Existential MSO over Two Successors Is Strictly Weaker than over Linear Orders

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Abstract

As is well-known a language of finite words, considered as labeled linear orders, is definable in monadic second-order logic (MSO) iff it is definable in the existential fragment of MSO, that is the quantifier alternation hierarchy collapses. Even more, it does not make a difference if we consider existential MSO over a linear order or a successor relation only. In this note we show that somewhat surprisingly the latter does not hold if we just add a second linear order and consider finite relational structures with two linear orders, so-called texts.

Key words: existential monadic second-order logic, Ehrenfeucht-Fraïssé game, Ajtai-Fagin game, texts, successor structures

1. Introduction

The fundamental result of Büchi and Elgot [Büc60, Elg61] states that the class of regular languages of finite words and the class of languages definable in monadic second-order logic (MSO) coincide. A consequence of its proof is that a language of finite words, considered as labeled linear orders, is definable in MSO iff it is definable in the existential fragment of MSO, that is the quantifier alternation hierarchy collapses. Even more, it does not make a difference if we consider existential MSO over a linear order or a successor relation only.

In contrast, Fagin [Fag75] showed that connectivity separates existential MSO from universal monadic second-order logic for the class of graphs. Furthermore, Matz, Schweikardt and Thomas [MST02] could show that for the class of grids and for the class of graphs the monadic quantifier alternation hierarchy is in fact strict, answering a question of Fagin. Matz [Mat98] also showed that for grids existential MSO (over two successor relations) can be separated from existential MSO over the transitive closure of the successor relations which is strictly weaker than the full MSO.

In this research note we show that for certain classes of finite structures with two linear orders the situation is in between the situation of words and grids. More precisely,

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even though existential MSO over two linear orders is as expressive as the full MSO it can be separated from existential MSO over two successor relations only. We give a language and show that it separates the two fragments by using a technique of Ajtai and Fagin [AF90]. The class of finite structures with two linear orders has been considered in the literature where it is known as the class of texts.

Texts have been introduced by Ehrenfeucht and Rozenberg [ER90]. The theory of texts originates in the theory of 2-structures (cf. [EHR99]) and it turns out that texts represent an important subclass of 2-structures, so-called T-structures [ER93]. Moreover, Ehrenfeucht and Rozenberg proposed texts as a well-suited model for natural texts that may carry in its tree-like structure grammatical information [ER93, p.264].

A number of authors [EtPR94, HtP96, HtP97] have investigated classes of text languages such as the families of context-free, equational or recognizable text languages and developed a language theory. In particular, the result of Büchi and Elgot on the coincidence of recognizable and MSO-definable languages in MSO has been extended to texts [HtP97]. Again, for certain classes of texts the quantifier alternation hierarchy collapses and MSO over two linear orders is expressively equivalent to its existential fragment (cf. [Mat07]).

2. Preliminaries

Texts. In the following let Δ be a finite alphabet. We consider texts as introduced by Ehrenfeucht and Rozenberg [ER90]. A text is a word over Δ equipped with an additional linear order; more precisely it is defined as follows:

Definition 1. A *text* over Δ is a tuple $(V, \lambda, \leq_1, \leq_2)$ where $\leq_1$ and $\leq_2$ are linear orders over the finite but non-empty domain V and $\lambda : V \to \Delta$ is a labeling function.

We consider texts as relational structures where the relations are given by the labeling and by $\leq_1$ and $\leq_2$. We collect all texts in $\mathrm{TXT}(\Delta)$, where as usual we identify isomorphic texts. For this reason we assume that for a text $(V, \lambda, \leq_1, \leq_2)$ we have $V = [n] := \{1, \ldots, n\}$ for some positive integer n and that the first order $\leq_1$ coincides with the usual order on $[n]$. We may thus represent a text with domain $[n]$ by the pair $(\lambda(1) \ldots \lambda(n), (i_1, \ldots, i_n))$ where the sequence $(i_1, \ldots, i_n)$ represents the successor structure of $\leq_2$. When visualizing a text in a picture we will often omit $\leq_1$ and assume the nodes to be ordered from the left to the right.

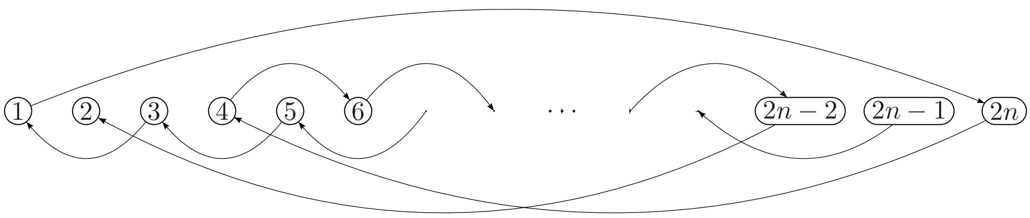
Let us start with defining an algebraic structure on the set of texts following [HtP97]. A *biorder* is a pair of two linear orders over a common finite domain, i.e. a text without labeling. Again we identify isomorphic biorders and assume that the domain equals $[n]$ for some positive integer n. Consequently we represent a biorder with domain $[n]$ by its successor structure $(i_1, \ldots, i_n)$. When visualizing a biorder we again often omit its first order. Each biorder π with domain $[n]$ defines an operation on texts – we obtain a new text $\pi(\tau_1, \ldots, \tau_n)$ by substituting given texts $\tau_1, \ldots, \tau_n$ into the nodes of the biorder. That is, we consider the disjoint union of the domains where, given two elements of the union, if they correspond to the same text τ_i, their order is determined by τ_i, otherwise, if they correspond to τ_i and τ_j for some $i \neq j$, then their order is given by the order of i and j in π. The texts $\tau_1, \ldots, \tau_n$ then become intervals of the new text in both the first and the second order. These kind of operations for graphs is known as modular decomposition and has been rediscovered several times (cf. [MR84]).

Example 2. There are two biorders $h = (1, 2)$, $v = (2, 1)$ of cardinality two.

Consider the texts $\tau_1 = (ab, (2, 1))$, $\tau_2 = (cd, (1, 2))$ and $\tau_3 = (ca, (2, 1))$. Then $v(\tau_1, \tau_2) = (abcd, 3421)$ and $h(v(\tau_1, \tau_2), \tau_3) = (abcdca, (3, 4, 2, 1, 6, 5))$.

A subset of the domain of some text being an interval of both orders is called a *clan*. A biorder is *primitive* if it has only trivial clans, i.e. the singletons and the domain itself. Clearly, the two biorders h and v of cardinality two are both primitive. Let in the following Σ be a set of primitive biorders and let $\mathrm{TXT}_\Sigma(\Delta)$ be the set of all texts generated from the singleton texts, i.e. from Δ, using Σ. If Σ comprises all primitive biorders, then $\mathrm{TXT}_\Sigma(\Delta) = \mathrm{TXT}(\Delta)$. We consider $\mathrm{TXT}_\Sigma(\Delta)$ as a Σ-algebra. Let $T_\Sigma(\Delta)$ be the set of terms over Σ with constants from Δ and let $\mathrm{txt} : T_\Sigma(\Delta) \to \mathrm{TXT}_\Sigma(\Delta)$ be the natural epimorphism assigning to each term over Σ and Δ its value. Applying the theory of 2-structures developed by Ehrenfeucht and Rozenberg [ER90] one obtains that $\mathrm{TXT}_\Sigma(\Delta)$ is the free algebra in the variety of all Σ-algebras where (the two biorders of cardinality two) h and v satisfy the associativity law [HtP97]. Thus, different preimages of a text $\tau \in \mathrm{TXT}(\Delta)$ under the natural epimorphism txt only differ with respect to these two associativity laws. Hoogeboom and ten Pas [HtP97] considered finite sets Σ and called in this case $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ a language of *bounded primitivity*. In particular, if $\Sigma = \{h, v\}$, then $\mathrm{TXT}_\Sigma(\Delta)$ is the free bisemigroup which has been considered by Ésik and Németh [ÉN04].

Example 3. Let $n \geq 3$ and let $\pi_n = (2n-1, 2n-3, \ldots, 1, 2n, 4, 6, 8, \ldots, 2n-2, 2)$ a biorder of length $2n$.



Observe that for any two vertices $i, i+1$ of π_n the smallest clan containing $i, i+1$ contains 1 and $2n$ since either $i \leq_2 1 \leq_2 2n \leq_2 i+1$ or $i+1 \leq_2 1 \leq_2 2n \leq_2 i$. Thus for any $n \geq 3$, π_n does not contain non-trivial clans and is hence primitive. This shows that the cardinality of the set of all primitive biorders is $\aleph_0$.

Monadic Second-Order Logic. We review classical MSO logic for texts for one thing over two linear orders $\leq_1$, $\leq_2$ for another over two successor relations S_1, S_2. For this we interpret a text $\tau = ([n], \lambda, \leq_1, \leq_2)$ as relational structures consisting of the domain $[n]$ together with the unary relations $\mathrm{Lab}_a = \{i \in [n] \mid \lambda(i) = a\}$ for all $a \in \Delta$ and the binary relations $\leq_1$ and $\leq_2$ (S_1 and S_2, respectively). Here $S_1 = \{(i, i+1) \in [n] \times [n]\}$ and $S_2 = \{(i, j) \in [n] \times [n] \mid i <_2 j \text{ and there is no } i <_2 k <_2 j\}$. The syntax of formulae of $\mathrm{MSO}(\leq_1, \leq_2)$ is given by the following grammar:

$$\varphi \quad ::= \quad x = y \mid \mathrm{Lab}_a(x) \mid x \leq_1 y \mid x \leq_2 y \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where x, y are first-order variables, X is a set variable and a ranges over Δ. The syntax of $\mathrm{MSO}(S_1, S_2)$ is defined just by replacing $\leq_1$ by S_1 and $\leq_2$ by S_2.

A closed formula φ, i.e. one without free variables, is a called a *sentence*. We write $\tau \models \varphi$ if φ holds in the text τ and denote $\mathscr{L}(\varphi) = \{\tau \in \mathrm{TXT}_\Sigma(\Delta) \mid \tau \models \varphi\}$. Let $Z \subseteq \mathrm{MSO}(\leq_1, \leq_2)$ (resp. $Z \subseteq \mathrm{MSO}(S_1, S_2)$). A text language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is *Z-definable* iff $L = \mathscr{L}(\varphi)$ for a sentence $\varphi \in Z$. First-order formulae, i.e. formulae containing only quantification over first-order variables are collected in $\mathrm{FO}(\leq_1, \leq_2)$ and $\mathrm{FO}(S_1, S_2)$, respectively. The class $\mathrm{EMSO}(\leq_1, \leq_2)$ (respectively $\mathrm{EMSO}(S_1, S_2)$) of existential monadic second-order logic consists of all formulae φ of the form $\exists X_1. \ldots \exists X_c. \psi$ where c is a natural number and $\psi \in \mathrm{FO}(\leq_1, \leq_2)$ (respectively $\psi \in \mathrm{FO}(S_1, S_2)$). Clearly, as the transitive closure of S_1 and S_2 can be expressed in $\mathrm{MSO}(S_1, S_2)$ we have that $\mathrm{MSO}(S_1, S_2)$ and $\mathrm{MSO}(\leq_1, \leq_2)$ are expressively equivalent.

Example 4. The following first-order formula states that the underlying biorder of a text is not primitive (x and y correspond to the first and last element with respect to $\leq_1$ of a clan).

$$\begin{aligned}&\exists x. \exists y.\ (\exists z.\ z <_1 x \ \vee\ y <_1 z) \ \wedge\ x <_1 y \ \wedge \\ &\qquad \wedge \forall x', y', z'.\ (x \leq_1 x', y' \leq_1 y \ \wedge\ x' \leq_2 z' \leq_2 y') \rightarrow (x \leq_1 z' \leq_1 y)\end{aligned}$$

Here, $\varphi \rightarrow \psi$ abbreviates $\neg\varphi \vee \psi$, $\forall z.\varphi$ abbreviates $\neg(\exists z.\neg\varphi)$, $x <_i y$ abbreviates $x \leq_i y \wedge (\neg x = y))$, $x \leq_1 z \leq_1 y$ abbreviates $x \leq_1 z \wedge z \leq_1 y$ and $x \leq_1 x', y' \leq_1 y$ abbreviates $x \leq_1 x' \leq_1 y \wedge x \leq_1 y' \leq_1 y$.

The *quantifier depth* $\mathrm{qd}(\varphi)$ of a first-order formula φ is recursively defined as follows. If φ does not contain quantifiers, then $\mathrm{qd}(\varphi) = 0$. Moreover, $\mathrm{qd}(\neg\varphi) = \mathrm{qd}(\varphi)$ and $\mathrm{qd}(\varphi \vee \psi) = \max(\mathrm{qd}(\varphi), \mathrm{qd}(\psi))$ and $\mathrm{qd}(\exists x.\varphi) = \mathrm{qd}(\varphi) + 1$.

In the following we will separate the expressive power of existential monadic second-order logic $\mathrm{EMSO}(\leq_1, \leq_2)$ from existential monadic second-order logic over two successors $\mathrm{EMSO}(S_1, S_2)$ relative to texts. That is, we will give a sentence $\varphi \in \mathrm{EMSO}(\leq_1, \leq_2)$ such that there is no sentence $\psi \in \mathrm{EMSO}(S_1, S_2)$ with $\tau \models \varphi$ iff $\tau \models \psi$ for all $\tau \in \mathrm{TXT}_\Sigma(\Delta)$. For this separation result we use the technique used in [FSV95] to show that graph connectivity is not definable in existential monadic second-order logic over graphs. For the convenience of the reader we first recall Hanf's Lemma [Han65] in the context of $\mathrm{FO}(S_1, S_2)$ over texts.

Hanf's Sphere Lemma. Let $\tau = ([n], \lambda, \leq_1, \leq_2)$ be a text and let $i, j \in [n]$. We say i and j are *adjacent* if $(i, j) \in S_1 \cup S_2$ or $(j, i) \in S_1 \cup S_2$. We say i and j have *distance* r, denoted $d(i, j) = r$, if the usual distance of i and j in the undirected graph given by the edge relation $E = \{(i, j), (j, i) \in [n] \times [n] \mid i, j \text{ are adjacent}\}$ is r. We consider τ as a graph with $\{1, 2\}$-colored edges given by the successor relations S_1, S_2. The sphere $\mathrm{Sph}_\tau(i, r)$ of radius r around i is the subgraph of τ induced by the vertices j with $d(i, j) \leq r$. Clearly, as each vertex has at most four adjacent vertices, for fixed r there is only a finite number of isomorphism types (r-types) of such spheres.

Let r, t be positive integers. We say two texts τ_1 and τ_2 are *(r, t)-equivalent* if for every r-type ι, either τ_1 and τ_2 have the same number of vertices with r-type ι or both have at least t vertices with r-type ι. Now Hanf's Lemma formulated for texts reads as follows.

Lemma 5 (Sphere Lemma [Han65]). *For any k there are $r_k, t_k \geq 0$ such that for any (r_k, t_k)-equivalent texts τ_1, τ_2 we have $\tau_1 \models \varphi$ iff $\tau_2 \models \varphi$ for all $\varphi \in \mathrm{FO}(S_1, S_2)$ with $\mathrm{qd}(\varphi) \leq k$.*

Remark. From the proof one obtains, that r_k can be chosen to be 3^k and t_k can be chosen to be $k \cdot 4^{3^k}$ (as we consider texts as graphs of degree 4).

Ajtai-Fagin Games. We now recall Ajtai-Fagin games which were initially introduced in [AF90] and also used in [FSV95]; see also [Fag96] for a survey on related results. We will adapt the definition to texts but we will only give an informal definition of the Ajtai-Fagin (c, k)-game over some $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ where c, k are positive integers. There are two players, called *Spoiler* and *Duplicator*. The rules of the game are as follows:

1. Duplicator selects some $\tau_1 \in L$.
2. Spoiler colors the vertices of τ_1 with colors from $1, \ldots, c$ (disjoint from Δ).
3. Duplicator selects some $\tau_2 \in \mathrm{TXT}_\Sigma(\Delta) \setminus L$ and colors it with colors from $1, \ldots, c$.
4. Spoiler and Duplicator play a k-round first-order Ehrenfeucht-Fraïssé game on the *colored* texts τ_1 and τ_2 (considered as colored graphs where the edges are given by the binary relations S_1, S_2).

We will not define Ehrenfeucht-Fraïssé games here as they are widely known, have been defined several times in the literature (see e.g. [FSV95, Tho97]) and as we will not need them explicitly. The only thing we need to know is that Duplicator has a winning strategy in the k-round first-order Ehrenfeucht-Fraïssé game iff $\tau_1 \models \varphi \Leftrightarrow \tau_2 \models \varphi$ for all $\varphi \in \mathrm{FO}(S_1, S_2)$ with $\mathrm{qd}(\varphi) \leq k$. Moreover, Spoiler has a winning strategy in the Ehrenfeucht-Fraïssé game iff Duplicator does not have one. The following result is Theorem 4.5 of [AF90] formulated now for texts.

Theorem 6 ([AF90]). *A language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is $\mathrm{EMSO}(S_1, S_2)$-definable iff there are $c, k \geq 1$ such that Spoiler has a winning strategy in the Ajtai-Fagin (c, k)-game over L.*

3. The Separating Language

Let us assume that $h, v \in \Sigma$ (cf. Example 2). We will use Theorem 6 and Lemma 5 to separate the expressive power of $\mathrm{EMSO}(S_1, S_2)$ from the expressive power of $\mathrm{EMSO}(\leq_1, \leq_2)$. In order to do this we will give a language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ definable in $\mathrm{EMSO}(\leq_1, \leq_2)$ and construct a winning strategy for Duplicator for the Ajtai-Fagin (c, k)-game over L for arbitrarily chosen c and k. When showing that the strategy constructed is indeed winning for Duplicator we use Lemma 5 and show that τ_1 and τ_2 of the strategy we construct are (r_k, t_k)-equivalent.

We will now define the separating language L by giving a property on the generating terms. We let $\tau \in L$ iff there is a term t over Σ and Δ such that $\mathrm{txt}(t) = \tau$, only the operations h, v occur in t and, moreover, if $v(t_1, t_2)$ is a subterm of t, then neither t_1 nor t_2 has a subterm of the form $h(t_1', t_2')$ for some terms t_1', t_2'.

Proposition 7. *The language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is $\mathrm{FO}(\leq_1, \leq_2)$-definable.*

PROOF. Let $\tau \in \mathrm{TXT}_\Sigma(\Delta)$. As pointed out on page 3, all terms in $T_\Sigma(\Delta)$ with value τ only differ with respect to the associativity of h and v. Following [HtP97], we call the term where the brackets are in right-most form the r-shape of τ, denoted $\mathrm{sh}(\tau)$. In [HtP97], a (uniform) 1-dimensional MSO interpretation of the r-shape in two disjoint copies of the text itself was given. More precisely, it was shown that $\mathrm{sh} : \mathrm{TXT}_\Sigma(\Delta) \to T_\Sigma(\Delta)$ is a 2-copying MSO-transduction without parameters (cf. [Cou94]). This yields a translation of formulae such that a formula φ over terms can be transformed into a formula ψ over texts with $\mathscr{L}(\psi) = \mathrm{sh}^{-1}(\mathscr{L}(\varphi))$. Now any set quantification in the interpreting formulae of [HtP97] only concerns intervals of the first order and hence can be transformed into first-order quantification by identifying an interval with the first and last element (cf. Example 4 where x and y represent the first and the last element of a clan). This yields a translation of formulae that transforms FO formulae over terms into formulae in $\mathrm{FO}(\leq_1, \leq_2)$ over texts. Clearly, it can be expressed in FO over terms that only the operations h and v occur in a term t and that if $v(t_1, t_2)$ is a subterm of t, then neither t_1 nor t_2 is of the form $h(t'_1, t'_2)$ for some terms t'_1, t'_2. But the preimage under sh of the set of terms defined in this way is exactly L, which is hence $\mathrm{FO}(\leq_1, \leq_2)$-definable. □

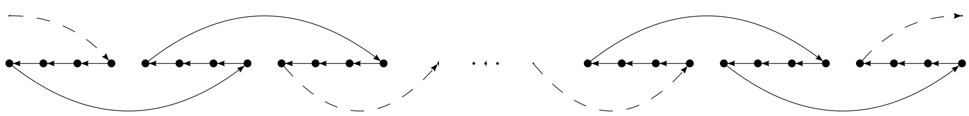
Proposition 8. *The language* $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ *is not* $\mathrm{EMSO}(S_1, S_2)$*-definable.*

PROOF. Suppose for contradiction that L is $\mathrm{EMSO}(S_1, S_2)$-definable. Then by Theorem 6 there are c, k such that Spoiler wins the Ajtai-Fagin (c, k)-game over L. We now construct a winning strategy for Duplicator – a contradiction.

Let r_k be given by Lemma 5 and let $n_c = c^{(r_k+1)^2}$. Moreover, let $a \in \Delta$. Duplicator chooses τ_1 to be the text given by the following term:

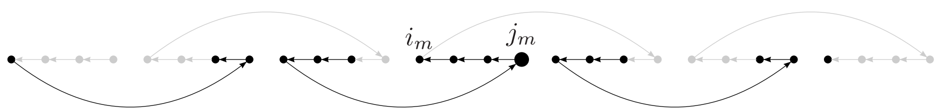
$$\underbrace{h(h(\ldots h}_{(2r_k+1)\cdot(n_c^2+2)}\ (\underbrace{v(v(\ldots v}_{r_k}(a,a)\ldots),a),\underbrace{v(v(\ldots v}_{r_k}(a,a)\ldots),a))\ldots,\underbrace{v(v(\ldots v}_{r_k}(a,a)\ldots),a))$$

For $r_k = 3$ a fragment of τ_1 looks as follows:

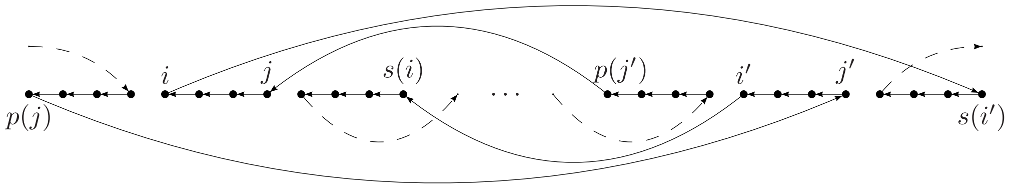


Let the domain of τ_1 be $\big[((2r_k+1)\cdot(n_c^2+2)+1)\cdot(r_k+1)\big]$. We will call a clan of τ_1 a *building block* if it corresponds to the subterms $v(v(\ldots v(a,a)\ldots),a)$, i.e. to the clan with the domain $\{m(r_k+1)+1, \ldots, (m+1)(r_k+1)\}$ for $0 \leq m \leq (2r_k+1)\cdot(n_c^2+2)$.

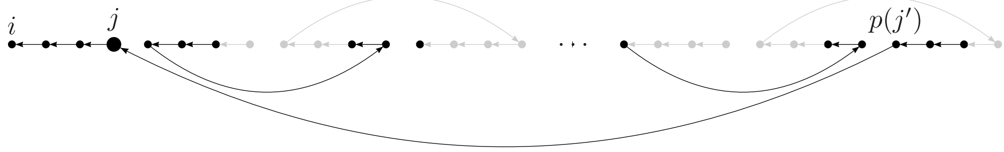
Now Spoiler colors τ_1 with c colors. Let us then consider the following sequence of pairs of vertices: $[i_m, j_m] = [(2mr_k+m)(r_k+1)+1, (2mr_k+m+1)(r_k+1)]$ for $1 \leq m \leq n_c^2+1$. These vertices are the least and greatest vertex with respect to $\leq_1$ of some building blocks. The following picture shows $\mathrm{Sph}_{\tau_1}(j_m, r_k)$ for $r_k = 3$.



Clearly, if we forget about the coloring, then for any $1 \leq m_1, m_2 \leq n_c^2 + 1$ we have that $\mathrm{Sph}_{\tau_1}(i_{m_1}, r_k)$ and $\mathrm{Sph}_{\tau_1}(i_{m_2}, r_k)$ ($\mathrm{Sph}_{\tau_1}(j_{m_1}, r_k)$ and $\mathrm{Sph}_{\tau_1}(j_{m_2}, r_k)$, respectively) are isomorphic, each of them consisting of $(r_k + 1)^2$ vertices. Thus the r_k-type with colors can be fully described by a sequence of colors of length $(r_k + 1)^2$. With $n_c := c^{(r_k+1)^2}$ there are $n_c^2 + 1$ such pairs. By the pigeon hole principal there must be two pairs $[i, j]$ and $[i', j']$ such that i has the same r_k-type as i' and j has the same r_k-type as j'. Now let $p(j)$ and $p(j')$ be the predecessor with respect to $\leq_2$ of j and j', respectively, and let $s(i)$ and $s(i')$ be the successor with respect to $\leq_2$ of i and i', respectively. To obtain τ_2, Duplicator now changes the edges in τ_1 such that $p(j)$ becomes the predecessor of j' and $p(j')$ becomes the predecessor of j as well as $s(i)$ becomes the successor of i' and $s(i')$ becomes the successor of i. The text τ_2 we obtain in this way is depicted in the following picture.



The sphere $\mathrm{Sph}_{\tau_2}(j, r_k)$ for $r_k = 3$ looks as in the following picture. It has the same r_k-type as $\mathrm{Sph}_{\tau_1}(j, r_k)$ since j' has the same r_k-type as j in τ_1. We will now argue that this holds true for all vertices in $[((2r_k + 1) \cdot (n_c + 2)^2 + 1) \cdot (r_k + 1)]$.



Let a *path* be a sequence $(v_1, \ldots, v_m)$ of adjacent but mutually distinct vertices and let its *length* be $m - 1$. Let $l \in [((2r_k + 1) \cdot (n_c + 2)^2 + 1) \cdot (r_k + 1)]$. Observe that for any vertex l' of $\mathrm{Sph}_{\tau_1}(l, r_k)$ the path from l to l' is unique since the circles in τ_1 all have length $\geq 2(r_k + 1)$ as for a circle one has to traverse at least two building blocks (here a circle is a path $(v_1, \ldots, v_m)$ with $m \geq 3$ such that v_1 and v_m are adjacent; its length is m). Let us consider a circle in τ_2. If it contains only adjacent vertices which are also adjacent in τ_1, then clearly the circle has length at least $2(r_k + 1)$. Now, observe that the shortest path between i and $s(i')$ of length at least 3 must have length $\geq 2r_k + 1$ as we have to traverse two building blocks. Similar arguments can be used for j' and $p(j')$. Between j and $p(j')$ the shortest path having length at least 3 uses the edge between $s(i)$ and i' (provided $r_k \geq 3$), since there are at least $2r_k - 1$ building blocks between them. Thus again we have to traverse two building blocks and conclude that the path is of length $2r_k + 3$. Again we may argue similarly for i' and $s(i)$. We conclude that any circle in τ_2 must have length at least $2(r_k + 1)$. Hence for any vertex l' of $\mathrm{Sph}_{\tau_2}(l, r_k)$ the path from l to l' is unique.

We want to show that $\mathrm{Sph}_{\tau_1}(l, r_k)$ and $\mathrm{Sph}_{\tau_2}(l, r_k)$ are isomorphic. Consider the graphs induced by the vertices of two paths in either $\mathrm{Sph}_{\tau_1}(l, r_k)$ or $\mathrm{Sph}_{\tau_2}(l, r_k)$, one from l to some l' and one from l to some l''. If there is an isomorphism between them mapping l to l, then $l' = l''$ as the successor relations are injective partial functions. Hence, it suffices to show that for the path from l to l' in $\mathrm{Sph}_{\tau_2}(l, r_k)$ there is a path from l to l'' in $\mathrm{Sph}_{\tau_1}(l, r_k)$ and vice versa such that their vertices induce isomorphic graphs.

Let us consider a path from l to some l' in $\mathrm{Sph}_{\tau_2}(l, r_k)$. If it contains only vertices also adjacent in τ_1, we find the same path in $\mathrm{Sph}_{\tau_1}(l, r_k)$. Otherwise observe that since its length is $\leq r_k$ the path contains at most one pair of vertices not adjacent in τ_1. We cut the path between these points obtaining two paths. Assume that the first is a path $p = (l, \ldots, i)$ leading from l to i and the second is a path $p' = (s(i'), \ldots, l')$ leading from $s(i')$ to l'. (We may argue similarly in the other cases.) Clearly, the first path can be found in $\mathrm{Sph}_{\tau_1}(l, r_k)$. Since i and i' have the same r_k-type we conclude that $s(i)$ and $s(i')$ have the same $(r_k - 1)$-type. Hence we can find in $\mathrm{Sph}_{\tau_1}(s(i), r_k - 1)$ a path p'' whose vertices induce a graph isomorphic to the graph induced by p'. Thus in $\mathrm{Sph}_{\tau_1}(l, r_k)$ we find the path pp'' which induces a graph isomorphic to the graph induced by the path of consideration. We may argue similarly to show that for each path in $\mathrm{Sph}_{\tau_1}(l, r_k)$ there is an isomorphic path in $\mathrm{Sph}_{\tau_2}(l, r_k)$.

Thus we can conclude that for any r_k-type ι we have that l has r_k-type ι in τ_1 iff l has r_k-type ι in τ_2. Hence τ_1 and τ_2 are (r_k, t_k)-equivalent and thus by Hanf's Lemma (Lemma 5) we have $\tau_1 \models \varphi$ iff $\tau_2 \models \varphi$ for all $\varphi \in \mathrm{FO}(S_1, S_2)$ with $\mathrm{qd}(\varphi) \leq k$. We conclude that Duplicator has a winning strategy in the k-round first-order Ehrenfeucht-Fraïssé game on τ_1 and τ_2.

It remains to show $\tau_2 \in \mathrm{TXT}_\Sigma(\Delta) \setminus L$. For this we partition τ_2 into five clans: $A = \tau_{2|[i-1]}$, the restriction of τ_2 to $\{1, \ldots, i-1\}$, $B = \tau_{2|\{k|i \leq k \leq j\}}$, $C = \tau_{2|\{k|j<k<i'\}}$, $D = \tau_{2|\{k|i' \leq k \leq j'\}}$ and $E = \tau_{2|\{k|j'<k\}}$. Clearly, each of the texts A, C and E is a member of L and is represented by some term, say t_A, t_C and t_E, respectively. Each of the terms t_A, t_C and t_E has the form $h(t_1, t_2)$ for some terms t_1 and t_2. Contrarily, B is isomorphic to D, is also a member of L, but represented by $t_B = v(v(\ldots v(a, a) \ldots), a)$. We conclude that the term

$$h\big(h(t_A, v(t_B, v(t_C, t_B)))\big), t_E\big)$$

has value τ_2. Since it contains the subterm $v(t_C, t_B)$ and since $\mathrm{TXT}_\Sigma(\Delta)$ is free in the variety of all Σ-algebras where h and v are associative, we conclude that $\tau_2 \in \mathrm{TXT}_\Sigma(\Delta) \setminus L$. □

Remark. If $\Sigma = \{h\}$ or $\Sigma = \{v\}$ we are essentially back in the word case and $\mathrm{MSO}(\leq_1, \leq_2)$ is as expressive as $\mathrm{EMSO}(S_1, S_2)$.

Application to Recognizable Sets of Texts. When defining recognizable subsets for certain classes of structures, there are at least two very different approaches.

First, the *algebraic approach*. There, given an algebraic structure on the class of interest the recognizable subsets are defined to be precisely the unions of congruence classes of finite index congruences. This notion corresponds to the famous Myhill-Nerode characterization of regular word languages and was first considered in its general setting by Mezei and Wright [MW67]. This definition was also the one used by Hoogeboom and ten Pas [HtP97] to define *recognizable* text languages $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ where the algebraic structure is given by the operations in Σ. Recall that each biorder in Σ with domain $[n]$ defines a n-ary operation on texts (cf. page 2 and 3). Hoogeboom and ten Pas showed that if Σ is finite, then a text language $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is recognizable iff it is $\mathrm{MSO}(\leq_1, \leq_2)$-definable. It follows that this is the case iff L is $\mathrm{EMSO}(\leq_1, \leq_2)$-definable (cf. [Mat07]).

The second approach, which we may call the *tiling approach*, was established by Thomas [Tho91]. It arises from the combinatorial structure of each element of the class, i.e. for texts e.g. from S_1, S_2 and the labeling, and corresponds to the characterization of regular languages by means of projections of local languages. Thomas defined so-called graph acceptors consisting of a finite set of states, a finite set of r-types colored by the states and a boolean combination of the form "there are $\geq n$ copies of the colored r-type ι". A structure is then accepted if it can be tiled coherently by the colored r-types fulfilling the constraint. This definition, however, is directly applicable only to classes of structures of bounded degree. For words Thomas therefore only considers the successor relation rather than the linear order. Similar for the class of texts the most natural way to bound the degree is to consider the successor relations S_1, S_2. The general result of Thomas [Tho91] then gives that $L \subseteq \mathrm{TXT}_\Sigma(\Delta)$ is accepted by some graph acceptor iff it is $\mathrm{EMSO}(S_1, S_2)$-definable.

We thus get from Proposition 7 and Proposition 8 the following corollary:

Corollary 9. *Let Σ be finite such that $h, v \in \Sigma$. Then the class of text languages accepted by some graph acceptor is a proper subclass of the class of recognizable text languages.*

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