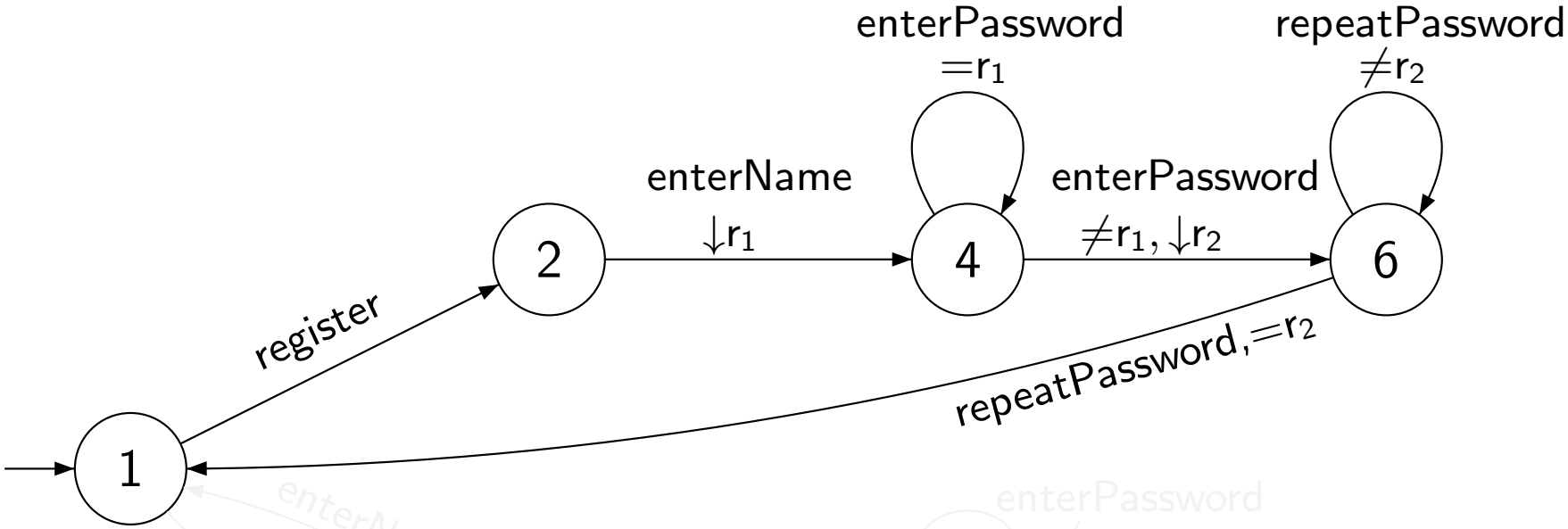

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$$(1,\emptyset,\{(r_1,r_2)\}) \xRightarrow{\text{register}} (2,\emptyset,\{r_1,r_2\})$$

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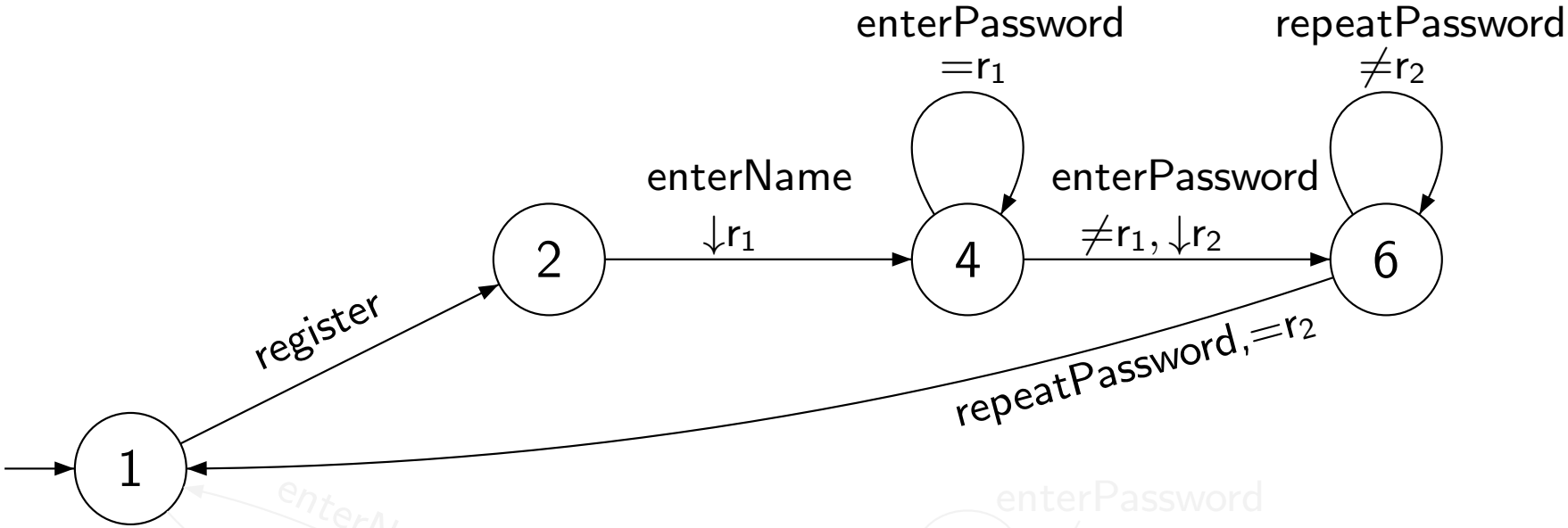

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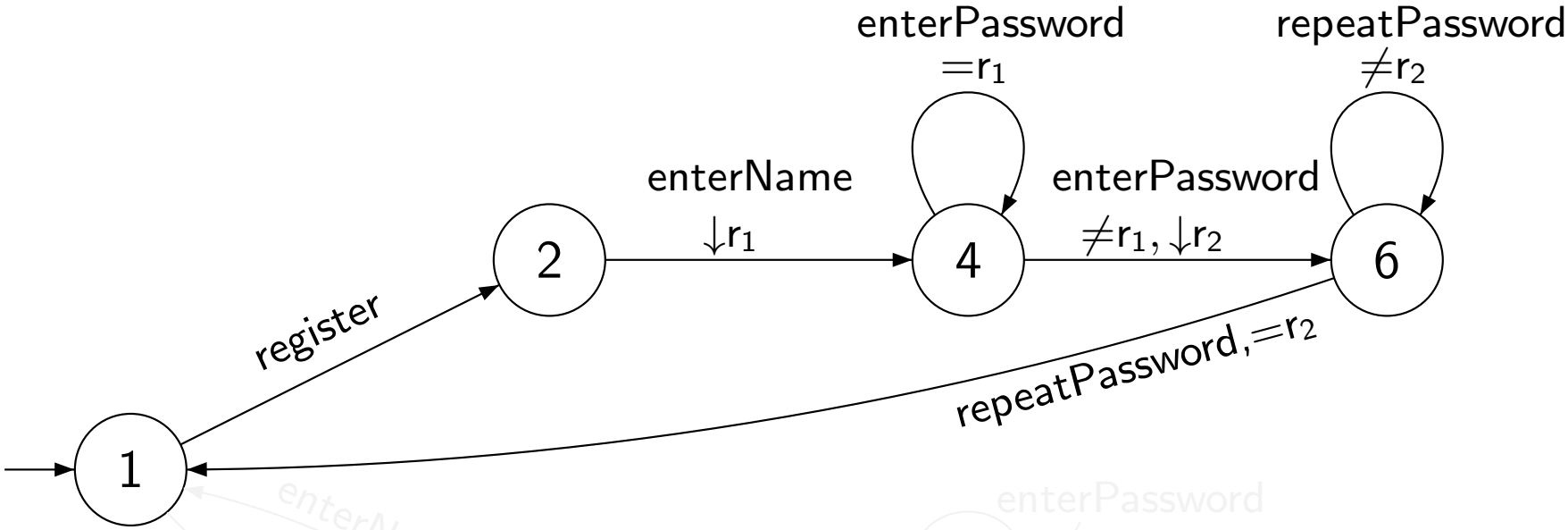

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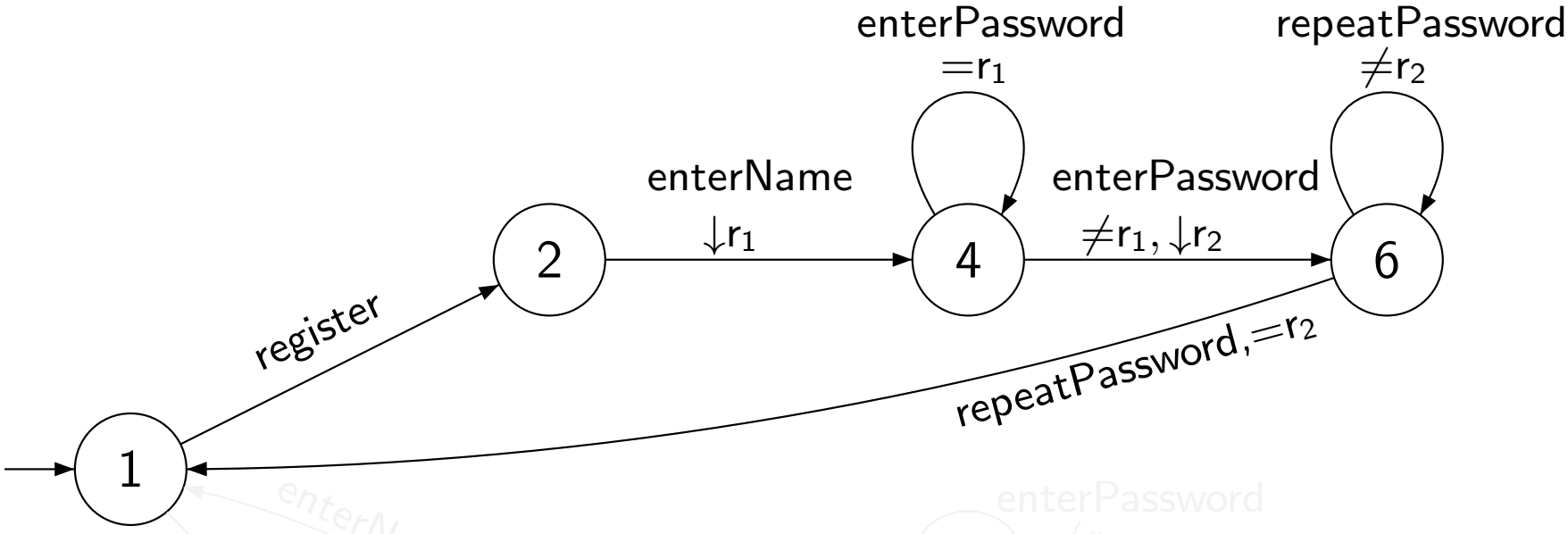

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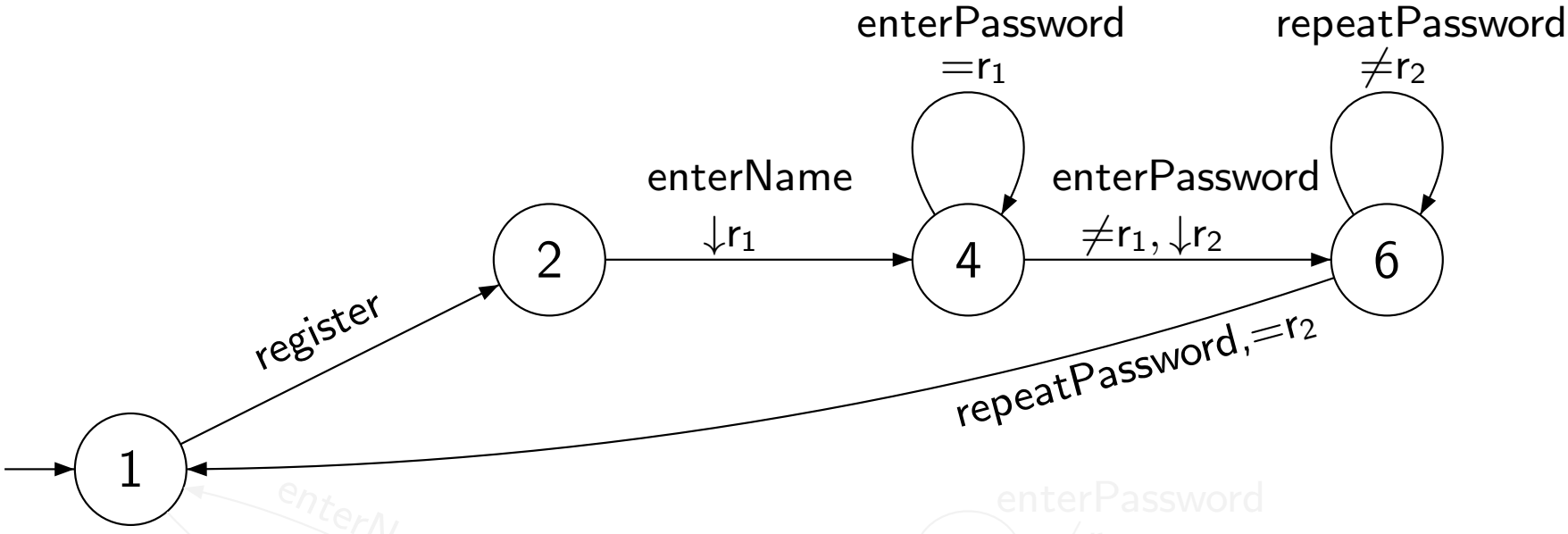

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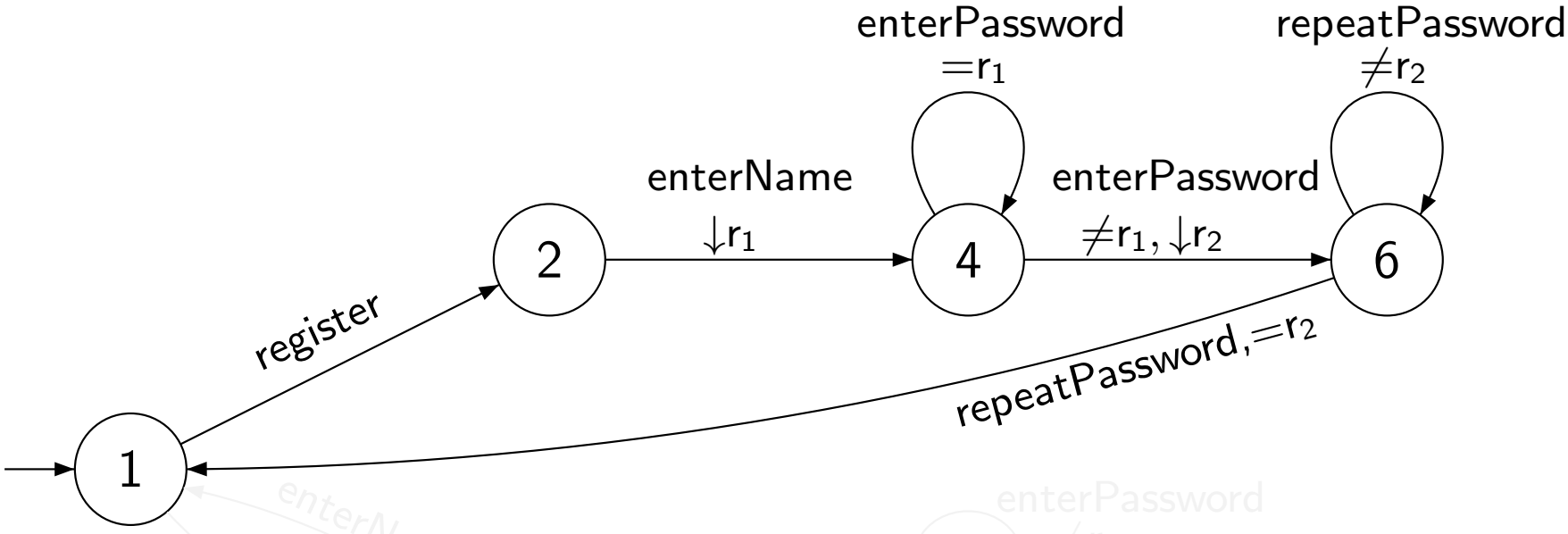

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The Non-Emptiness Problem

Instance: A register automaton $\mathcal{A}$.

Question: Does $\mathcal{A}$ accept a data word?

Theorem (Demri & Lazic, 2008)

The non-emptiness problem is PSPACE-complete.

The Language Inclusion Problem

Instance: Register automata $\mathcal{A}, \mathcal{B}$.

Question: Does $\mathcal{A}$ accept a subset of the language accepted by $\mathcal{B}$?

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Theorem (Neven, Schwentick & Vianu, 2004)

The language inclusion problem is undecidable.

Freeze LTL

Syntax

$$\varphi ::= a \mid =r \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{X}\varphi \mid \varphi \mathsf{U} \varphi \mid \downarrow r.\varphi$$

where $a \in \Sigma$ and r is a register variable in the set $\mathcal{R}$ or registers

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Semantics

$w = (a_1, d_1) \dots (a_n, d_n)$, $i \in \{1, \dots, n\}$, $\nu \in (D \cup \{\bot\})^{\mathcal{R}}$

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\begin{array}{lll}
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\end{array}
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Freeze LTL

Syntax

$$\varphi ::= a \mid =r \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{X}\varphi \mid \varphi \mathsf{U} \varphi \mid \downarrow r.\varphi$$

where $a \in \Sigma$ and r is a register variable in the set $\mathcal{R}$ or registers

Semantics

$w = (a_1, d_1) \dots (a_n, d_n)$, $i \in \{1, \dots, n\}$, $\nu \in (D \cup \{\bot\})^{\mathcal{R}}$

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\end{array}
$$

Freeze LTL

Examples

Let $w = (a_1, d_1)(a_2, d_2)(a_3, d_3)(a_4, d_4)(a_5, d_5)$ be a data word, let $\nu \in D \cup \{\bot\}$.

$(w, 1, \nu) \models a \quad \Leftrightarrow \quad a_1 = a$

Freeze LTL

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Freeze LTL

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(w, 1, \nu) &\models \downarrow r.b\mathsf{U}(b \wedge \neq r) && \Leftrightarrow\ \exists 1 < i \leq 5.a_i = b, d_i \neq d_1, \\
& && \quad \forall 1 < j < i.a_i = b
\end{aligned}
$$

Freeze LTL

Two useful abbreviations

Definition

$\mathsf{F}\varphi := \mathsf{true}\mathsf{U}\varphi$

$\mathsf{G}\varphi := \neg\mathsf{F}\neg\varphi$

Semantics

$w = (a_1, d_1)\ldots(a_n, d_n),\ i \in \{1, \ldots, n\},\ \nu \in (D \cup \{\bot\})^{\mathcal{R}}$

$$(w, i, \nu) \models F\varphi \quad \Leftrightarrow \quad \exists 1 < i \leq n.(w, i, \nu) \models \varphi$$
$$(w, i, \nu) \models \mathsf{G}\varphi \quad \Leftrightarrow \quad \forall 1 < i \leq n.(w, i, \nu) \models \varphi$$

The Satisfiability Problem

Instance: Freeze LTL formula φ.

Question: Does there exist a data word such that $(w, 1, \{\bot\}^{\mathcal{R}}) \models \varphi$?

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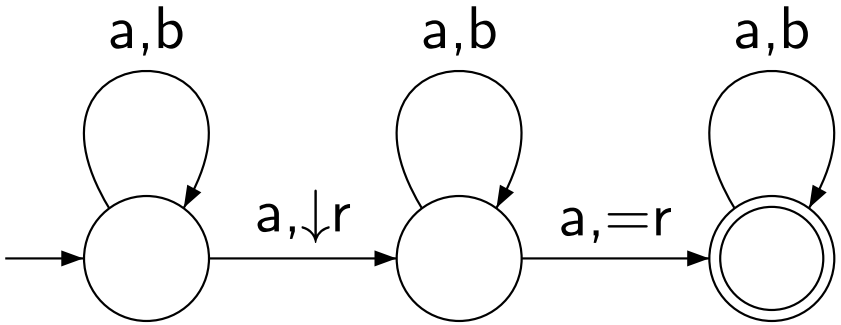
Can we use an automata-based approach to show decidability?

Freeze LTL

Can Formulas be translated to equivalent Register Automata?

Recognizable data languages are closed under **union**, **intersection**, but not closed under **complement**:



$L = \{(a_1, d_1) \dots (a_n, d_n) \in (\Sigma \times \mathrm{D})^+ \mid \exists i < j . a_i = a_j = a, d_i = d_j\}$

$(\Sigma \times \mathrm{D})^* \backslash L$ cannot be recognized by a register automaton

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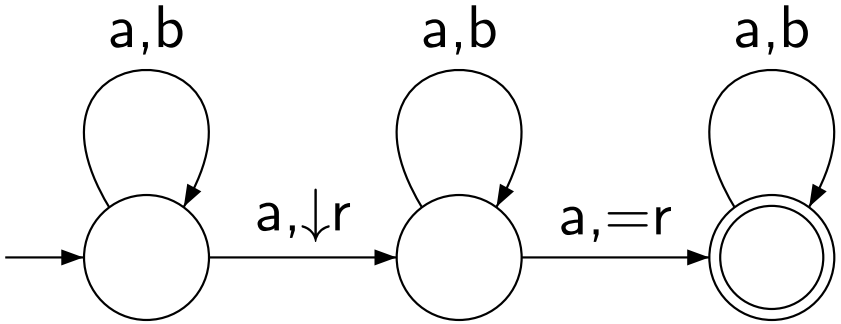

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Hence, the formula $\neg \mathrm{F}(a \wedge \downarrow r . \mathrm{F}(a \wedge = r))$ cannot be translated into an equivalent register automaton

Freeze LTL

Decision Problems

The Satisfiability Problem

Instance: Freeze LTL formula φ.

Question: Does there exist a data word such that $(w, 1, \{\bot\}^{\mathcal{R}}) \models \varphi$?

Theorem (Demri & Lazic, 2008)

The satisfiability problem is undecidable.

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Proof idea: Automata-based approach!

- Every Freeze LTL formula with at most one register can be translated into an equivalent **alternating** single-register automaton.

Freeze LTL

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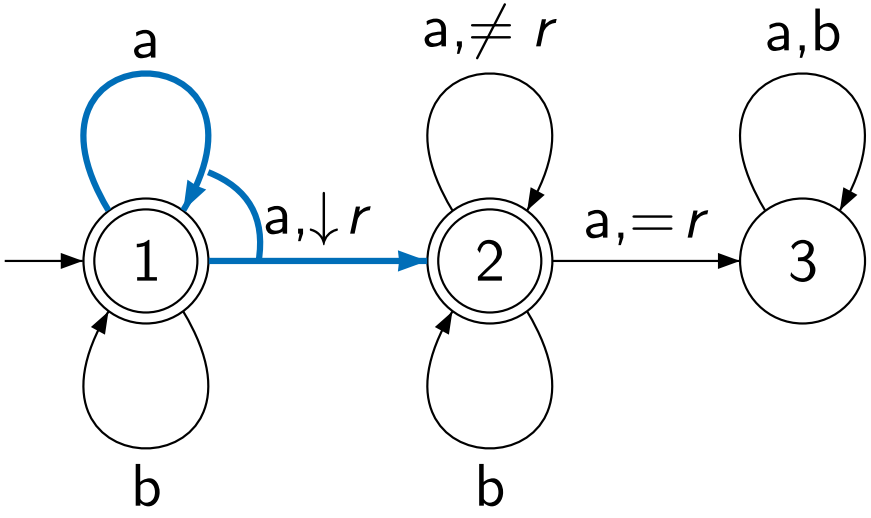
- Every Freeze LTL formula with at most one register can be translated into an equivalent **alternating** single-register automaton.
- The non-emptiness problem for alternating single-register automata is decidable.

Alternating Single-Register Automata



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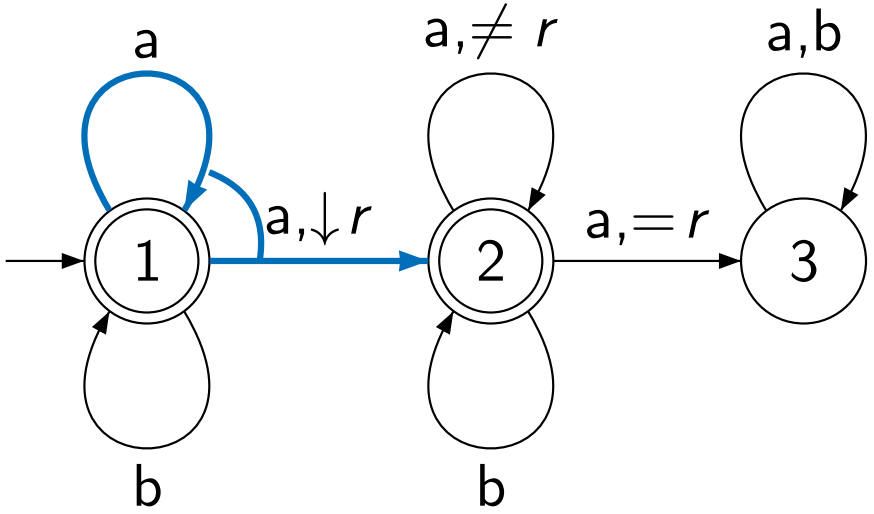
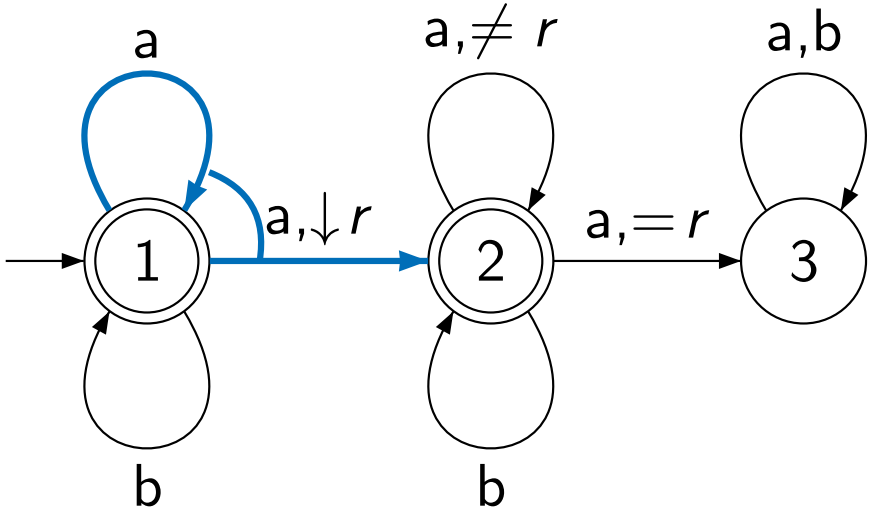


	a	b
1	$1 \wedge \downarrow r.2$	1
2	$(2 \wedge \neq r) \vee (3 \wedge = r)$	2
3	3	3

Alternating Single-Register Automata



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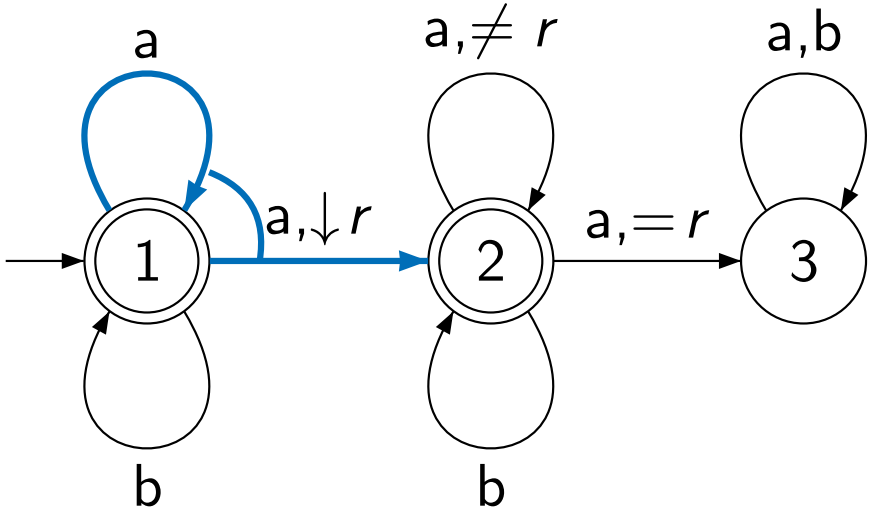
$$\{(1, \perp)\} \xrightarrow[d_1]{\mathsf{b}}$$

Alternating Single-Register Automata



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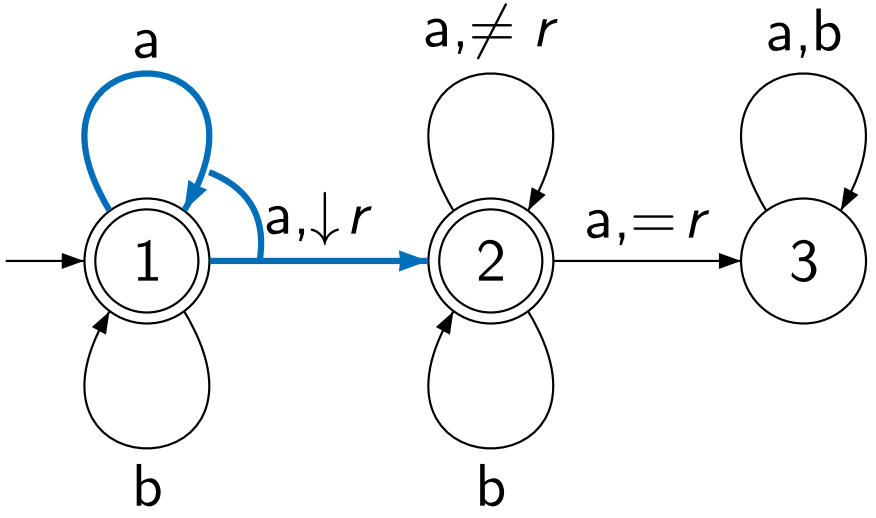

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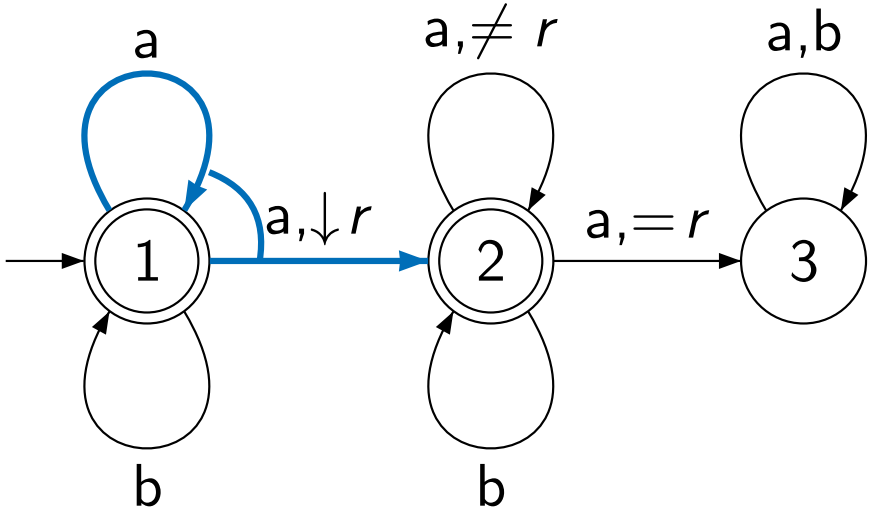
Alternating Single-Register Automata



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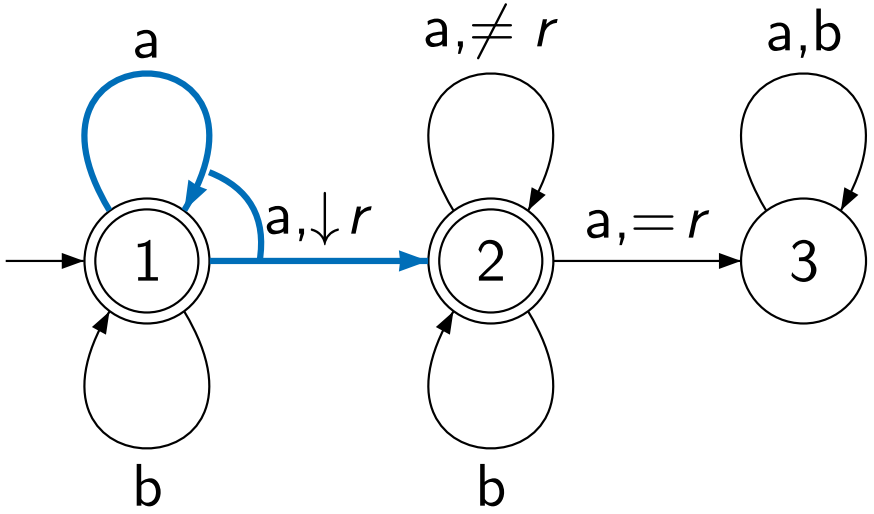

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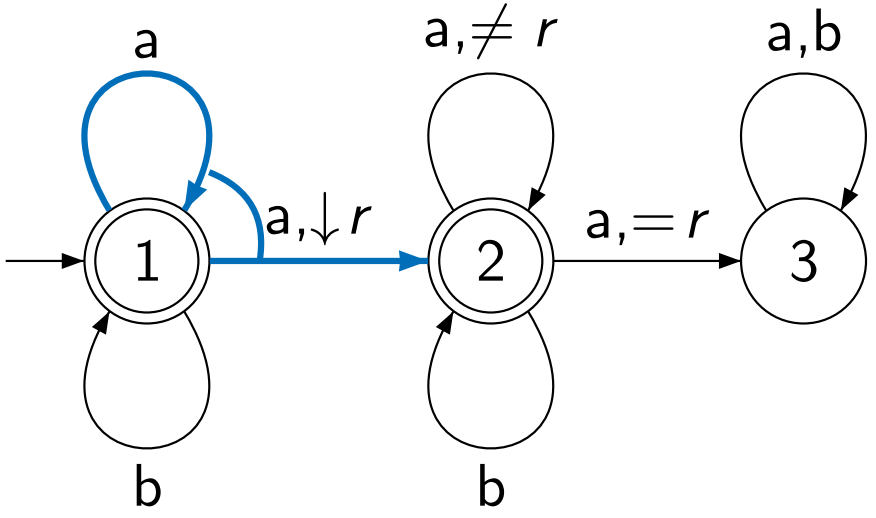

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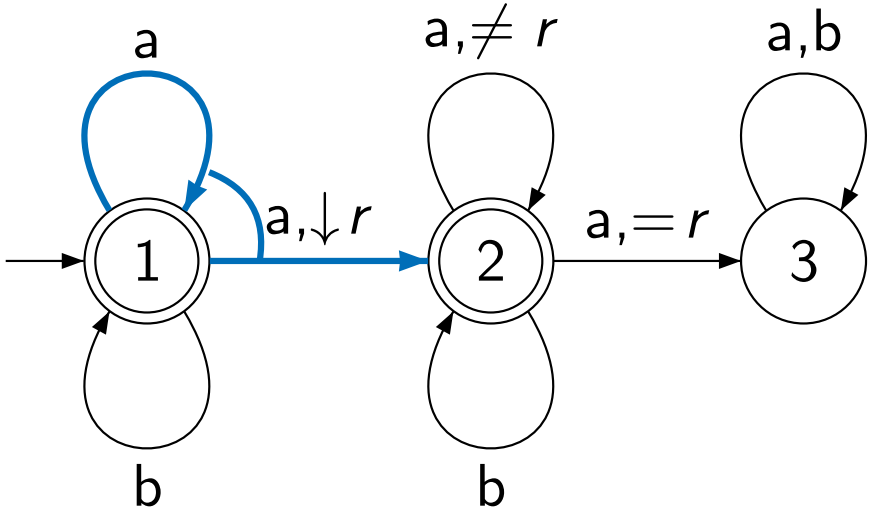

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$$\{(1,\perp),(2,d_2),(2,d_3),(2,d_1),(3,d_3)\} \xrightarrow[d_4]{a} \ldots$$

Alternating Single-Register Automata

Decision Problems

Theorem (Demri & Lazic, 2008)

The non-emptiness problem for alternating single-register automata is decidable.

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Two proof variants

- Reduction to the non-emptiness problem of counter automata with insertion errors
- The semantical graph is a well-structured transition system

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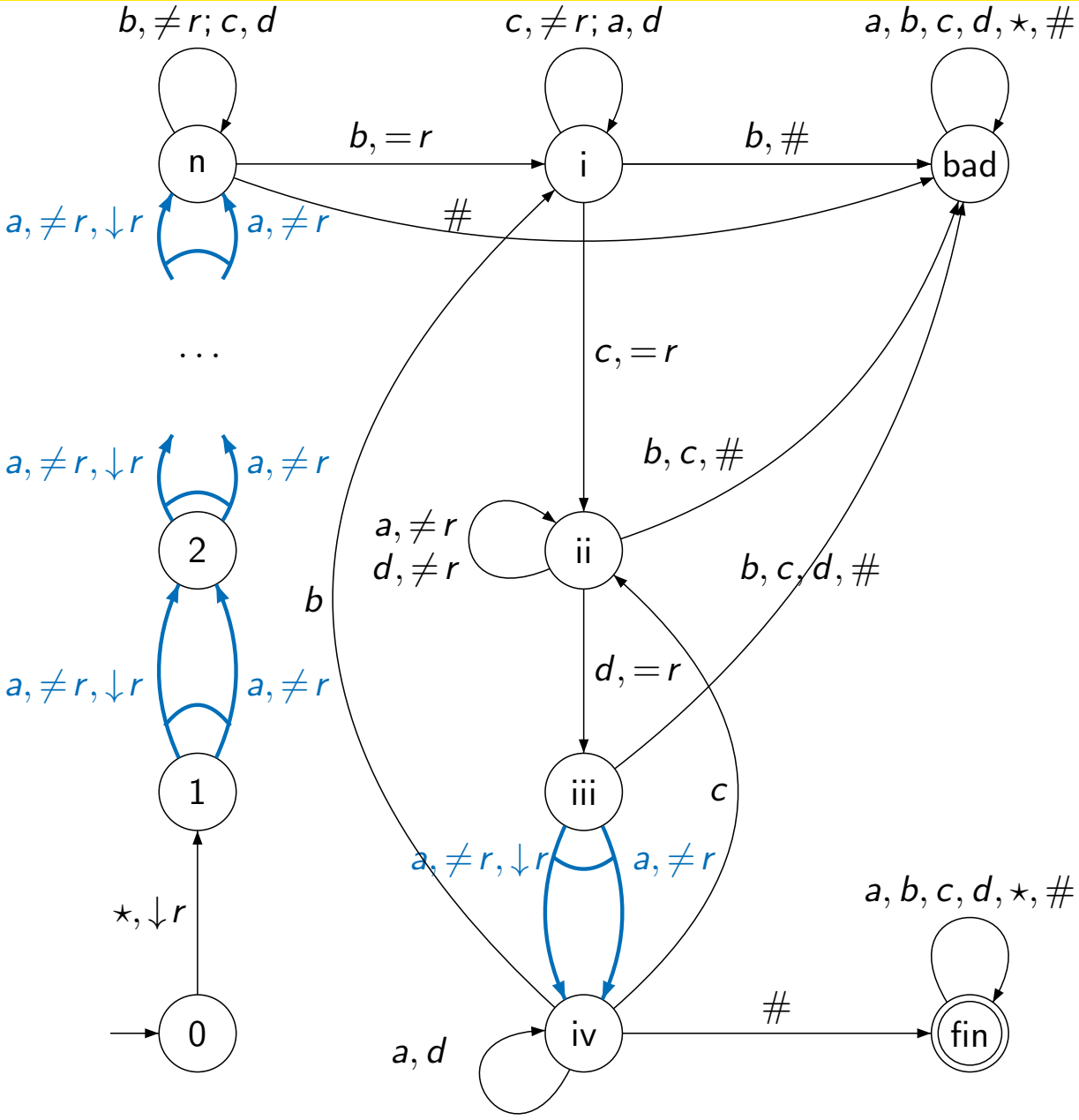
Bad complexity!

The problem is Ackermann-complete.

Alternating Single-Register Automata

Quiz: Which fast-growing function does this family of automata simulate?



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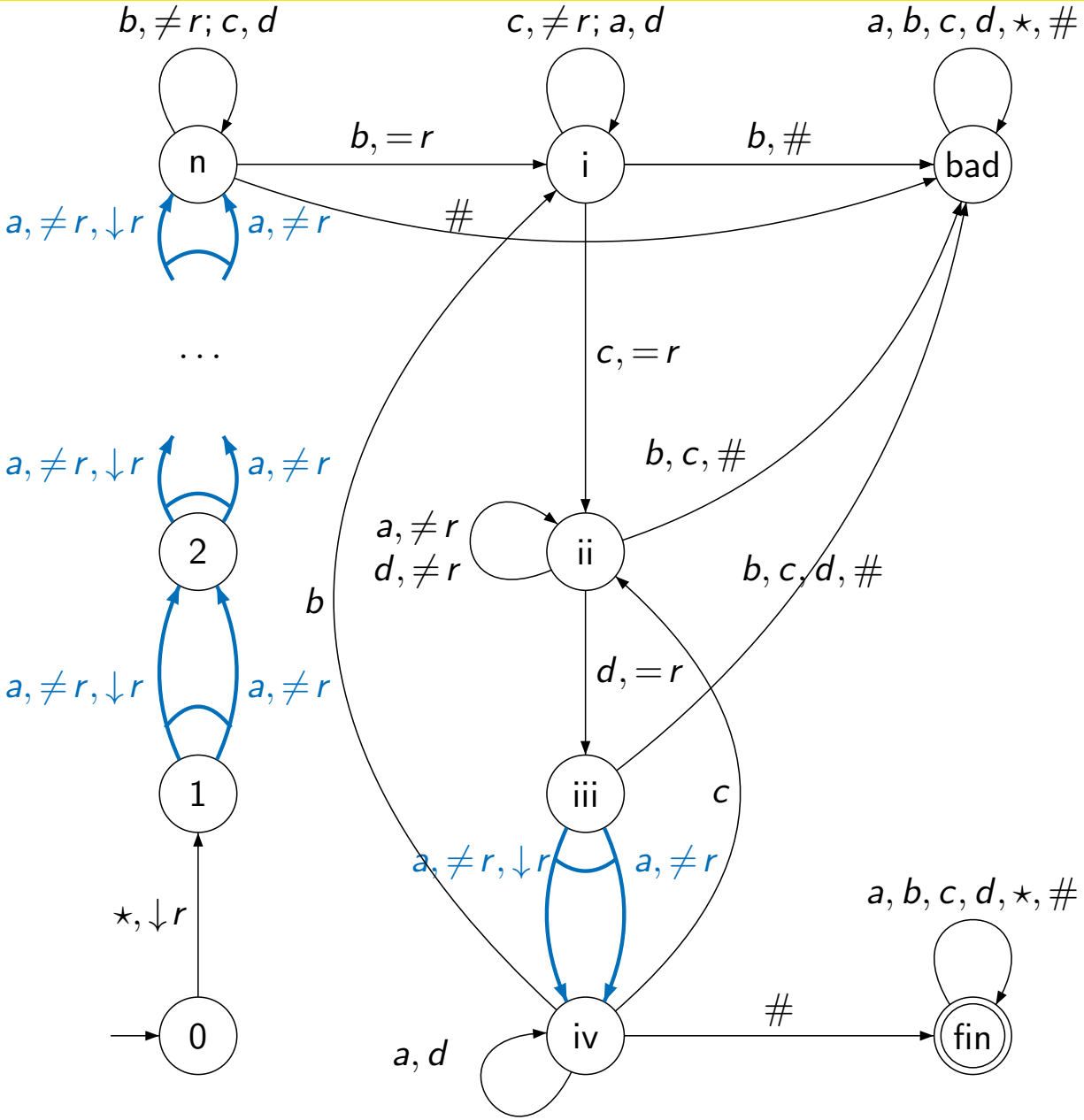

$\{(0, \perp)\}$
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$\downarrow a, 3$
$\ldots$

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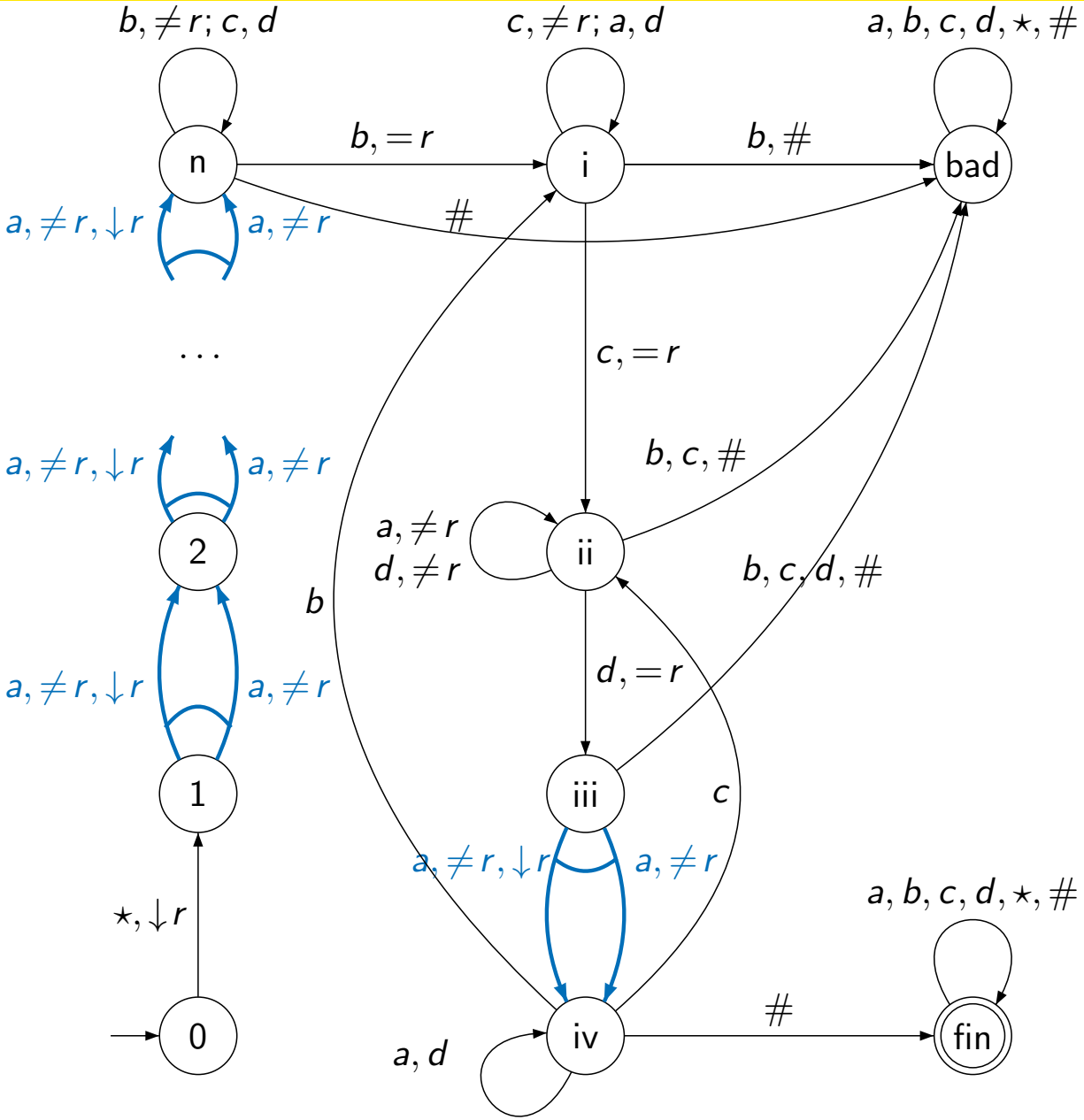

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$\downarrow a, 3$
...

A run is successful only if
a set only contains state fin!

Literature

- Demri & Lazic: *LTL with the Freeze Quantifier and Register Automata.* ACM Transactions on Computational Logic, 2008.
- Neven, Schwentick & Vianu: *Finite State Machines for Strings over Infinite Alphabets*. ACM Transactions on Computational Logic, 2001.
- Figueira: *Alternating register automata on finite words and trees.* Logical Methods in Computer Science, 2012.