Weighted Logics for Nested Words and Algebraic Formal Power Series *

Christian Mathissen **

Institut für Informatik, Universität Leipzig
D-04009 Leipzig, Germany
mathissen@informatik.uni-leipzig.de

Abstract. Nested words, a model for recursive programs proposed by Alur and Madhusudan, have recently gained much interest. In this paper we introduce quantitative extensions and study nested word series which assign to nested words elements of a semiring. We show that regular nested word series coincide with series definable in weighted logics as introduced by Droste and Gastin. For this, we establish a connection between nested words and series-parallel-biposets. Applying our result, we obtain a characterization of algebraic formal power series in terms of weighted logics. This generalizes a result of Lautemann, Schwentick and Thérien on context-free languages.

1 Introduction

Model checking of finite state systems has become an established method for automatic hardware and software verification and led to numerous verification programs used in industrial application. In order to verify recursive programs, it is necessary to model them as pushdown systems rather than finite automata. This has motivated Alur and Madhusudan [2, 3] to define the classes of nested word languages and visibly pushdown languages which is a proper subclass of the class of context-free languages and exceeds the regular languages. These classes gained much interest and set a starting point for a new research field (see e.g. [1, 4] among many others).

The goal of this paper will be: 1. to introduce a quantitative automaton model and a quantitative logic for nested words that are equally expressive, 2. to establish a connection between nested words and series-parallel-biposets which were studied by Ésik and Németh [11] and others, 3. to give a characterization of the important class of algebraic formal power series by means of weighted logics.

In order to be able to model quantitative aspects, extensions of existing models were investigated, such as weighted automata or probabilistic pushdown automata. In this paper we introduce and investigate weighted nested word automata which we propose as a quantitative model for sequential programs with

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recursive procedure calls. Due to the fact that we define them over arbitrary semirings they are very flexible and can model, for example, probabilistic or stochastic systems. As the first main result of this paper we characterize their expressiveness using weighted logics, generalizing a result of Alur and Madhusudan. Weighted logics were introduced by Droste and Gastin [7]. They enriched the classical language of monadic second-order logic with values from a semiring in order to add quantitative expressiveness. For example one may now express how often a certain property holds, how much execution time a process needs or how reliable it is. The result of Droste and Gastin has been extended to infinite words, trees, texts, pictures and traces [9, 10, 17, 16, 18]. Moreover, a restriction of Łukasiewicz multi-valued logic coincides with this weighted logics [19].

For our result we establish a new connection between series-parallel-biposets and nested words. The class of sp-biposets forms the free bisemigroup which was investigated by Hashiguchi et al. (e.g. [12]). Moreover, a language theory for series-parallel-biposets was developed by Ésik and Németh [11]. We anticipate that the connection between nested words and sp-biposets can be utilized to obtain further results. We give an indication in the conclusions at the end of the paper.

Using projections of nested word series and applying the above mentioned result, we obtain the second main result, a characterization of algebraic formal power series. These form an important generalization of context-free languages. Algebraic formal power series were already considered initially by Chomsky and Schützenberger [5] and have since been intensively studied by Kuich and others. For a survey see [13] or [14]. Here we are able to give a characterization of algebraic formal power series in terms of weighted logics, generalizing a result of Lautemann, Schwentick and Thérien [15] on context-free languages.

2 Weighted Automata on Nested Words

Definition 2.1 (Alur & Madhusudan [3]). *A* nested word *(over a finite alphabet Δ) is a pair (w, ν) such that $w \in \Delta^+$ and ν is a binary* nesting relation *on $\{1, \dots, |w|\}$ that satisfies (a) if $\nu(i,j)$, then $i < j$, (b) if $\nu(i,j)$* and *$\nu(i,j')$, then $j = j'$, (c) if $\nu(i,j)$* and *$\nu(i',j)$, then $i = i'$ and (d) if $\nu(i,j)$* and *$\nu(i',j')$* and *$i < i'$, then $j < i'$* or *$j' < j$.*

If $\nu(i,j)$ we say i is a call position *and j is a* return position.

We collect all nested words over Δ in $\mathrm{NW}(\Delta)$. Let $nw = (w, \nu) \in \mathrm{NW}(\Delta)$ where $w = a_1 \dots a_n$. The *factor* $nw[i,j]$ for $i \leq j$ is the restriction of nw to positions between i and j; more formally $nw[i,j] = (a_i \dots a_j, \nu[i,j])$ where $\nu[i,j] = \{(k,l) \mid (k+i-1, l+i-1) \in \nu, 1 \leq k, l \leq j-i+1\}$.

Nested words have been introduced in order to model executions of recursive programs, as well as nested data structures such as XML documents. Here we model quantitative behavior of systems such as runtime or the probability of an execution of a randomized program. That is we assign to a nested word a quantity expressing, for example, runtime or probability.

Example 2.2. 1. Probabilistic automata have been used to model fault-tolerant systems or to model randomized programs. Consider the randomized recursive pseudo-procedure `bar` (see next page) where `flip(Y)` means flipping a fair coin `Y`. Consider furthermore the alphabet $\Delta = \{r, w, b, call, ret\}$ of atomic events which stand for read, write, beep, call and return. Then the nested word $nw = (w, \nu)$ defined by $w = r.call.r.b.ret.w.w.ret$ and $\nu = \{(2, 5)\}$ models an execution of `bar`. We calculate the probability of the execution by multiplying the probability of each atomic action, i.e. $1 \cdot 1/2 \cdot 1 \cdot 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/64$.

2. As Alur and Madhusudan point out XML documents or bibtex databases can naturally be modeled as nested words where the nesting relation captures open and close tags [3]. Suppose we model bibtex databases as nested words. Then we may assign to a nested word e.g. the number of technical reports it stores.

```
proc bar(){
read(x);
flip(Y);if(Y==head)
      beep;
    else
      bar();
flip(Y);while(Y==head)
      write(x);
      flip(Y);
exit;}
```

To be as flexible as possible we take the quantities we assign to a nested word from a commutative semiring. A *commutative semiring* $\mathbb{K}$ is an algebraic structure $(\mathbb{K}, +, \cdot, 0, 1)$ such that $(\mathbb{K}, +, 0)$ and $(\mathbb{K}, \cdot, 1)$ are commutative monoids, multiplication distributes over addition and 0 is absorbing. For example the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ form a commutative semiring. An important example is also the max-plus semiring $(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$ which has been used to model real-time systems or discrete event systems. This semiring possesses the property that any finitely generated submonoid of $(\mathbb{K}, +, 0)$ is finite. Such semirings are called *additively locally finite*. Another important example of an additively locally finite semiring is the probabilistic semiring $([0, 1], \max, \cdot, 0, 1)$. We call a semiring *locally finite* if any finitely generated subsemiring is finite. Examples include any Boolean algebra such as the trivial Boolean algebra $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ as well as $(\mathbb{R}_+ \cup \{\infty\}, \max, \min, 0, \infty)$ and the fuzzy semiring $([0, 1], \max, \min, 0, 1)$.

Definition 2.3. *A* weighted nested word automaton (WNWA for short) *is a quadruple* $\mathcal{A} = (Q, \iota, \delta, \kappa)$ *where* $\delta = (\delta_c, \delta_i, \delta_r)$ *such that*

1. Q *is a finite set of states,*
2. $\delta_c, \delta_i : Q \times \Delta \times Q \to \mathbb{K}$ *are the* call *and* internal transition functions*,*
3. $\delta_r : Q \times Q \times \Delta \times Q \to \mathbb{K}$ *is the* return transition function*,*
4. $\iota, \kappa : Q \to \mathbb{K}$ *are the* initial *and* final distribution.

A *run* of $\mathcal{A}$ on $nw = (a_1 \dots a_n, \nu)$ is a *sequence of states* $r = (q_0, \dots, q_n)$; we also write $r : q_0 \overset{nw}{\to} q_n$. The *weight* of r *at position* $1 \leq j \leq n$ is given by

$$\mathrm{wgt}_{\mathcal{A}}(r, j) = \begin{cases} \delta_c(q_{j-1}, a_j, q_j) & \text{if } \nu(j, i) \text{ for some } j < i \leq n \\ \delta_r(q_{i-1}, q_{j-1}, a_j, q_j) & \text{if } \nu(i, j) \text{ for some } 1 \leq i < j \\ \delta_i(q_{j-1}, a_j, q_j) & \text{otherwise.} \end{cases}$$

The *weight* $\mathrm{wgt}_{\mathcal{A}}(r)$ of r is defined by $\mathrm{wgt}_{\mathcal{A}}(r) = \prod_{1\le j\le n} \mathrm{wgt}_{\mathcal{A}}(r,j)$ and the *behavior* $\|\mathcal{A}\|\colon \mathrm{NW}(\Delta) \to \mathbb{K}$ of $\mathcal{A}$ is given by

$$\|\mathcal{A}\|\,(nw) = \sum\nolimits_{q_0,q_n\in Q} \iota(q_0)\cdot \sum\nolimits_{r:q_0 \overset{nw}{\to} q_n} \mathrm{wgt}_{\mathcal{A}}(r)\cdot\kappa(q_n).$$

A function $S : \mathrm{NW}(\Delta) \to \mathbb{K}$ is called a *nested word series*. As for formal power series we write (S, nw) for $S(nw)$. We define the *scalar multiplication* $\cdot$ and the *sum* $+$ pointwise, i.e. for $k \in \mathbb{K}$ and nested word series S_1, S_2 we let $(k\cdot S_1, nw) = k\cdot(S_1, nw)$ and $(S_1 + S_2, nw) = (S_1, nw) + (S_2, nw)$ for all $nw \in \mathrm{NW}(\Delta)$. A series S is *regular* if there is a WNWA $\mathcal{A}$ with $\|\mathcal{A}\|= S$. For the Boolean semiring $\mathbb{B}$ WNWA are equivalent to (unweighted) nested word automata [3]. A language $L \subseteq \mathrm{NW}(\Delta)$ is recognized by a nested word automaton iff its characteristic function $\mathbb{1}_L : \mathrm{NW}(\Delta) \to \mathbb{B}$ is regular.

Example 2.4. The procedure `bar` of Example 2.2 can be modeled by a WNWA over $\mathbb{K} = ([0,1], \max, \cdot, 0, 1)$ with four states $\{q_1,\ldots,q_4\}$. The transitions (only the ones with weight $\neq 0$) are given as follows. We let $\iota(q_1) = 1$ and $\kappa(q_4) = 1$. Moreover,

$$\delta_i(q_1, r, q_2) = 1, \qquad \delta_i(q_2, b, q_3) = \delta_i(q_3, w, q_3) = \delta_i(q_3, ret, q_4) = 1/2,$$
$$\delta_c(q_2, call, q_1) = 1/2, \qquad \delta_r(q_2, q_3, ret, q_3) = 1/2.$$

Intuitively, each of the states corresponds to a line in the procedure `bar` which is the next to be executed: q_1 to line 2, q_2 to line 3, q_3 to line 7 and q_4 is only reached at the end of an execution. Consider the nested word nw of Example 2.2(a). There is exactly one run $r : q_1 \overset{nw}{\to} q_4$ with $\mathrm{wgt}(r) \neq 0$. Observe that the automaton assigns $1/64$ to nw.

3 Weighted Logics for Nested Words

We now introduce a formalism for specifying nested word series. For this we interpret a nested word $nw = (a_1 \ldots a_n, \nu)$ as a relational structure consisting of the domain $\mathrm{dom}(nw) = \{1,\ldots,n\}$ together with the unary relations $\mathrm{Lab}_a = \{i \in \mathrm{dom}(w) \mid a_i = a\}$ for all $a \in \Delta$, the binary relation ν and the usual $\le$ relation on $\mathrm{dom}(nw)$.

First, we recall classical MSO logic. Formulae of MSO are inductively built from the atomic formulae $x = y$, $\mathrm{Lab}_a(x)$, $x \le y$, $\nu(x, y)$, $x \in X$ using negation $\neg$, the connective $\vee$ and the quantifications $\exists x.$ and $\exists X.$ (where x, y range over individuals and X over sets). Let $\varphi \in \mathrm{MSO}$, let $\mathrm{Free}(\varphi)$ be the set of free variables, let $\mathcal{V} \supseteq \mathrm{Free}(\varphi)$ be a finite set of variables and let γ be a $(\mathcal{V}, nw)$-*assignment* (assigning variables of $\mathcal{V}$ an element or a set of $\mathrm{dom}(nw)$, resp.). For $i \in \mathrm{dom}(nw)$ and $T \subseteq \mathrm{dom}(nw)$ we denote by $\gamma[x \to i]$ and $\gamma[X \to T]$ the $(\mathcal{V} \cup \{x\}, nw)$-assignment (resp. $(\mathcal{V} \cup \{X\}, nw)$-assignment) which equals γ on $\mathcal{V} \setminus \{x\}$ (resp. $\mathcal{V} \setminus \{X\}$) and assumes i for x (resp. T for X). We let $\mathscr{L}_{\mathcal{V}}(\varphi) = \{(nw, \gamma) \mid (nw, \gamma) \models \varphi\}$ and $\mathscr{L}(\varphi) = \mathscr{L}_{\mathrm{Free}(\varphi)}(\varphi)$.

Let $Z \subseteq$ MSO. A language $L \subseteq \mathrm{NW}(\Delta)$ is *Z-definable* if $L = \mathscr{L}(\varphi)$ for a sentence $\varphi \in Z$. First-order formulae, that is formulae containing only quantification over individuals, are collected in FO. Monadic second-order logic and (unweighted) nested word automata turned out to be equally expressive (Alur and Madhusudan [3,2]).

We now turn to weighted MSO logics as introduced in [7]. Formulae of MSO($\mathbb{K}$) are built from the atomic formulae k (for $k \in \mathbb{K}$), $x = y$, $\mathrm{Lab}_a(x)$, $x \leq y$, $\nu(x, y)$, $x \in X$, $\neg(x = y)$, $\neg \mathrm{Lab}_a(x)$, $\neg(x \leq y)$, $\neg(\nu(x, y))$, $\neg(x \in X)$ using the connectives $\vee$, $\wedge$ and the quantifications $\exists x.$, $\exists X.$, $\forall x.$, $\forall X.$. Let $\varphi \in$ MSO($\mathbb{K}$) and $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$. The weighted semantics $[\![\varphi]\!]_{\mathcal{V}}$ of φ is a function which assigns to each pair (nw, γ) an element of $\mathbb{K}$. For $k \in \mathbb{K}$ we put $[\![k]\!]_{\mathcal{V}}(nw, \gamma) = k$. For all other atomic formulae φ we let $[\![\varphi]\!]_{\mathcal{V}}$ be the characteristic function $\mathbb{1}_{\mathscr{L}_{\mathcal{V}}(\varphi)}$. Moreover:

$$
\begin{aligned}
[\![\varphi \vee \psi]\!]_{\mathcal{V}}(nw, \gamma) &= [\![\varphi]\!]_{\mathcal{V}}(nw, \gamma) + [\![\psi]\!]_{\mathcal{V}}(nw, \gamma), \\
[\![\varphi \wedge \psi]\!]_{\mathcal{V}}(nw, \gamma) &= [\![\varphi]\!]_{\mathcal{V}}(nw, \gamma) \cdot [\![\psi]\!]_{\mathcal{V}}(nw, \gamma), \\
[\![\exists x.\varphi]\!]_{\mathcal{V}}(nw, \gamma) &= \sum\nolimits_{i \in \mathrm{dom}(nw)} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}(nw, \gamma[x \to i]), \\
[\![\exists X.\varphi]\!]_{\mathcal{V}}(nw, \gamma) &= \sum\nolimits_{T \subseteq \mathrm{dom}(nw)} [\![\varphi]\!]_{\mathcal{V} \cup \{X\}}(nw, \gamma[X \to T]), \\
[\![\forall x.\varphi]\!]_{\mathcal{V}}(nw, \gamma) &= \prod\nolimits_{i \in \mathrm{dom}(nw)} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}(nw, \gamma[x \to i]), \\
[\![\forall X.\varphi]\!]_{\mathcal{V}}(nw, \gamma) &= \prod\nolimits_{T \subseteq \mathrm{dom}(nw)} [\![\varphi]\!]_{\mathcal{V} \cup \{X\}}(nw, \gamma[X \to T]).
\end{aligned}
$$

In the following, we shortly write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$.

Remark. A formula $\varphi \in$ MSO($\mathbb{K}$) which does not contain a subformula $k \in \mathbb{K}$ can also be interpreted as an unweighted formula. Moreover, if $\mathbb{K}$ is the Boolean semiring $\mathbb{B}$, then it is easy to see that weighted logics and classical MSO logic coincide. In this case k is either 0 (false) or 1 (true).

Example 3.1. 1. As in Example 2.2 suppose we model bibtex databases as nested words. Moreover, assume that tecrep $\in \Delta$ marks the beginning of an entry containing a technical report. Now, let $\mathbb{K} = \mathbb{N}$ be the semiring of the natural numbers. Then $[\![\exists x. \mathrm{Lab}_{\mathrm{tecrep}}(x)]\!](nw)$ counts the number of technical reports of the bibtex database modeled by nw.

2. The *nesting depth* of a position of a nested word is the number of open call positions (i.e. where the corresponding return position has not occurred yet). The nesting depth of a nested word is the maximum over all positions. Let $\mathbb{K} = (\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$. Define

$$\mathrm{open}(x) := \forall y.(y \leq x \wedge \mathrm{call}(y)) \to 1 \vee (y \leq x \wedge \mathrm{return}(y)) \to -1$$

where $\mathrm{call}(x) := \exists y.\nu(x, y)$ and $\mathrm{return}(x) := \exists y.\nu(y, x)$ (the precise definition of $\to$ is given below). Then $[\![\exists x.\mathrm{open}(x)]\!]$ assigns to a nested word its nesting depth.

Let $Z \subseteq \text{MSO}(\mathbb{K})$. A series $S : \text{NW}(\Delta) \to \mathbb{K}$ is *Z-definable* if $S = [\![\varphi]\!]$ for a sentence $\varphi \in Z$. Already for words, examples [7] show that unrestricted application of universal quantification does not preserve regularity as the resulting series may grow to fast. Therefore we now define different fragments of $\text{MSO}(\mathbb{K})$. For the fragment RMSO for words, which we do not define here, Droste and Gastin [7] showed that a formal power series is regular iff it is RMSO-definable. Unfortunately, RMSO is a semantic restriction and it is not clear if it can be decided. In order to have a decidable fragment, we syntactically define the fragment sRMSO. For this we follow the approach of [8]. Given a classical MSO formula φ we assign to it formulae φ^+ and φ^- such that $[\![\varphi^+]\!] = \mathbb{1}_{\mathscr{L}(\varphi)}$ and $[\![\varphi^-]\!] = \mathbb{1}_{\mathscr{L}(\neg\varphi)}$. The problem that arises is that by definition, e.g. $\vee$ is interpreted as $+$. Hence, for a formula $\varphi \vee \psi$ one might not end up with a sum which equals 0 or 1. One possible solution is to evaluate φ only if ψ evaluates to 0. Similar for $\exists x.$ and $\exists X.$. This leads to the following definition:

1. If φ is of the form $x = y$, $\text{Lab}_a(x)$, $x \leq y$, $\nu(x, y)$, $x \in X$, then $\varphi^+ = \varphi$ and $\varphi^- = \neg\varphi$.
2. If $\varphi = \neg\psi$ then $\varphi^+ = \psi^-$ and $\varphi^- = \psi^+$.
3. If $\varphi = \psi \vee \psi'$, then $\varphi^+ = \psi^+ \vee (\psi^- \wedge \psi'^+)$ and $\varphi^- = \psi^- \wedge \psi'^-$.
4. If $\varphi = \exists x.\psi$, then $\varphi^+ = \exists x.\psi^+ \wedge \forall y.(y < x \wedge \psi(y))^-$ and $\varphi^- = \forall x.\psi^-$.

In order to disambiguate set quantification, we have to define a linear order on the subsets of the domain of a nested word or equivalently on nested words (of fixed length) over the alphabet $\{0, 1\}$. We take the lexicographic order $<$ which is given by the following formula: $X < Y := \exists y.y \in Y \wedge \neg y \in X \wedge \forall z.[z < y \to (z \in X \leftrightarrow z \in Y)]^+$. We proceed:

5. If $\varphi = \exists X.\psi$, then $\varphi^+ = \exists X.\psi^+ \wedge \forall Y.(Y < X \wedge \psi(Y))^-$ and $\varphi^- = \forall X.\psi^-$.

Formulae of the form φ^+ or φ^- for some $\varphi \in \text{MSO}$ are called *syntactically unambiguous*. In the following, we shortly write $\varphi \to \psi$ for $\varphi^- \vee (\varphi^+ \wedge \psi)$ for any two weighted formulae φ, ψ where φ does not contain subformulae of form k ($k \in \mathbb{K}$).

We define aUMSO, the collection of *almost unambiguous* formulae, to be the smallest subset of $\text{MSO}(\mathbb{K})$ containing all constants k ($k \in \mathbb{K}$) and all syntactically unambiguous formulae which is closed under conjunction and disjunction.

Definition 3.2. *A weighted formula φ is in* sRMSO (syntactically restricted MSO) *if: 1. Whenever it contains a subformula $\forall X.\psi$, then ψ is syntactically unambiguous. 2. Whenever it contains a subformula $\forall x.\psi$, then $\psi \in$* aUMSO.

Let now wUMSO, the collection of *weakly unambiguous* formulae, be the smallest subset of $\text{MSO}(\mathbb{K})$ containing all constants k ($k \in \mathbb{K}$) and all syntactically unambiguous formulae which is closed under conjunction, disjunction and existential quantification.

Definition 3.3. *A weighted formula* φ *is in* swRMSO (syntactically weakly restricted MSO) *if: 1. Whenever it contains a subformula* $\forall X.\psi$*, then* ψ *is syntactically unambiguous. 2. Whenever it contains a subformula* $\forall x.\psi$*, then* $\psi \in$ wUMSO.

Clearly, sRMSO $\subset$ swRMSO $\subset$ MSO($\mathbb{K}$). The first main result of this paper is the characterization of regular nested word series using weighted logics.

Theorem 3.4. *Let* $\mathbb{K}$ *be a commutative semiring and let* $S : \mathrm{NW}(\Delta) \to \mathbb{K}$*. Then:*

(a) The series S *is regular iff* S *is* sRMSO*-definable.*
(b) If $\mathbb{K}$ *is additively locally finite, then* S *is regular iff* S *is* swRMSO*-definable.*
(c) If $\mathbb{K}$ *is locally finite, then* S *is regular iff* S *is* MSO*-definable.*

We prove the result in the next section by interpreting nested words in sp-biposets, which we first investigate. The results are interesting in their own rights.

4 Nested Words and SP-Biposets

A bisemigroup is a set together with two associative operations. Several authors investigated the free bisemigroup as a fundamental, two-dimensional extension of classical automaton theory, see e.g. Ésik and Németh [11] and Hashiguchi et al. (e.g. [12]). Ésik and Németh considered as a representation for the free bisemigroup the so-called *sp-biposets*, a certain class of biposets. A Δ*-labeled biposet* is a finite nonempty set V of vertices equipped with two partial orders $\leq_h$ and $\leq_v$ and a labeling function $\lambda : V \to \Delta$. Let $p_1 = (V_1, \lambda_1, \leq_h^1, \leq_v^1), p_2 = (V_2, \lambda_2, \leq_h^2, \leq_v^2)$ be biposets. We define $p_1 \circ_h p_2 = (V_1 \uplus V_2, \lambda_1 \cup \lambda_2, \leq_h, \leq_v)$ by letting $\leq_h = \leq_h^1 \cup \leq_h^2 \cup (V_1 \times V_2)$ and $\leq_v = \leq_v^1 \cup \leq_v^2$. The operation $\circ_v$ is defined dually. Clearly, both products are associative. The set of biposets generated from the singletons by $\circ_h$ and $\circ_v$ is denoted SPB(Δ). Its elements are called *sp-biposets*.

Weighted parenthesizing automata operating on sp-biposets generalizing the automata of Ésik and Németh [11] were defined in [16].

Definition 4.1. *A* weighted parenthesizing automaton *(*WPA *for short) over* Δ *is a tuple* $\mathcal{P} = (\mathcal{H}, \mathcal{V}, \Omega, \mu, \mu_{op}, \mu_{cl}, \lambda, \gamma)$ *where*

1. $\mathcal{H}$,$\mathcal{V}$ *are finite disjoint sets of* horizontal *and* vertical states*, respectively,*
2. Ω *is a finite set of* parentheses *(to help the intuition we write* $(_s$ *or* $)_s$ *for* $s \in \Omega$*),*
3. $\mu : (\mathcal{H} \times \Delta \times \mathcal{H}) \cup (\mathcal{V} \times \Delta \times \mathcal{V}) \to \mathbb{K}$ *is the* transition function*,*
4. $\mu_{op}, \mu_{cl} : (\mathcal{H} \times \Omega \times \mathcal{V}) \cup (\mathcal{V} \times \Omega \times \mathcal{H}) \to \mathbb{K}$ *are the* opening *and* closing parenthesizing functions *and*
5. $\lambda, \gamma : \mathcal{H} \cup \mathcal{V} \to \mathbb{K}$ *are the* initial *and* final weight functions.

A *run* r of $\mathcal{P}$ is a word over the alphabet $(\mathcal{H} \cup \mathcal{V}) \times (\Delta \cup \Omega) \times (\mathcal{H} \cup \mathcal{V})$ defined inductively as follows. We also define its *label* $\mathrm{lab}(r)$, its *weight* $\mathrm{wgt}_{\mathcal{P}}(r)$, its *initial state* $\mathrm{init}(r)$ and its *final state* $\mathrm{fin}(r)$.

1. (q_1, a, q_2) is a run for all $(q_1, q_2) \in (\mathcal{H} \times \mathcal{H}) \cup (\mathcal{V} \times \mathcal{V})$ and $a \in \Delta$. We set

$$\mathrm{lab}((q_1, a, q_2)) = a \in \mathrm{SPB}(\Delta), \qquad \mathrm{wgt}_{\mathcal{P}}((q_1, a, q_2)) = \mu(q_1, a, q_2),$$
$$\mathrm{init}((q_1, a, q_2)) = q_1 \text{ and } \mathrm{fin}((q_1, a, q_2)) = q_2.$$

2. Let r_1 and r_2 be runs such that $\mathrm{fin}(r_1) = \mathrm{init}(r_2) \in \mathcal{H}$ (resp. $\mathcal{V}$). Then $r = r_1 r_2$ is a run having

$$\mathrm{lab}(r) = \mathrm{lab}(r_1) \circ_h \mathrm{lab}(r_2) \quad (\text{resp. } \mathrm{lab}(r) = \mathrm{lab}(r_1) \circ_v \mathrm{lab}(r_2)),$$
$$\mathrm{wgt}_{\mathcal{P}}(r) = \mathrm{wgt}_{\mathcal{P}}(r_1) \cdot \mathrm{wgt}_{\mathcal{P}}(r_2), \mathrm{init}(r) = \mathrm{init}(r_1) \text{ and } \mathrm{fin}(r) = \mathrm{fin}(r_2).$$

3. Let r be a run resulting from 2. such that $\mathrm{fin}(r), \mathrm{init}(r) \in \mathcal{H}$ (resp. $\mathcal{V}$). Let $q_1, q_2 \in \mathcal{V}$ (resp. $\mathcal{H}$) and $s \in \Omega$. Then $r' = (q_1, (_s, \mathrm{init}(r))\ r\ (\mathrm{fin}(r),)_s, q_2)$ is a run. We set

$$\mathrm{lab}(r') = \mathrm{lab}(r), \mathrm{wgt}_{\mathcal{P}}(r') = \mu_{\mathrm{op}}((q_1, (_s, \mathrm{init}(r))) \cdot \mathrm{wgt}_{\mathcal{P}}(r) \cdot \mu_{\mathrm{cl}}((\mathrm{fin}(r),)_s, q_2)),$$
$$\mathrm{init}(r') = q_1 \text{ and } \mathrm{fin}(r') = q_2.$$

Let $p \in \mathrm{SPB}(\Delta)$. If r is a run of $\mathcal{P}$ with $\mathrm{lab}(r) = p$, $\mathrm{init}(r) = q_1$, $\mathrm{fin}(r) = q_2$, we write $r : q_1 \xrightarrow{p} q_2$. Since we do not allow repeated application of rule 3, there are only finitely many runs with label p. The behavior of $\mathcal{P}$ is a function $\|\mathcal{P}\|\colon \mathrm{SPB}(\Delta) \to \mathbb{K}$ with

$$(\|\mathcal{P}\|, p) = \sum\nolimits_{q_1, q_2 \in \mathcal{H} \cup \mathcal{V}} \lambda(q_1) \cdot \sum\nolimits_{r : q_1 \xrightarrow{p} q_2} \mathrm{wgt}_{\mathcal{P}}(r) \cdot \gamma(q_2).$$

An sp-biposet series, i.e. a function $S : \mathrm{SPB}(\Delta) \to \mathbb{K}$, is *regular* if there is a WPA $\mathcal{P}$ such that $\|\mathcal{P}\| = S$. For sp-biposets, MSO($\mathbb{K}$) can be defined similar as for nested words. Moreover, for any sp-biposet $p = (V, \lambda, \leq_h, \leq_v)$ the union $\leq := \leq_h \cup \leq_v$ gives a linear order [11]. Using this linear order, we can define syntactically unambiguous formulae as for nested words and then sRMSO and swRMSO. From Theorems 6.3 and 5.6 of [16] we obtain (here without proof due to space constraints) the following result which generalizes the Büchi-type result of Ésik and Németh on the coincidence of MSO-definable and regular languages of sp-biposets [11].

Theorem 4.2. *Let* $\mathbb{K}$ *be a commutative semiring and let* $S : \mathrm{SPB}(\Delta) \to \mathbb{K}$. *Then:*

(a) The series S *is regular iff* S *is* sRMSO*-definable.*
(b) If $\mathbb{K}$ *is additively locally finite, then* S *is regular iff* S *is* swRMSO*-definable.*
(c) If $\mathbb{K}$ *is locally finite, then* S *is regular iff* S *is* MSO*-definable.*

We note that under the assumptions of Theorem 4.2, given an sRMSO (resp. swRMSO, resp. MSO) formula φ we can effectively construct a WPA $\mathcal{P}$ such that $[\![\varphi]\!] = \|\mathcal{P}\|$ (and vice versa).

We will now derive similar results for nested words as for sp-biposets by interpreting the different structures within each other. For this, we utilize definable transductions as introduced by Courcelle [6]. Let σ_1 and $\sigma_2 = ((R_i)_{i\in I}, \rho)$ be two relational signatures where $\rho : I \to \mathbb{N}_+$ assigns to each relation symbol R_i a positive arity and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of finite σ_1- and σ_2-structures, respectively.

Definition 4.3. *A* (σ_1, σ_2)-1-copying definition scheme *(without parameter) is a tuple* $\mathcal{D} = (\vartheta, \delta, (\varphi_i)_{i\in I})$ *of formulae in* $\mathrm{MSO}(\sigma_1)$ *such that* $\mathrm{Free}(\vartheta) = \emptyset$*,* $\mathrm{Free}(\delta) = \{x_1\}$ *and* $\mathrm{Free}(\varphi_i) = \{x_1, \ldots, x_{\rho(i)}\}$*.*

Let $\mathcal{D}$ be a (σ_1, σ_2)-1-copying definition scheme. For each $s_1 \in \mathcal{C}_1$ such that $s_1 \models \vartheta$ define the σ_2-structure $\mathbf{def}_{\mathcal{D}}(s_1) = s_2 = (\mathrm{dom}(s_2), (R_i^{s_2})_{i\in I})$ where $\mathrm{dom}(s_2) = \{v \in \mathrm{dom}(s_1) \mid (s_1, v) \models \varphi\}$ and $R_i^{s_2} = \{(v_1, \ldots, v_r) \in \mathrm{dom}(s_2)^r \mid (s_1, v_1, \ldots, v_r) \models \varphi_i\}$ with $r = \rho(i)$. Now a partial function $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ is *definable* if there is a definition scheme $\mathcal{D}$ such that $\Phi = \mathbf{def}_{\mathcal{D}}$.

Clearly, $\mathrm{MSO}(\mathbb{K})$ can be defined for $\mathcal{C}_1$ and $\mathcal{C}_2$ along the lines as for nested words. In order to disambiguate a formula, we need a linear order on each $s \in \mathcal{C}_1$ (resp. $\mathcal{C}_2$). For the next proposition we therefore assume that there are binary relation symbols $\leq_1 \in \sigma_1$ and $\leq_2 \in \sigma_2$ such that the interpretation of $\leq_i$ in s is a linear order for any $s \in \mathcal{C}_i$ $(i = 1, 2)$. Using these linear orders, we can define syntactically unambiguous formulae and then $\mathrm{sRMSO}(\mathbb{K})$ and $\mathrm{swRMSO}(\mathbb{K})$ over σ_1 and σ_2. Now, let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a partial function with domain $\mathrm{dom}(\Phi)$ and let $S : \mathcal{C}_2 \to \mathbb{K}$. Define $\Phi^{-1}(S)$ by letting $(\Phi^{-1}(S), s_1) = (S, \Phi(s_1))$ for all $s_1 \in \mathrm{dom}(\Phi)$ and $(\Phi^{-1}(S), s_1) = 0$ otherwise.

Proposition 4.4. *Let* $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ *be a definable function. If* $S : \mathcal{C}_2 \to \mathbb{K}$ *is* MSO*-definable (resp.* sRMSO*-definable,* swRMSO*-definable), then so is* $\Phi^{-1}(S)$*.*

Remark. It suffices that $\leq_i$ can be defined by a formula $\varphi_i(x, y)$ such that $[\![\varphi_i]\!] = \mathbb{1}_{\mathscr{L}(\varphi_i)}$. This is the case for sp-biposets where we let $\varphi(x, y) = x \leq_v y \vee x \leq_h y$.

Now we define two embeddings of nested words into sp-biposets Φ_v, Φ_h : $\mathrm{NW}(\Delta) \to \mathrm{SPB}(\Delta)$ as follows. Let $nw = (w, \nu) \in \mathrm{NW}(\Delta)$ where $w = a_1 \ldots a_n$. If $\nu = \emptyset$, then let $\Phi_h(nw) = a_1 \circ_h \ldots \circ_h a_n$ and $\Phi_v(nw) = a_1 \circ_v \ldots \circ_v a_n$. If $\nu \neq \emptyset$, let i be the minimal call position and j the corresponding return position. Let $nw' = nw[i+1, j-1]$ and $nw'' = nw[j+1, n]$. Suppose for the moment that $i + 1 \leq j - 1$ and $j + 1 \leq n$. Then

$$\Phi_h(nw) = a_1 \circ_h \ldots \circ_h a_{i-1} \circ_h (a_i \circ_v \Phi_v(nw') \circ_v a_j) \circ_h \Phi_h(nw''),$$
$$\Phi_v(nw) = a_1 \circ_v \ldots \circ_v a_{i-1} \circ_v (a_i \circ_h \Phi_h(nw') \circ_h a_j) \circ_v \Phi_v(nw'').$$

If $i+1 > j-1$ or $j+1 > n$, we just ignore the terms $\Phi_h(nw')$, $\Phi_h(nw'')$, $\Phi_v(nw')$ and $\Phi_v(nw'')$, respectively, in the definition above. We identify the domain of $\Phi(nw)$ with $\{1, \ldots, n\}$ in the obvious way. Observe that Φ_h and Φ_v are injective.

Lemma 4.5. *Let* $nw = (a_1 \dots a_n, \nu) \in \mathrm{NW}(\Delta)$, $\Phi_h(nw) = (\{1, \dots, n\}, \lambda, \leq_h, \leq_v)$. *Moreover, let* $1 \leq k \leq i < j \leq l \leq n$ *with* $(k, l) \in \nu$ *such that there is no* $(k', l') \in \nu$ *with* $k < k' \leq i \leq j \leq l' < l$. *Then* $i \leq_h j$ *iff* k *has even nesting depth (cf. Ex. 3.1).*

Recall that for an sp-biposet $p = (V, \lambda, \leq_h, \leq_v)$ we let $\leq := \leq_h \cup \leq_v$. A *clan* of p is an interval $[i, j] = \{k \in V \mid i \leq k \leq j\}$ which can not be distinguished from outside, i.e. if for all $i \leq k, k' \leq j$ and $l < i$ or $j < l$ we have $k \leq_v l$ iff $k' \leq_v l$ and $k \leq_h l$ iff $k' \leq_h l$ and $l \leq_v k$ iff $l \leq_v k'$. A *prime clan* is a clan that does not overlap with any other, i.e. there is no clan $[k, l]$ such that $k < i < l < j$ or $i < k < j < l$.

Lemma 4.6. *Let* $nw = (a_1 \dots a_n, \nu) \in \mathrm{NW}(\Delta)$, $\Phi_h(nw) = (\{1, \dots, n\}, \lambda, \leq_h, \leq_v)$. *Then* $(i, j) \in \nu$ *iff* $i < j$, $[i, j]$ *is a prime clan and not* $i = 1$, $j = n$ *and* $1 \leq_h n$.

Since the conditions of Lemma 4.6 and Lemma 4.5 can be expressed in MSO (actually the latter can be expressed in FO), we obtain:

Corollary 4.7. *The (partial) functions* $\Phi_h, \Phi_v, \Phi_h^{-1}, \Phi_v^{-1}$ *are definable.*

We will now show that not only the formulae can be translated, but that WPA can simulate WNWA and vice versa. More precisely:

Proposition 4.8. *A series* $S : \mathrm{NW}(\Delta) \to \mathbb{K}$ *is regular iff* $(\Phi_h^{-1})^{-1}(S)$ *is regular.*

Proof (Sketch). (*If*). Let $\mathcal{P} = (\mathcal{H}, \mathcal{V}, \Omega, \mu, \mu_{\mathrm{op}}, \mu_{\mathrm{cl}}, \lambda, \gamma)$ be a WPA. There is a WNWA $\mathcal{A} = (Q, \iota, \delta, \kappa)$ with state space $Q = (\mathcal{H} \uplus \mathcal{V}) \times (\Omega \uplus \{i\})$ such that $\|\mathcal{A}\| = \Phi_h^{-1}(\|\mathcal{P}\|)$. Intuitively, in the first component one simulates the states of the WPA and in the second component one stores the most recent open bracket. This has to be updated when reading a return position using the look-back ability of the nested word automaton.

(*Only if*). Let $\mathcal{A} = (Q, \iota, \delta, \kappa)$ be a WNWA. There is a WPA where $\mathcal{H}, \mathcal{V}$ are disjoint copies of $Q \times (\{c, i\} \uplus \Delta)$ and $\Omega = Q$ such that $(\|\mathcal{P}\|, \Phi_h(nw)) = (\|\mathcal{A}\|, nw)$ for all $nw \in \mathrm{NW}(\Delta)$. Intuitively, in the first component one simulates the states of the WNWA, in the second component one either selects if the next transition is a call or an internal transition or one stores the letter to simulate a return transition in the next bracket. Look-back behavior is simulated storing a state in the opening bracket and closing it at the appropriate return position. □

Theorem 3.4 now follows from Cor. 4.7, Prop. 4.4 and Thm. 4.2 together with Prop. 4.8.

We note that Proposition 4.8 also holds for non-commutative semirings. Moreover, we note that under the assumptions of Theorem 3.4, given an sRMSO (resp. swRMSO, resp. MSO) formula φ we can effectively construct a WNWA $\mathcal{A}$ such that $[\![\varphi]\!] = \|\mathcal{A}\|$ (and vice versa). Furthermore from Corollaries 5.7 and 5.8 of [16] we obtain

Corollary 4.9. *Let* $\mathbb{K}$ *be a computable field (resp. computable locally finite semiring). It is decidable whether two sentences* $\varphi, \psi \in$ sRMSO *(resp.* MSO*) satisfy* $[\![\varphi]\!] = [\![\psi]\!]$.

5 Algebraic Formal Power Series

In this section we consider algebraic formal power series and show that they arise as the projections of regular nested word series. Algebraic formal power series were already considered initially by Chomsky and Schützenberger [5] and have since been intensively studied by Kuich and others. For a survey see [13] or [14].

A formal power series is a mapping $S : \Delta^+ \to \mathbb{K}$. Given two formal power series S_1, S_2, their *Cauchy product*, denoted $S_1 \cdot S_2$, is given by $(S_1 \cdot S_2, w) = \sum_{w_1 w_2 = w} (S_1, w_1)(S_2, w_2)$. Let $\mathbb{1}_u$ denote the characteristic series of a word $u \in \Delta^+$.

Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of variables. A polynomial P over $(\Delta \cup \mathcal{X})$ with values in $\mathbb{K}$ is a mapping $P : (\Delta \cup \mathcal{X})^+ \to \mathbb{K}$ such that its support is finite, i.e. the set $\mathrm{supp}(P) = \{w \in (\Delta \cup \mathcal{X})^+ \mid (P, w) \neq 0\}$ is finite. A collection of polynomials P_i for $i = 1, \ldots, n$ is called an *algebraic system* with variables in $\mathcal{X}$. The supports of the P_i's are, thus, finite sets consisting of words of the form $u_1 X_{i_1} \ldots u_k X_{i_k} u_{k+1}$ where $u_j \in \Delta^*$ and $X_{i_j} \in \mathcal{X}$. A collection $(S_i)_{1 \leq i \leq n}$ of formal power series $S_i : \Delta^+ \to \mathbb{K}$ is the solution of the algebraic system $(P_i)_{1 \leq i \leq n}$ if

$$S_i = \sum_{u_1 X_{i_1} \ldots u_k X_{i_k} u_{k+1} \in \mathrm{supp}(P_i)} P_i(u_1 X_{i_1} \ldots u_k X_{i_k} u_{k+1}) \mathbb{1}_{u_1} \cdot S_{i_1} \cdots \mathbb{1}_{u_k} \cdot S_{i_k} \cdot \mathbb{1}_{u_{k+1}}.$$

for $1 \leq i \leq n$. An algebraic system $(P_i)_{1 \leq i \leq n}$ is *proper* if $(P_i, X_j) = 0$ for all $1 \leq i, j \leq n$. Proper algebraic systems have a unique solution [14]. If a formal power series S is the (component of) a solution of a proper algebraic system, then S is called an *algebraic formal power series*. Over the trivial Boolean algebra $\mathbb{B}$ these series correspond exactly to the ε-free context-free languages (the bijection is given by supp).

We now consider the projections of regular nested word series and show that they give rise exactly to the algebraic series. The projection $\pi(nw)$ of a nested word $nw = (w, \nu)$ is simply the word w, i.e. we forget the nesting relation. This projection is canonically generalized to nested word series S by letting $\pi(S) : w \mapsto \sum_{w = \pi(nw)} (S, nw)$.

Proposition 5.1. *Let $S : \mathrm{NW}(\Delta) \to \mathbb{K}$ be a regular nested word series. Then $\pi(S)$ is an algebraic formal power series.*

Proof (Sketch). Let $\mathcal{A} = (Q, \iota, \delta, \kappa)$ be a WNWA such that $\| \mathcal{A} \| = S$ and let $q_1, q_2 \in Q$. We define polynomials $P_{q_1,q_2} : (\Delta \cup Q^2)^* \to \mathbb{K}$ as follows (in order to obtain a more compact presentation we consider here the empty word ε): For any $a, b, c, d \in \Delta$ and $q_1, \ldots, q_8 \in Q$ let $(P_{q_1,q_1}, \varepsilon) = 1$, $(P_{q_1,q_2}, a) = \delta_i(q_1, a, q_2)$, $(P_{q_1,q_2}, a(q_3, q_4)b) = \delta_i(q_1, a, q_3) \cdot \delta_i(q_4, b, q_2) + \delta_c(q_1, a, q_3) \cdot \delta_r(q_1, q_4, b, q_2)$, $(P_{q_1,q_2}, a(q_3, q_4)b(q_5, q_6)c) = \delta_c(q_1, a, q_3) \cdot \delta_r(q_1, q_4, b, q_5) \cdot \delta_i(q_6, c, q_2) + \delta_i(q_1, a, q_3) \cdot \delta_c(q_4, b, q_5) \cdot \delta_r(q_4, q_6, c, q_2)$ and $(P_{q_1,q_2}, a(q_3, q_4)b(q_5, q_6)c(q_7, q_8)d) = \delta_c(q_1, a, q_3) \cdot \delta_r(q_1, q_4, b, q_5) \cdot \delta_c(q_6, c, q_7) \cdot \delta_r(q_6, q_8, d, q_2)$. Finally, let $(P_{q_1,q_2}, w) = 0$ in any other case. This gives a so-called strict algebraic system with variables in Q^2

having a necessarily unique solution $(S_{q_1,q_2})_{q_1,q_2\in Q}$ which is algebraic [14]. By induction on the length of w one gets

$$(S_{q_1,q_2}, w) = \sum\nolimits_{\pi(nw)=w} \sum\nolimits_{r:q_1 \overset{nw}{\to} q_2} \mathrm{wgt}_{\mathcal{A}}(r).$$

Now use that algebraic series are closed under sum and scalar multiplication. □

Our aim is to give a logical characterization for algebraic formal power series in the spirit of Lautemann, Schwentick and Thérien [15]. They showed that the context-free languages are precisely the languages which can be defined by sentences of the form $\exists\nu.\varphi$ where φ is a first-order formula and ν a binary predicate ranging over nesting relations. Let φ be a weighted MSO formula over nested words, $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$, $w \in \Delta^+$ and γ a $(\mathcal{V}, w)$-assignment. We define the semantics $[\![\exists\nu.\varphi]\!]^{\mathrm{nest}} : \Delta^+ \to \mathbb{K}$ by letting

$$[\![\exists\nu.\varphi]\!]^{\mathrm{nest}}(w,\gamma) = \sum\nolimits_{\text{nesting rel. } \nu} [\![\varphi]\!]((w,\nu),\gamma).$$

Using Theorem 3.4, we may now reformulate Proposition 5.1.

Corollary 5.2. *Let $\varphi \in$ sRMSO be a sentence, then $[\![\exists\nu.\varphi]\!]^{nest}$ is an algebraic formal power series.*

Conversely, we can construct a nested word automaton $\mathcal{A}$ such that $\pi(\|\mathcal{A}\|)$ is the solution of a given algebraic system in Greibach normal form [14], i.e. we require $\mathrm{supp}(P_i) \subseteq \Delta \cup \Delta\mathcal{X} \cup \Delta\mathcal{X}\mathcal{X}$. Elements of $\Delta\mathcal{X}\mathcal{X}$ produce call transitions, elements in $\Delta\mathcal{X}$ internal transitions and elements in Δ return transitions. Therefore we conclude:

Proposition 5.3. *Let $R : \Delta^+ \to \mathbb{K}$ be an algebraic formal power series. Then there is a regular nested word series $S : \mathrm{NW}(\Delta) \to \mathbb{K}$ such that $\pi(S) = R$.*

Even stronger, we can restrict the series to first-order definable ones:

Proposition 5.4. *Let $S : \mathrm{NW}(\Delta) \to \mathbb{K}$ be a regular nested word series. Then there is a first-order sentence $\varphi \in$ sRMSO such that $[\![\exists\nu.\varphi]\!]^{nest} = \pi(S)$.*

Proof (Hint). Let $\mathcal{A}$ be a weighted nested word automaton such that $\|\mathcal{A}\| = S$. Construct an algebraic system as in the proof of Proposition 5.1. This system has a form as required in the proof of [15]. The rest of the proof follows [15]; one has to ensure not to count weights twice and to obtain $\varphi \in$ sRMSO. □

In the following theorem we summarize what we obtained in this section.

Theorem 5.5. *Let $\mathbb{K}$ be a commutative semiring and let $S : \Delta^+ \to \mathbb{K}$. Then the following are equivalent:*

1. *S is an algebraic formal power series.*
2. *$S = \pi(R)$ for some regular nested words series $R : \mathrm{NW}(\Delta) \to \mathbb{K}$.*
3. *There is a first-order sentence $\varphi \in$ sRMSO such that $[\![\exists\nu.\varphi]\!]^{nest} = S$.*

Let $S : \{a\}^+ \to \mathbb{N}$ be an algebraic formal power series. One can show that $(S, a^n) \leq c^n$ for some constant c and all $n \in \mathbb{N}$. Thus, in item 3 of the last result we may not replace sRMSO by MSO since $([\![\forall x.\exists y.1]\!], a^n) = n^n$.

Again we note that given an algebraic $S : \Delta^+ \to \mathbb{K}$ we can effectively construct a sentence $\varphi \in \mathrm{FO} \cap \mathrm{sRMSO}$ such that $S = [\![\exists\nu.\varphi]\!]^{\mathrm{nest}}$ and vice versa.

Conclusion and Consequences. We introduced a quantitative automaton model and a quantitative logic for nested words and showed that they are equally expressive. This generalizes the logical characterization in the unweighted case as given in [3]. Moreover, we established a new connection between nested words and sp-biposets. Presumably, the logical characterization of regular nested word series could also be obtained by structural induction. However, the connection between sp-biposets and nested words enables us to also obtain a generalization of the second main result of [15]. In this paper another logical characterization of context-free languages is given where quantification over nesting relations is now replaced by quantification over tree-definable orders. It is easy to see that there is a definable bijection between sp-biposets and the class of words together with a tree-definable order [11]. Thus, using the connection between nested words and sp-biposets we can conclude that every algebraic formal power series can be defined by a formula $\exists\nu.\varphi$ where $\varphi \in \mathrm{FO} \cap \mathrm{sRMSO}$ and ν ranges now over tree-definable orders. The converse can be shown by simulating weighted parenthesizing automata by weighted pushdown automata (as defined in [14]).

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