Weighted Tree Automata over DM-Monoids

Zoltán Fülöp[1] and Andreas Maletti[2] and Heiko Vogler[2]

[1] Department of Computer Science
University of Szeged

[2] Department of Computer Science
TECHNISCHE UNIVERSITÄT DRESDEN

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Distributive Multioperator-Monoid

Definition (Kuich '98)

$(A, +, 0, \Omega)$ distributive M-monoid (short DM-monoid), if

- $(A, +, 0)$ commutative monoid
- (A, Ω) algebra
- for every $\omega \in \Omega$

$$\omega(\dots, 0, \dots) = 0$$

- for every $\omega \in \Omega$

$$\omega(\dots, a + b, \dots) = \omega(\dots, a, \dots) + \omega(\dots, b, \dots)$$

Combining Operations

Definition

Monoid $(A, +, 0)$ and $\omega\colon A \to A$ and $\omega', \omega''\colon A^k \to A$

- $(\omega' + \omega'')\colon A^k \to A$

$$(\omega' + \omega'')(a_1, \dots, a_k) = \omega'(a_1, \dots, a_k) + \omega''(a_1, \dots, a_k)$$

$$\begin{array}{c} \omega' + \omega'' \\ \diagup \qquad \diagdown \\ a_1 \quad \cdots \quad a_k \end{array} = \begin{array}{c} \omega' \\ \diagup \qquad \diagdown \\ a_1 \quad \cdots \quad a_k \end{array} + \begin{array}{c} \omega'' \\ \diagup \qquad \diagdown \\ a_1 \quad \cdots \quad a_k \end{array}$$

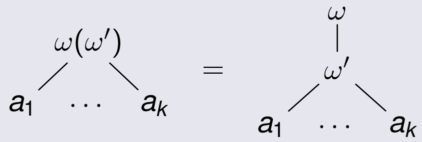
Combining Operations

Definition

Monoid $(A, +, 0)$ and $\omega\colon A \to A$ and $\omega', \omega''\colon A^k \to A$

- $(\omega' + \omega'')\colon A^k \to A$
- $\omega(\omega')\colon A^k \to A$

$$(\omega(\omega'))(a_1, \ldots, a_k) = \omega(\omega'(a_1, \ldots, a_k))$$



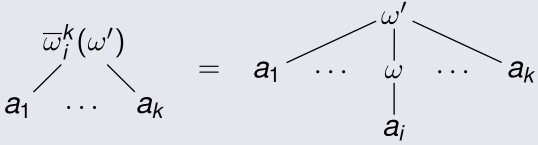
Combining Operations

Definition

Monoid $(A, +, 0)$ and $\omega\colon A \to A$ and $\omega', \omega''\colon A^k \to A$

- $(\omega' + \omega'')\colon A^k \to A$
- $\omega(\omega')\colon A^k \to A$
- $\overline{\omega}_i^k(\omega')\colon A^k \to A$

$$(\overline{\omega}_i^k(\omega'))(a_1, \ldots, a_k) = \omega'(a_1, \ldots, a_{i-1}, \omega(a_i), a_{i+1}, \ldots, a_k)$$



Properties of DM-Monoids

Definition

DM-monoid $(A, +, 0, \Omega)$ is

- sum closed,
  if $\omega' + \omega'' \in \Omega^{(k)}$ for every $\omega', \omega'' \in \Omega^{(k)}$
- $(1, \star)$-composition closed,
  if $\omega(\omega') \in \Omega^{(k)}$ for every $\omega \in \Omega^{(1)}$ and $\omega' \in \Omega^{(k)}$
- $(\star, 1)$-composition closed,
  if $\overline{\omega}_i^k(\omega') \in \Omega^{(k)}$ for every $1 \leq i \leq k$, $\omega \in \Omega^{(1)}$ and $\omega' \in \Omega^{(k)}$
- unary-composition closed,
  if $(1, \star)$- and $(\star, 1)$-composition closed.

A Generic Example

Example (Kuich '98)

Let $(A, +, \cdot, 0, 1)$ semiring. Let $\Omega^{(k)} = \{\, \overline{a}^k \mid a \in A \,\}$ where

$$\overline{a}^k \colon A^k \to A$$
$$(a_1, \ldots, a_k) \mapsto a_1 \cdot \ldots \cdot a_k \cdot a$$

$(A, +, 0, \Omega)$ is sum and unary-composition closed DM-monoid

Uniform Mapping

Definition

$\psi \colon T_\Sigma(Z) \to \mathsf{Ops}(A)$ uniform, if $(\psi, t) \in \mathsf{Ops}^n(A)$ with $n = |t|_Z$

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$\psi \in A\langle\langle T_{\Sigma} \rangle\rangle$ are uniform (“classical tree series”)

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Definition

Let $(A, +, 0)$ monoid. Monoid $(\mathsf{Umaps}(\Sigma, Z, A), +^{\mathsf{u}}, \widetilde{0^{\mathsf{u}}})$ with

- $\mathsf{Umaps}(\Sigma, Z, A) = \{\, \psi \colon T_\Sigma(Z) \to \mathsf{Ops}(A) \mid \psi \text{ uniform} \,\}$
- $(\psi +^{\mathsf{u}} \psi', t) = (\psi, t) + (\psi', t)$
- $(\widetilde{0^{\mathsf{u}}}, t) = 0$

Weighted Tree Automaton

Definition (M. '04)

$(Q, \Sigma, Z, \underline{A}, F, \mu, \nu)$ weighted tree automaton (short: wta), if

- $Q = \{1, \ldots, n\}$ finite set
- Σ ranked alphabet
- $Z = \{z_1, \ldots, z_m\}$ finite set (of *variables*)
- $\underline{A} = (A, +, 0, \Omega)$ DM-monoid
- $F\colon Q \to \Omega^{(1)}$ (*final weights*)

$$F = \begin{pmatrix} F_1 \\ \ldots \\ F_n \end{pmatrix} \quad \text{with } F_1, \ldots, F_n \in \Omega^{(1)}$$

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- $F \colon Q \to \Omega^{(1)}$ (*final weights*)
- $\mu = (\mu_k)_{k \in \mathbb{N}}$ with $\mu_k \colon \Sigma^{(k)} \to \left(\Omega^{(k)}\right)^{Q^k \times Q}$

$$\mu_k(\sigma) = \begin{pmatrix} \mu_k(\sigma)_{(1\cdots 11),1} & \cdots & \mu_k(\sigma)_{(1\cdots 11),n} \\ \mu_k(\sigma)_{(1\cdots 12),1} & \cdots & \mu_k(\sigma)_{(1\cdots 12),n} \\ \cdots & \cdots & \cdots \\ \mu_k(\sigma)_{(n\cdots nn),1} & \cdots & \mu_k(\sigma)_{(n\cdots nn),n} \end{pmatrix}$$

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- $\nu\colon Z \to \left(\Omega^{(1)}\right)^{Q}$

$$\nu(z) = \begin{pmatrix} \nu(z)_1 \\ \dots \\ \nu(z)_n \end{pmatrix} \quad \text{with } \nu(z)_1, \dots, \nu(z)_n \in \Omega^{(1)}$$

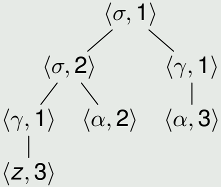
Semantics of Wta

Definition

Let $M = (Q, \Sigma, Z, \underline{A}, F, \mu, \nu)$ wta.

- $R_M = T_{\langle \Sigma, Q \rangle}(\langle Z, Q \rangle)$ runs of M
- $R_M(t) = \{\, r \in R_M \mid \pi_1(r) = t \,\}$ runs of M on t
- $R_M(t, q) = \{\, r \in R_M(t) \mid \pi_2(r(\varepsilon)) = q \,\}$ runs of M on t ending in q

Example



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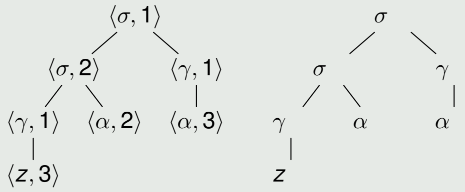


Semantics of Wta (cont'd)

Definition

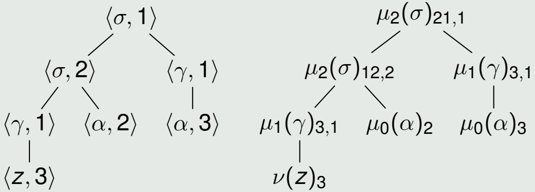
weight of run r

$$c_M(\langle \sigma, q\rangle(r_1, \dots, r_k)) = \mu_k(\sigma)_{(q_1,\dots,q_k),q}(c_M(r_1), \dots, c_M(r_k))$$

$$c_M(\langle z, q\rangle) = \nu(z)_q$$

where $q_i = \pi_2(r_i(\varepsilon))$

Example



Semantics of Wta (cont'd)

Definition

Uniform mapping recognized by M in state q

$$(S(M)_q, t) = \sum_{r \in R_M(t,q)}^{\mathsf{u}} c_M(r)$$

Uniform mapping recognized by M

$$(S(M), t) = \sum_{q \in Q}^{\mathsf{u}} F_q((S(M)_q, t))$$

Rational Operations

Definition

The following operations on $\mathsf{Umaps}(\Sigma, Z, A)$ are rational:

- sum $+^{\mathsf{u}}$
- top concatenation $\mathsf{top}_{\sigma,\omega}$ ($\sigma \in \Sigma^{(k)}$ and $\omega \in \Omega^{(k)}$)

$$(\mathsf{top}_{\sigma,\omega}(\psi_1, \ldots, \psi_k), \sigma(t_1, \ldots, t_k)) = \omega((\psi_1, t_1), \ldots, (\psi_k, t_k))$$

$$(\mathsf{top}_{\sigma,\omega}(\psi_1, \ldots, \psi_k), \delta(t_1, \ldots, t_n)) = 0$$

$$(\mathsf{top}_{\sigma,\omega}(\psi_1, \ldots, \psi_k), z) = 0$$

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- z-catenation $\cdot_z$ ($z \in Z$)

$$
\begin{aligned}
&(\psi \cdot_z \psi', t) \\
= &\sum^{\mathsf{u}}_{\substack{s,t_1,\ldots,t_k \in T_\Sigma(Z) \\ t = s[z \leftarrow (t_1,\ldots,t_k)]}} \Big((\psi, s) \circ_{W,V} \big((\psi', t_1), \ldots, (\psi', t_k) \big) \Big)
\end{aligned}
$$

where $W = \mathsf{pos}_Z(s)$ and $V = \mathsf{pos}_z(s)$

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- z-catenation $\cdot_z$ ($z \in Z$)
- z-iteration $[\cdot]^*_z$ ($z \in Z$) by $(\psi^*_z, t) = (\psi_z^{\mathsf{height}(t)+1}, t)$

$$\psi_z^0 = \widetilde{0^{\mathsf{u}}} \quad \text{and} \quad \psi_z^{n+1} = (\psi \cdot_z \psi_z^n) +^{\mathsf{u}} \mathsf{id}.z$$

Rational Expressions

Definition

$\mathsf{Rat}(\Sigma, Z, \underline{A})$ smallest R s.t.

- $\omega.z \in R$ and $[\![\omega.z]\!] = \omega.z$ ($z \in Z$ and $\omega \in \Omega^{(1)}$)
- $\mathsf{top}_{\sigma,\omega}(r_1, \ldots, r_k) \in R$ if $\sigma \in \Sigma^{(k)}$ and $\omega \in \Omega^{(k)}$

$$[\![\mathsf{top}_{\sigma,\omega}(r_1, \ldots, r_k)]\!] = \mathsf{top}_{\sigma,\omega}([\![r_1]\!], \ldots, [\![r_k]\!])$$

- $(r_1 + r_2) \in R$ and $[\![r_1 + r_2]\!] = [\![r_1]\!] +^{\mathsf{u}} [\![r_2]\!]$
- $(r_1 \cdot_z r_2) \in R$ and $[\![r_1 \cdot_z r_2]\!] = [\![r_1]\!] \cdot_z [\![r_2]\!]$
- Provided that $([\![r]\!], z) = 0$ then $r_z^* \in R$ and $[\![r_z^*]\!] = [\![r]\!]_z^*$

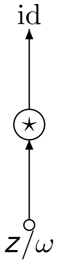
Variables

Lemma

$\omega.z \in \underline{A}^{\mathsf{rec}}\langle\langle T_\Sigma(Z)\rangle\rangle$ *($z \in Z$ and $\omega \in \Omega^{(1)}$)*

Proof.

Trivial! □



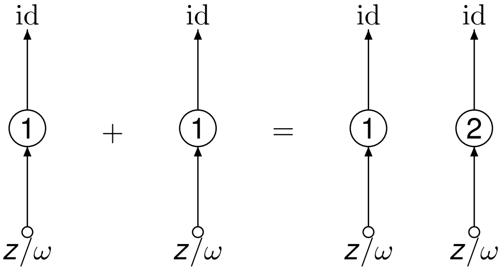
Sum

Lemma

$\underline{A}^{\mathsf{rec}}\langle\langle T_\Sigma(Z)\rangle\rangle$ *closed under uniform sum* $+^{\mathsf{u}}$.

Proof.

Classic union construction. □



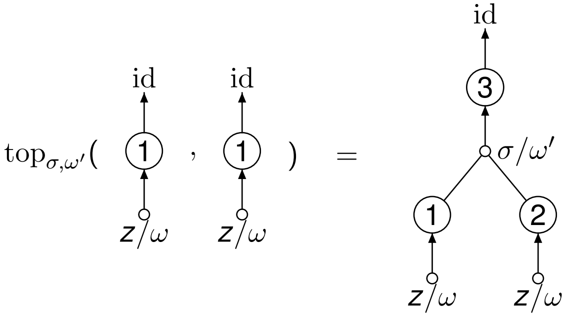
Top Concatenation

Lemma

$\underline{A}^{\mathrm{rec}}\langle\langle T_\Sigma(Z)\rangle\rangle$ *is closed under* $\mathrm{top}_{\sigma,\omega'}$ *(*$\sigma \in \Sigma^{(k)}$*,* $\omega' \in \Omega^{(k)}$*)*

Proof.

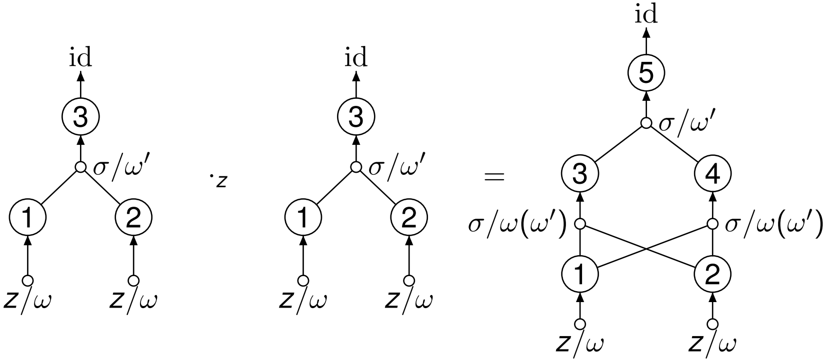
Construction illustrated below. □



z-Catenation

Lemma

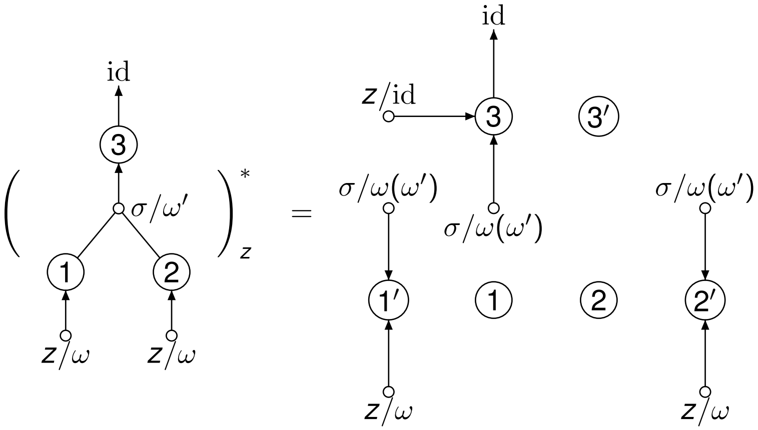
$\underline{A}^{\mathsf{rec}}\langle\langle T_\Sigma(Z)\rangle\rangle$ is closed under $\cdot_z$ ($z \in Z$)



z-Iteration

Lemma

$\underline{A}^{\text{rec}}\langle\langle T_\Sigma(Z)\rangle\rangle$ is closed under $\cdot_z^*$ ($z \in Z$)



Main Corollary

Corollary

Let $\underline{A}$ $(1,\star)$*-composition and sum closed DM-monoid*

$$\underline{A}^{\mathsf{rat}}\langle\!\langle T_{\Sigma}(Q_{\infty})\rangle\!\rangle \subseteq \underline{A}^{\mathsf{rec}}\langle\!\langle T_{\Sigma}(Q_{\infty})\rangle\!\rangle$$

Corollary

Let $\underline{A}$ $(1,\star)$*-composition and sum closed DM-monoid*

$$\underline{A}^{\mathsf{rec}}\langle\!\langle T_{\Sigma}(Q_{\infty})\rangle\!\rangle \subseteq \underline{A}^{\mathsf{rat}}\langle\!\langle T_{\Sigma}(Q_{\infty})\rangle\!\rangle$$

Main Theorem

Theorem

Let $\underline{A}$ $(1,\star)$*-composition and sum closed DM-monoid*

$$\underline{A}^{\mathsf{rec}}\langle\langle T_{\Sigma}(Q_{\infty})\rangle\rangle = \underline{A}^{\mathsf{rat}}\langle\langle T_{\Sigma}(Q_{\infty})\rangle\rangle$$

References

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Thank you for your attention!