The Power of Tree Series Transducers of Type I and II

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Abstract. The power of tree series transducers of type I and II is studied for IO as well as OI tree series substitution. More precisely, it is shown that the IO tree series transformations of type I (respectively, type II) are characterized by the composition of homomorphism top-down IO tree series transformations with bottom-up (respectively, linear bottom-up) IO tree series transformations. On the other hand, polynomial OI tree series transducers of type I and II and top-down OI tree series transducers are equally powerful.

1 Introduction

In [1] (restricted) top-down tree transducers were generalized to tree series transducers [2,3], in which each transition carries a weight taken from a semiring. It was shown in Corollary 14 of [1] that nondeleting and linear top-down tree series transformations preserve recognizable tree series [4,5,6]. In a sequel [7], Kuich also showed that nondeleting, linear top-down tree series transformations are closed under composition (see Theorem 2.4 in [7]). He built on those two properties the theory of full abstract families of tree series [7]. These results leave an unexplained gap because nondeletion is not required for these results in tree transducer theory; *i. e.*, linear top-down tree transformations with regular look-ahead [8] preserve recognizable tree languages and are closed under composition. Consequently, the survey [9] poses Question 2, which asks for the power of tree series transducers which allow look-ahead and copying of output trees [2,3].

In the unweighted case, linear top-down tree transducers with regular look-ahead are as powerful as linear bottom-up tree transducers, which was shown in Theorem 5.13 of [8]. Moreover, the power of generalized finite-state tree transducers (respectively, top-down tree transducers with regular look-ahead) is characterized by the composition of a homomorphism and a bottom-up (respectively, linear bottom-up) tree transformation (see Theorems 5.10 and 5.15 of [8]). In this paper we show that these results generalize nicely to tree series transducers. In particular, we show that the linear tree series transducers of type II, which are the canonical generalization of top-down tree transducers with regular look-ahead, compute exactly the class of linear bottom-up tree series transformations

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(see Theorem 4). Similarly, we study the canonical extension of generalized finite-state tree transducers, which are called tree series transducers of type I in the sequel. We show that the class of tree series transformations of type I (respectively, of type II) coincides with the composition of the class of homomorphism top-down tree series transformations with the class of bottom-up (respectively, linear bottom-up) tree series transformations (see Theorem 3). Altogether we obtain the analogue of the diagram presented on page 228 of [8] for IO tree series transformations over commutative and $\aleph_0$-complete semirings.

Finally, we investigate tree series transducers of type I and II using OI-substitution and thereby address Question 2 as originally posed in [9]. It turns out that polynomial tree series transducers of type I and II and top-down tree series transducers are equally powerful.

2 Preliminaries

We use $\mathbb{N}$ to represent the nonnegative integers $\{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. In the sequel, let $k, n \in \mathbb{N}$ and $[k]$ be an abbreviation for $\{\, i \in \mathbb{N} \mid 1 \leqslant i \leqslant k \,\}$. A set Σ which is nonempty and finite is also called an *alphabet*, and the elements thereof are called *symbols*. As usual, Σ^* denotes the set of all finite sequences of symbols of Σ (also called Σ-words). Given $w \in \Sigma^*$, the *length of* w is denoted by $|w|$, and for every $1 \leqslant i \leqslant |w|$ the i-th symbol in w is denoted by w_i (*i. e.*, $w = w_1 \cdots w_{|w|}$).

A *ranked alphabet* is an alphabet Σ together with a mapping $\mathrm{rk}_\Sigma \colon \Sigma \longrightarrow \mathbb{N}$, which associates to each symbol a *rank*. We use the denotation Σ_k to represent the set of symbols (of Σ) which have rank k. Furthermore, we use the set $\mathrm{X} = \{\, \mathrm{x}_i \mid i \in \mathbb{N}_+ \,\}$ of *(formal) variables* and the finite subset $\mathrm{X}_k = \{\, \mathrm{x}_i \mid i \in [k] \,\}$. Given a ranked alphabet Σ and $V \subseteq \mathrm{X}$, the set of *Σ-trees indexed by V*, denoted by $T_\Sigma(V)$, is inductively defined to be the smallest set T such that (i) $V \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T$ also $\sigma(t_1, \dots, t_k) \in T$. Since we generally assume that $\Sigma \cap \mathrm{X} = \emptyset$, we write α instead of $\alpha()$ whenever $\alpha \in \Sigma_0$. Moreover, we also write T_Σ to denote $T_\Sigma(\emptyset)$.

For every $t \in T_\Sigma(\mathrm{X})$, we denote by $|t|_x$ the number of occurrences of $x \in \mathrm{X}$ in t. Given a finite $I \subseteq \mathbb{N}_+$ and family $(\, t_i \,)_{i \in I}$ of $t_i \in T_\Sigma(\mathrm{X})$, the expression $t[t_i]_{i \in I}$ denotes the result of substituting in t every x_i by t_i for every $i \in I$. If $I = [n]$, then we simply write $t[t_1, \dots, t_n]$. Let $V \subseteq \mathrm{X}$ be finite. We say that $t \in T_\Sigma(\mathrm{X})$ is *linear in* V (respectively, *nondeleting in* V), if every $x \in V$ occurs at most once (respectively, at least once) in t. The set of all Σ-trees, which are linear and nondeleting in V, is denoted by $\widehat{T_\Sigma}(V)$.

A *semiring* is an algebraic structure $\mathcal{A} = (A, +, \cdot, 0, 1)$ consisting of a commutative monoid $(A, +, 0)$ and a monoid $(A, \cdot, 1)$ such that $\cdot$ distributes over $+$ and 0 is absorbing with respect to $\cdot$. The semiring is called commutative, if $\cdot$ is commutative. As usual we use $\sum_{i \in I} a_i$ (respectively, $\prod_{i \in I} a_i$ for $I \subseteq \mathbb{N}$) for sums (respectively, products) of families $(\, a_i \,)_{i \in I}$ of $a_i \in A$ where for only finitely many $i \in I$ we have $a_i \neq 0$ (respectively, $a_i \neq 1$). For products the order of the factors is given by the order $0 \leqslant 1 \leqslant \cdots$ on the index set I. We say that $\mathcal{A}$ is

$\aleph_0$-*complete*, whenever it is possible to define an infinitary sum operation $\sum_I$ for each countable index set I (*i. e.*, $\mathrm{card}(I) \leqslant \aleph_0$) such that for every family $(a_i)_{i \in I}$ of $a_i \in A$

(i) $\sum_I (a_i)_{i \in I} = a_{j_1} + a_{j_2}$, if $I = \{j_1, j_2\}$ with $j_1 \neq j_2$,
(ii) $\sum_I (a_i)_{i \in I} = \sum_J (\sum_{I_j} (a_i)_{i \in I_j})_{j \in J}$, whenever $I = \bigcup_{j \in J} I_j$ for some countable J and $I_{j_1} \cap I_{j_2} = \emptyset$ for all $j_1 \neq j_2$, and
(iii) $\left(\sum_I (a_i)_{i \in I}\right)\left(\sum_J (b_j)_{j \in J}\right) = \sum_{I \times J} (a_i b_j)_{(i,j) \in I \times J}$ for all countable J and families $(b_j)_{j \in J}$ of $b_j \in A$.

In the sequel, we simply write the accustomed $\sum_{i \in I} a_i$ instead of the cumbersome $\sum_I (a_i)_{i \in I}$, and we implicitly assume $\sum_I$ to be given whenever we speak about an $\aleph_0$-complete semiring.

Let S be a set and $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a semiring. A *(formal) power series* φ is a mapping $\varphi\colon S \longrightarrow A$. Given $s \in S$, we denote $\varphi(s)$ also by (φ, s) and write the series as $\sum_{s \in S} (\varphi, s)\, s$. The *support* of φ is $\mathrm{supp}(\varphi) = \{\, s \in S \mid (\varphi, s) \neq 0 \,\}$. Power series with finite support are called *polynomials*, and power series with at most one support element are also called *singletons*. We denote the set of all power series by $A\langle\!\langle S \rangle\!\rangle$ and the set of polynomials by $A\langle S \rangle$. We call $\varphi \in A\langle\!\langle S \rangle\!\rangle$ *boolean*, if $(\varphi, s) = 1$ for every $s \in \mathrm{supp}(\varphi)$. The boolean singleton with empty support is denoted by $\widetilde{0}$. Power series $\varphi, \varphi' \in A\langle\!\langle S \rangle\!\rangle$ are added componentwise; *i. e.*, $(\varphi + \varphi', s) = (\varphi, s) + (\varphi', s)$ for every $s \in S$, and the power series φ is multiplied with a coefficient $a \in A$ componentwise; *i. e.*, $(a \cdot \varphi, s) = a \cdot (\varphi, s)$ for every $s \in S$.

In this paper, we consider only power series in which the set S is a set of trees. Such power series are also called *tree series*. Let Δ be a ranked alphabet. A tree series $\varphi \in A\langle\!\langle T_\Delta(\mathrm{X}) \rangle\!\rangle$ is said to be *linear* (respectively, *nondeleting*) *in* $V \subseteq \mathrm{X}$, if every $t \in \mathrm{supp}(\varphi)$ is linear (respectively, nondeleting) in V. Let $\mathcal{A}$ be an $\aleph_0$-complete semiring, $\varphi \in A\langle\!\langle T_\Delta(\mathrm{X}) \rangle\!\rangle$, $I \subseteq \mathbb{N}_+$ be finite, and $(\psi_i)_{i \in I}$ be a family of $\psi_i \in A\langle\!\langle T_\Delta(\mathrm{X}) \rangle\!\rangle$. The *pure IO tree series substitution* (for short: IO-substitution) *(of $(\psi_i)_{i \in I}$ into φ)* [10,2], denoted by $\varphi \longleftarrow (\psi_i)_{i \in I}$, is defined by

$$\varphi \longleftarrow (\psi_i)_{i \in I} = \sum_{\substack{t \in T_\Delta(\mathrm{X}),\\ (\forall i \in I)\colon t_i \in T_\Delta(\mathrm{X})}} (\varphi, t) \cdot \prod_{i \in I} (\psi_i, t_i)\, t[t_i]_{i \in I} \enspace .$$

Let Q be an alphabet and $V \subseteq \mathrm{X}$. We write $Q(V)$ for $\{\, q(v) \mid q \in Q, v \in V \,\}$. We use the notation $|w|_x$ and the notions of linearity and nondeletion in V accordingly also for $w \in Q(\mathrm{X})^*$. Let $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a semiring and Σ and Δ be ranked alphabets. A *(type I) tree representation μ (over Q, Σ, Δ, and $\mathcal{A}$)* [2,9] is a family $(\mu_k(\sigma))_{k \in \mathbb{N}, \sigma \in \Sigma_k}$ of matrices $\mu_k(\sigma) \in A\langle\!\langle T_\Delta(\mathrm{X}) \rangle\!\rangle^{Q \times Q(\mathrm{X}_k)^*}$ such that for every $(q, w) \in Q \times Q(\mathrm{X}_k)^*$ it holds that $\mu_k(\sigma)_{q,w} \in A\langle\!\langle T_\Delta(\mathrm{X}_{|w|}) \rangle\!\rangle$, and we have $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ for only finitely many $(q, w) \in Q \times Q(\mathrm{X}_k)^*$. A tree representation μ is said to be

- *polynomial* (respectively, *boolean*), if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ the tree series $\mu_k(\sigma)_{q,w}$ is polynomial (respectively, boolean),

- *of type II* (respectively, *top-down*), if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ the tree series $\mu_k(\sigma)_{q,w}$ is linear (respectively, linear and nondeleting) in $\mathrm{X}_{|w|}$,
- *linear* (respectively, *nondeleting*), if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ both $\mu_k(\sigma)_{q,w}$ is linear (respectively, nondeleting) in $\mathrm{X}_{|w|}$, and w is linear (respectively, nondeleting) in X_k,
- *bottom-up*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $(q, w) \in Q \times Q(\mathrm{X}_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ we have that $w = q_1(\mathrm{x}_1) \cdots q_k(\mathrm{x}_k)$ for some $q_1, \dots, q_k \in Q$,
- *td-deterministic*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q \in Q$ there exists at most one $(w, t) \in Q(\mathrm{X}_k)^* \times T_\Delta(\mathrm{X})$ such that $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$, and
- *bu-deterministic*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $w \in Q(\mathrm{X}_k)^*$ there exists at most one $(q, t) \in Q \times T_\Delta(\mathrm{X})$ such that $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$.

Usually when we specify a tree representation μ, we just specify some entries of $\mu_k(\sigma)$ and implicitly assume the remaining entries to be $\widetilde{0}$. A *tree series transducer* [2,9] is a sextuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ consisting of

- an alphabet Q of *states*,
- ranked alphabets Σ and Δ, also called *input* and *output ranked alphabet*,
- a semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$,
- a vector $F \in A\langle\langle \widehat{T_\Delta}(\mathrm{X}_1) \rangle\rangle^Q$ of *final outputs*, and
- a tree representation μ over Q, Σ, Δ, and $\mathcal{A}$.

Tree series transducers inherit the properties from their tree representation; *e. g.*, a tree series transducer with a polynomial bottom-up tree representation is called a polynomial bottom-up tree series transducer. Additionally, we say that M is a *td-homomorphism* (respectively, *bu-homomorphism*), if $Q = \{\star\}$, $F_\star = 1\,\mathrm{x}_1$, and μ is td-deterministic (respectively, bu-deterministic).

For the definition of the IO tree series transformation induced by M we need IO-substitution, and consequently, $\mathcal{A}$ should be $\aleph_0$-complete. Hence let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tree series transducer over the $\aleph_0$-complete semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$. Then M induces a mapping $\|M\| \colon A\langle\langle T_\Sigma \rangle\rangle \longrightarrow A\langle\langle T_\Delta \rangle\rangle$ as follows. For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$ we define the mapping $h_\mu \colon T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle^Q$ componentwise for every $q \in Q$ by

$$h_\mu\big(\sigma(t_1, \dots, t_k)\big)_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \longleftarrow \big(\, h_\mu(t_{i_j})_{q_j} \,\big)_{j \in [n]} \enspace .$$

Then for every $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ the *(IO) tree series transformation computed by M* is

$$\|M\|(\varphi) = \sum_{t \in T_\Sigma} (\varphi, t) \cdot \sum_{q \in Q} F_q \longleftarrow \big(h_\mu(t)_q\big) \enspace .$$

By $\mathrm{TOP}(\mathcal{A})$ we denote the class of tree series transformations computable by top-down tree series transducers over the semiring $\mathcal{A}$. Similarly, we use p–$\mathrm{TOP}(\mathcal{A})$ [respectively, b–$\mathrm{TOP}(\mathcal{A})$, l–$\mathrm{TOP}(\mathcal{A})$, n–$\mathrm{TOP}(\mathcal{A})$, d–$\mathrm{TOP}(\mathcal{A})$, and h–$\mathrm{TOP}(\mathcal{A})$]

for the classes of tree series transformations computable by polynomial (respectively, boolean, linear, nondeleting, td-deterministic, and td-homomorphism) top-down tree series transducers over the semiring $\mathcal{A}$. Combinations of restrictions are handled in the usual manner; *i. e.*, let x–TOP($\mathcal{A}$) and y–TOP($\mathcal{A}$) be two classes of top-down tree series transformations, then

$$xy\text{–TOP}(\mathcal{A}) = x\text{–TOP}(\mathcal{A}) \cap y\text{–TOP}(\mathcal{A}) \ .$$

The same nomenclature using the stem $\mathrm{TOP}^{\mathrm{R}}$ (respectively, GST and BOT) is applied to type II (respectively, type I and bottom-up) tree series transducers, where for bottom-up tree series transducers the properties beginning with "td" are replaced by the corresponding ones starting with "bu". For example, hn–BOT($\mathcal{A}$) denotes the class of tree series transformations computable by nondeleting bu-homomorphism bottom-up tree series transducers over the semiring $\mathcal{A}$.

We write $\circ$ for function composition; so if $\tau_1 \colon A\langle\!\langle T_\Sigma \rangle\!\rangle \longrightarrow A\langle\!\langle T_\Delta \rangle\!\rangle$ and $\tau_2 \colon A\langle\!\langle T_\Delta \rangle\!\rangle \longrightarrow A\langle\!\langle T_\Gamma \rangle\!\rangle$ then $(\tau_1 \circ \tau_2)(\varphi) = \tau_2\big(\tau_1(\varphi)\big)$ for every $\varphi \in A\langle\!\langle T_\Sigma \rangle\!\rangle$. This composition is extended to classes of functions in the usual manner.

3 IO Tree Series Substitution

In this section we first show how to simulate a tree series transducer M of type I or II by means of the composition of a td-homomorphism top-down tree series transducer M_1 and a bottom-up tree series transducer M_2. Thereby we obtain a limitation of the power of tree series transducers of types I and II. The idea of the construction is to simply create sufficiently many copies of subtrees of the input tree by M_1. Then multiple visits of M to one input subtree such as, for example, $q(\mathrm{x}_1)$ and $p(\mathrm{x}_1)$ can be simulated by $q(\mathrm{x}_1)$ and $p(\mathrm{x}_6)$ where x_6 refers to a copy of x_1 created by M_1. More precisely, we first compute the maximal number of visits to one subtree spawned by one rule application. Let mx be that number. We create a new output alphabet from the input alphabet Σ of M by keeping the symbols of Σ but changing their rank to mx-times their rank in Σ. Reading $\sigma(t_1, \dots, t_k)$ in the input, M_1 simply outputs $\sigma(u_1, \dots, u_1, \dots, u_k, \dots, u_k)$ where u_i is the translation of t_i for every $i \in [k]$. Then we can simulate M without visiting input subtrees twice because enough copies are available. Altogether this yields that at each node of the output tree of M_1 each direct input subtree is visited at most once and such a tree series transducer can be simulated by a bottom-up tree series transducer M_2.

In the sequel, we use the notation $[y]$ where y is one of the abbreviations of restrictions (*i. e.*, $y \in \{\mathrm{p}, \mathrm{b}, \mathrm{l}, \mathrm{n}, \mathrm{d}, \mathrm{h}\}$) in equalities and inequalities to mean that this restriction is optional; *i. e.*, throughout the statement $[y]$ can be substituted by the empty word or by y. For example, [d]–TOP($\mathcal{A}$) $\subseteq$ [d]–$\mathrm{TOP}^{\mathrm{R}}(\mathcal{A})$ states that each tree series transformation computable by top-down (respectively, td-deterministic top-down) tree series transducers is also computable by tree series transducers of type II (respectively, td-deterministic tree series transducers of type II).

Lemma 1 (Decomposition). *Let $\mathcal{A}$ be a commutative, $\aleph_0$-complete semiring.*

$$[\mathrm{p}][\mathrm{b}][\mathrm{l}]\text{–GST}(\mathcal{A}) \subseteq [\mathrm{l}]\mathrm{bhn}\text{–TOP}(\mathcal{A}) \circ [\mathrm{p}][\mathrm{b}][\mathrm{l}]\text{–BOT}(\mathcal{A}) \tag{1}$$

$$[\mathrm{p}][\mathrm{b}][\mathrm{l}]\text{–TOP}^{\mathrm{R}}(\mathcal{A}) \subseteq [\mathrm{l}]\mathrm{bhn}\text{–TOP}(\mathcal{A}) \circ [\mathrm{p}][\mathrm{b}]\mathrm{l}\text{–BOT}(\mathcal{A}) \tag{2}$$

Proof. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tree series transducer. We construct a td-homomorphism top-down tree series transducer M_1 and a bottom-up tree series transducer M_2 such that $\|M\| = \|M_1\| \circ \|M_2\|$. Let $w \in Q(\mathrm{X})^*$. Recall that by $|w|_x$ we denote the number of occurrences of $x \in \mathrm{X}$ in w. Let

$$\mathrm{mx} = \max\bigl(\{1\} \cup \{\, |w|_{\mathrm{x}_j} \mid k, j \in \mathbb{N}, \sigma \in \Sigma_k, (q, w) \in Q \times Q(\mathrm{X})^*, \mu_k(\sigma)_{q,w} \neq \widetilde{0}\,\}\bigr)$$

and for every $k \in \mathbb{N}$ we let $\Gamma_{k\cdot\mathrm{mx}} = \Sigma_k$ and $\Gamma_n = \emptyset$ for every $n \in \mathbb{N}$ that is not a multiple of mx. We note that $\mathrm{mx} = 1$ if M is linear. We construct $M_1 = (\{\star\}, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ with $(F_1)_\star = 1\,\mathrm{x}_1$ and for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$

$$(\mu_1)_k(\sigma)_{\star,\underbrace{\star(\mathrm{x}_1)\cdots\star(\mathrm{x}_1)}_{\text{mx times}}\cdots\underbrace{\star(\mathrm{x}_k)\cdots\star(\mathrm{x}_k)}_{\text{mx times}}} = 1\,\sigma(\mathrm{x}_1, \ldots, \mathrm{x}_{k\cdot\mathrm{mx}}) \enspace .$$

Clearly, M_1 is a boolean, nondeleting, homomorphism tree series transducer, which is linear whenever M is so. In this case M_1 just computes the identity.

Let $\perp \notin Q$ be a new state, $Q' = Q \cup \{\perp\}$, and $d \in \mathbb{N}$ be the maximal integer such that $\Sigma_d \neq \emptyset$. For every $n \in [d]$ let $I_n \in \mathbb{N}^{[d]}$, where $\mathbb{N}^{[d]}$ is the set of all mappings from $[d]$ to $\mathbb{N}$ (alternatively a vector with d entries of $\mathbb{N}$), be $I_n(n') = 0$ for every $n' \in [d] \setminus \{n\}$ and $I_n(n) = 1$. Moreover, let $I = \sum_{i\in[d]} I_i$. For every $k \in \mathbb{N}$ we define $\mathrm{ren}_k \colon Q(\mathrm{X})^* \times \mathbb{N}^{[d]} \longrightarrow Q'(\mathrm{X})^*$ for every $f \in \mathbb{N}^{[d]}$ inductively on $Q(\mathrm{X})^*$ by

$$\begin{aligned}\mathrm{ren}_k(\varepsilon, f) = {} & \perp(\mathrm{x}_{f(1)}) \cdots \perp(\mathrm{x}_{\mathrm{mx}}) \perp(\mathrm{x}_{\mathrm{mx}+f(2)}) \cdots \perp(\mathrm{x}_{2\cdot\mathrm{mx}}) \\ & \cdots \\ & \perp(\mathrm{x}_{(k-2)\cdot\mathrm{mx}+f(k-1)}) \cdots \perp(\mathrm{x}_{(k-1)\cdot\mathrm{mx}}) \perp(\mathrm{x}_{(k-1)\cdot\mathrm{mx}+f(k)}) \cdots \perp(\mathrm{x}_{k\cdot\mathrm{mx}})\end{aligned}$$

and for every $q \in Q$, $i \in [d]$, $w \in Q(\mathrm{X}_d)^*$ by

$$\mathrm{ren}_k(q(\mathrm{x}_i)\cdot w, f) = q(\mathrm{x}_{(i-1)\cdot\mathrm{mx}+f(i)})\cdot \mathrm{ren}_k(w, f + I_i) \enspace .$$

Secondly, let $M_2' = (Q', \Gamma, \Delta, \mathcal{A}, F_2, \mu_2')$ with $(F_2)_q = F_q$ for every $q \in Q$ and $(F_2)_\perp = \widetilde{0}$ and $(\mu_2')_{k\cdot\mathrm{mx}}(\sigma)_{q,\mathrm{ren}_k(w,I)} = \mu_k(\sigma)_{q,w}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$. Finally, let $\alpha \in \Sigma_0$ be arbitrary and

$$(\mu_2')_{k\cdot\mathrm{mx}}(\sigma)_{\perp,\perp(\mathrm{x}_1)\cdots\perp(\mathrm{x}_{k\cdot\mathrm{mx}})} = 1\,\alpha \enspace .$$

Note that M_2' need not be bottom-up because there may be $(\mu_2')_k(\sigma)_{q,w} \neq \widetilde{0}$ where w is of the form $w_1\cdot q_1(\mathrm{x}_{j_1}) q_2(\mathrm{x}_{j_2})\cdot w_2$ with $j_1 > j_2$; *i. e.*, the variables in w do not occur in the order $\mathrm{x}_1, \ldots, \mathrm{x}_k$. By a straightforward reordering of the symbols $q_i(\mathrm{x}_j)$ in w and a corresponding substitution of variables in $(\mu_2')_k(\sigma)_{q,w}$, we can, however, turn M_2' into a bottom-up tree series transducer M_2.

We furthermore note that if M is of type II, then M_2' is actually a linear tree series transducer of type II, and consequently, M_2 is linear as well. Finally, it is also obvious that M_2 is polynomial (respectively, boolean, linear), whenever M is so. Clearly, the homomorphism property (of M) is not preserved because we added the extra state $\perp$. □

Now let us investigate the opposite direction; *i. e.*, the composition of a td-homomorphism top-down tree series transducer M_1 and a bottom-up tree series transducer M_2. The idea of the construction is quite straightforward; we translate the output of M_1 with the help of M_2. Therefore, we need to generalize the mapping h_{μ_2} to trees with variables. Roughly speaking, we supply h_{μ_2} with a mapping that assigns a state to each variable. A successful computation may commence at a variable only in the state assigned to the variable. So let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tree series transducer over an $\aleph_0$-complete semiring $\mathcal{A}$, and let $V \subseteq \mathbb{N}_+$. For every $\bar{q} \in Q^V$ we define the mapping $h_\mu^{\bar{q}} \colon T_\Sigma(\mathrm{X}) \longrightarrow A\langle\langle T_\Delta(\mathrm{X})\rangle\rangle^Q$ as follows.

- For every $j \in \mathbb{N}_+$ and $q \in Q$

$$h_\mu^{\bar{q}}(\mathrm{x}_j)_q = \begin{cases} \widetilde{0} & \text{if } j \in V, \bar{q}_j \neq q \ , \\ 1\,\mathrm{x}_j & \text{otherwise} \ . \end{cases}$$

- For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma(\mathrm{X})$, and $q \in Q$

$$h_\mu^{\bar{q}}\big(\sigma(t_1, \ldots, t_k)\big)_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \longleftarrow (\, h_\mu(t_{i_j})_{q_j}\,)_{j \in [n]} \ .$$

We just write $\bar{q}_1 \cdots \bar{q}_n$ for $\bar{q}$ whenever $V = [n]$ for some $n \in \mathbb{N}$.

Lemma 2 (Composition). *Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.*

$$\text{[l]h–TOP}(\mathcal{A}) \circ \text{[p][l][h]–BOT}(\mathcal{A}) \subseteq \text{[p][l][h]–GST}(\mathcal{A}) \tag{3}$$

$$\text{[l]h–TOP}(\mathcal{A}) \circ \text{[p][h]l–BOT}(\mathcal{A}) \subseteq \text{[p][l][h]–TOP}^{\mathrm{R}}(\mathcal{A}) \tag{4}$$

$$\text{[l]h–TOP}(\mathcal{A}) \circ \text{[p][h]nl–BOT}(\mathcal{A}) \subseteq \text{[p][l][h]–TOP}(\mathcal{A}) \tag{5}$$

Proof. Let $M_1 = (\{\star\}, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ be a homomorphism top-down tree series transducer and $M_2 = (Q, \Gamma, \Delta, \mathcal{A}, F, \mu_2)$ be a bottom-up tree series transducer. We construct a tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ as follows. For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$ let

$$\mu_k(\sigma)_{q,w} = h_{\mu_2}^{q_1 \cdots q_n}\big((\mu_1)_k(\sigma)_{\star, \star(\mathrm{x}_{i_1}) \cdots \star(\mathrm{x}_{i_n})}\big)_q \ .$$

By the definition of top-down tree representations, $(\mu_1)_k(\sigma)_{\star, \star(\mathrm{x}_{i_1}) \cdots \star(\mathrm{x}_{i_n})}$ is nondeleting and linear in X_n. Whenever M_2 is linear (respectively, nondeleting and linear), then M will be of type II (respectively, top-down). The proof of preservation of the additional properties is left to the reader. □

Putting Lemmata 1 and 2 together, we obtain the following characterization of the power of tree series transducers of type I and II.

Theorem 3. *Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.*

$$\text{[p][l]–GST}(\mathcal{A}) = \text{[l]bh–TOP}(\mathcal{A}) \circ \text{[p][l]–BOT}(\mathcal{A}) \tag{6}$$

$$\text{[p][l]–TOP}^{\mathrm{R}}(\mathcal{A}) = \text{[l]bh–TOP}(\mathcal{A}) \circ \text{[p]l–BOT}(\mathcal{A}) \tag{7}$$

Proof. The statements follow directly from Lemmata 1 and 2. □

Finally, we turn to the question concerning linear tree series transformations of type II. So far, we have seen that $\text{l–TOP}^{\mathrm{R}}(\mathcal{A}) = \text{lbh–TOP}(\mathcal{A}) \circ \text{l–BOT}(\mathcal{A})$ and hence $\text{l–BOT}(\mathcal{A}) \subseteq \text{l–TOP}^{\mathrm{R}}(\mathcal{A})$. The converse [*i. e.*, $\text{l–TOP}^{\mathrm{R}}(\mathcal{A}) \subseteq \text{l–BOT}(\mathcal{A})$] can be seen from our remark in the proof of Lemma 1. There we noted that if the input transducer M is linear, then the first transducer M_1 of the composition just computes the identity; thus $\text{l–TOP}^{\mathrm{R}}(\mathcal{A}) \subseteq \text{l–BOT}(\mathcal{A})$. Hence we derived the following theorem.

Theorem 4. *Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.*

$$\text{l–TOP}^{\mathrm{R}}(\mathcal{A}) = \text{l–BOT}(\mathcal{A}) \tag{8}$$

The inclusions are displayed graphically for commutative and $\aleph_0$-complete semirings $\mathcal{A}$ in Fig. 1, where all line segments are directed upwards, so that, *e. g.*, $\text{l–TOP}(\mathcal{A}) \subseteq \text{l–BOT}(\mathcal{A})$. However, none of the inclusions needs to be strict.

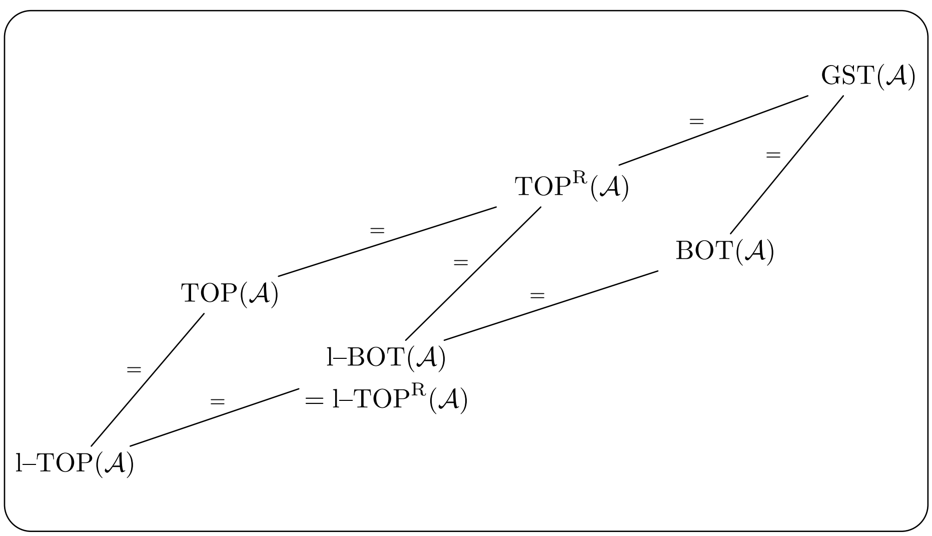

Fig. 1. Hierarchy of IO tree series transformations.

4 OI Tree Series Substitution

Throughout the survey [9] the used tree series substitution is OI tree series substitution [11,1]. It is clear that as long as the tree representation of a tree series transducer is linear and nondeleting (*i. e.*, the tree series transducer is top-down), the use of OI tree series substitution instead of pure IO tree series substitution does not yield different results. If these conditions are, however, not required (*e. g.*, for tree series transducers of types I and II), the results may diverge, and this section examines the ramifications of using OI tree series substitution. Thereby we answer Question 2 as originally posed in [9].

Let $\mathcal{A} = (A, +, \cdot, 0, 1)$ be an $\aleph_0$-complete semiring, $k \in \mathbb{N}$, $\delta \in \Delta_k$ be a symbol of a ranked alphabet Δ, and $\psi_1, \ldots, \psi_k \in A\langle\!\langle T_\Sigma(\mathrm{X})\rangle\!\rangle$. We define

$$\delta(\psi_1, \ldots, \psi_k) = \sum_{t_1, \ldots, t_k \in T_\Delta(\mathrm{X})} (\psi_1, t_1) \cdots (\psi_k, t_k)\, \delta(t_1, \ldots, t_k) \enspace .$$

Let $I \subseteq \mathbb{N}_+$ be finite, $\varphi \in A\langle\!\langle T_\Delta(\mathrm{X})\rangle\!\rangle$, and $(\, \psi_i \,)_{i \in I}$ be a family of tree series $\psi_i \in A\langle\!\langle T_\Sigma(\mathrm{X})\rangle\!\rangle$. The *OI tree series substitution (of $(\, \psi_i \,)_{i \in I}$ into φ)* [11,1] (for short: OI-substitution), denoted by $\varphi[\psi_i]_{i \in I}$, is inductively defined as follows.

- For every $j \in \mathbb{N}_+$
$$\mathrm{x}_j[\psi_i]_{i \in I} = \begin{cases} \psi_j & \text{if } j \in I \enspace , \\ 1\, \mathrm{x}_j & \text{otherwise} \enspace . \end{cases}$$
- For every $k \in \mathbb{N}$, $\delta \in \Delta_k$, and $t_1, \ldots, t_k \in T_\Delta(\mathrm{X})$
$$\delta(t_1, \ldots, t_k)[\psi_i]_{i \in I} = \delta(t_1[\psi_i]_{i \in I}, \ldots, t_k[\psi_i]_{i \in I}) \enspace .$$

Finally, $\varphi[\psi_i]_{i \in I} = \sum_{t \in T_\Sigma(\mathrm{X})} (\varphi, t) \cdot t[\psi_i]_{i \in I}$. We write $\varphi[\psi_1, \ldots, \psi_n]$ for $\varphi[\psi_i]_{i \in [n]}$.

The semantics of tree series transducers using OI-substitution is defined next. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tree series transducer. We define the mapping $h_\mu^{\mathrm{OI}} \colon T_\Sigma \longrightarrow A\langle\!\langle T_\Delta\rangle\!\rangle^Q$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$ componentwise for every $q \in Q$ by

$$h_\mu^{\mathrm{OI}}\big(\sigma(t_1, \ldots, t_k)\big)_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \big[h_\mu^{\mathrm{OI}}(t_{i_j})_{q_j}\big]_{j \in [n]} \enspace .$$

The *OI tree series transformation computed by M*, denoted by $\|M\|^{\mathrm{OI}}$, is then defined for every $\varphi \in A\langle\!\langle T_\Sigma\rangle\!\rangle$ by

$$\|M\|^{\mathrm{OI}}(\varphi) = \sum_{t \in T_\Sigma} (\varphi, t) \cdot \sum_{q \in Q} F_q\big[h_\mu^{\mathrm{OI}}(t)_q\big] \enspace .$$

We denote the class of OI tree series transformations computable by a class of tree series transducers by the OI-subscripted denotation of the class of tree series transformations computable by the same class of transducers using IO-substitution. For example, p–$\mathrm{TOP}^{\mathrm{R}}_{\mathrm{OI}}(\mathcal{A})$ denotes the class of all OI tree series transformations computable by polynomial tree series transducers of type II (over the semiring $\mathcal{A}$).

Firstly, we show that tree series transducers of type II and top-down tree series transducers are equally powerful supposed that OI-substitution is used. Roughly speaking, this is due to the fact that the tree series to be substituted for a deleted variable has absolutely no influence on the result of the substitution; *i. e.*, $\varphi[\psi_i]_{i \in I} = \varphi[\psi_i]_{i \in \mathrm{var}(\varphi)}$ where $\mathrm{var}(\varphi) = \bigcup_{t \in \mathrm{supp}(\varphi)} \mathrm{var}(t)$.

Lemma 5. *For every $\aleph_0$-complete semiring $\mathcal{A}$*

$$[\mathrm{p}][\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\text{–}\mathrm{TOP}^{\mathrm{R}}_{\mathrm{OI}}(\mathcal{A}) = [\mathrm{p}][\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\text{–}\mathrm{TOP}_{\mathrm{OI}}(\mathcal{A}) \ . \tag{9}$$

Proof. Clearly, each top-down tree series transducer is also of type II, so it just remains to prove $y\text{–}\mathrm{TOP}^{\mathrm{R}}_{\mathrm{OI}}(\mathcal{A}) \subseteq y\text{–}\mathrm{TOP}_{\mathrm{OI}}(\mathcal{A})$. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tree series transducer and $j \in \mathbb{N}_+$ be the maximal integer such that there exist $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, $w \in Q(\mathrm{X})^*$, and $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$ such that $j \leqslant |w|$ and $|t|_{\mathrm{x}_j} = 0$. We construct a tree series transducer $M' = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu')$ such that $\|M\| = \|M'\|$ and for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$ all tree series in the range of $\mu'_k(\sigma)$ will be nondeleting in $\mathrm{X}_k \cap (\mathrm{X} \setminus \mathrm{X}_{j-1})$. Clearly, since $\max\{\, |w| \mid k \in \mathbb{N}, \sigma \in \Sigma_k, \mu_k(\sigma)_{q,w} \neq \widetilde{0} \,\}$ is finite, iteration of this construction yields the desired result.

For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$, if $j > n$ then we let $\mu'_k(\sigma)_{q,w} = \mu_k(\sigma)_{q,w}$ and otherwise we set

$$\begin{aligned}\mu'_k(\sigma)_{q,w} = & \sum_{\substack{t \in T_\Delta(\mathrm{X}),\\ |t|_{\mathrm{x}_j} \geqslant 1}} (\mu_k(\sigma)_{q,w}, t)\, t \,+ \\ & + \sum_{\substack{w' \in Q(\mathrm{X}_k)^{n+1},\\ w = w'_1 \cdots w'_{j-1} w'_{j+1} \cdots w'_{n+1},\\ t \in T_\Delta(\mathrm{X} \setminus \{\mathrm{x}_j\})}} (\mu_k(\sigma)_{q,w'}, t)\, t[\mathrm{x}_1, \ldots, \mathrm{x}_j, \mathrm{x}_j, \ldots, \mathrm{x}_n] \ .\end{aligned}$$

Clearly, $\mu'_k(\sigma)_{q,w}$ is nondeleting in $X_k \cap (\mathrm{X} \setminus \mathrm{X}_{j-1})$ as is every other tree series in the range of $\mu'_k(\sigma)$ for arbitrary $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$. Moreover, $\|M\| = \|M'\|$ because for every $\varphi, \varphi' \in A\langle\langle T_\Delta(\mathrm{X})\rangle\rangle$ and family $(\,\psi_i\,)_{i \in I}$ of $\psi_i \in A\langle\langle T_\Delta(\mathrm{X})\rangle\rangle$ we have that $(\varphi + \varphi')[\psi_i]_{i \in I} = \varphi[\psi_i]_{i \in I} + \varphi'[\psi_i]_{i \in I}$ and $\varphi[\psi_i]_{i \in I} = \varphi[\psi_i]_{i \in I \setminus \{j\}}$, whenever $j \notin \mathrm{var}(\varphi)$. $\square$

Similarly, we can show that nonlinearity can be resolved by naming multiple occurrences of the same variable apart. Let $\mathrm{ren}\colon T_\Delta(\mathrm{X}) \times \mathbb{N}_+ \times \mathbb{N}_+ \longrightarrow T_\Delta(\mathrm{X})$ be the mapping such that $\mathrm{ren}(t, j, n)$ is the tree obtained by renaming the first occurrence (with respect to a depth-first left-to-right traversal of t) of x_j in t to x_j, the second occurrence of x_j to x_n, the third occurrence of x_j to x_{n+1}, and so on. Then roughly speaking, the construction is based on the observation $t[\psi_1, \ldots, \psi_k] = \mathrm{ren}(t, j, k+1)[\psi_1, \ldots, \psi_k, \psi_j, \ldots, \psi_j]$ for every $t \in T_\Delta(\mathrm{X}_k)$ and $j \in [k]$.

Lemma 6. *For every $\aleph_0$-complete semiring $\mathcal{A}$*

$$[\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\mathrm{p}\text{–}\mathrm{TOP}^{\mathrm{R}}_{\mathrm{OI}}(\mathcal{A}) = [\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\mathrm{p}\text{–}\mathrm{GST}_{\mathrm{OI}}(\mathcal{A}) \ . \tag{10}$$

Proof. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial tree series transducer and $j \in \mathbb{N}_+$ be the maximal integer such that there exist $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, $w \in Q(\mathrm{X})^*$, and $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$ such that $j \leqslant |w|$ and $|t|_{\mathrm{x}_j} > 1$. We construct a tree series transducer $M' = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu')$ such that $\|M\| = \|M'\|$ and for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$ all tree series in the range of $\mu'_k(\sigma)$ will be linear in $\mathrm{X}_k \cap (\mathrm{X} \setminus \mathrm{X}_{j-1})$. Clearly, since $\max\{\, |w| \mid k \in \mathbb{N}, \sigma \in \Sigma_k, \mu_k(\sigma)_{q,w} \neq \widetilde{0} \,\}$ is finite, iteration of this construction yields the desired result.

For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$, if $j > n$ then we let $\mu'_k(\sigma)_{q,w} = \mu_k(\sigma)_{q,w}$, and otherwise we set

$$\mu'_k(\sigma)_{q,w} = \sum_{\substack{t \in T_\Delta(\mathrm{X}),\\ |t|_{\mathrm{x}_j} \leqslant 1}} (\mu_k(\sigma)_{q,w}, t)\, t + \\ + \sum_{\substack{w' \in Q(\mathrm{X}_k)^*,\\ t \in \mathrm{supp}(\mu_k(\sigma)_{q,w'}), |t|_{\mathrm{x}_j} > 1,\\ w'' = (w'_j)^{|t|_{\mathrm{x}_j} - 1}, w = w'w''}} (\mu_k(\sigma)_{q,w'}, t)\ \mathrm{ren}(t, j, |w'| + 1) \ ,$$

Due to the fact that M is polynomial, μ' is a tree representation. Moreover, $\mu'_k(\sigma)_{q,w}$ is linear in $X_k \cap (\mathrm{X} \setminus \mathrm{X}_{j-1})$ as is every other tree series in the range of $\mu'_k(\sigma)$ for arbitrary $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$. Moreover, $\|M\| = \|M'\|$ because for every $n \in \mathbb{N}$, $i \in [n]$, $t \in T_\Delta(\mathrm{X}_n)$, and $\psi_1, \ldots, \psi_n \in A\langle T_\Delta(\mathrm{X})\rangle$ we have that

$$t[\psi_1, \ldots, \psi_n] = \mathrm{ren}(t, i, n+1)[\psi_1, \ldots, \psi_n, \underbrace{\psi_i, \ldots, \psi_i}_{|t|_{\mathrm{x}_i} - 1}] \ .$$

The proof proceeds along the lines of the one of Lemma 5 and is therefore omitted. ☐

The proof of the previous result breaks down whenever M is not polynomial. However, if there exists a constant $n \in \mathbb{N}$ such that for every $j \in \mathbb{N}_+$, $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, $w \in Q(\mathrm{X}_k)^*$, and $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$ we have that $|t|_{\mathrm{x}_j} \leqslant n$, then the result holds and can be proved in essentially the same manner.

Theorem 7. *For every $\aleph_0$-complete semiring $\mathcal{A}$*

$$[\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\mathrm{p}\text{–}\mathrm{TOP}(\mathcal{A}) = [\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\mathrm{p}\text{–}\mathrm{TOP}_{\mathrm{OI}}(\mathcal{A}) \\ = [\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\mathrm{p}\text{–}\mathrm{TOP}^{\mathrm{R}}_{\mathrm{OI}}(\mathcal{A}) = [\mathrm{b}][\mathrm{l}][\mathrm{n}][\mathrm{d}][\mathrm{h}]\mathrm{p}\text{–}\mathrm{GST}_{\mathrm{OI}}(\mathcal{A}) \ . \tag{11}$$

Proof. The theorem is an immediate consequence of Lemmata 5 and 6. ☐

Hence Question 2 of [9] can be answered by stating that polynomial tree series transducers of type I, polynomial tree series transducers of type II, and polynomial top-down tree series transducers are all equally powerful with respect to OI-substitution.

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