Weighted Picture Automata and Weighted Logics*

Ina Mäurer

Institut für Informatik, Universität Leipzig
Augustusplatz 10-11, D-04109 Leipzig, Germany,
`maeurer@informatik.uni-leipzig.de`

Abstract. The theory of two-dimensional languages, generalizing formal string languages, was motivated by problems arising from image processing and models of parallel computing. Weighted automata and series over pictures map pictures to some semiring and provide an extension to a quantitative setting. We establish a notion of a weighted MSO logics over pictures. The semantics of a weighted formula will be a picture series. We introduce weighted 2-dimensional online tessellation automata (W2OTA) extending the common automata-theoretic model for picture languages. We prove that the class of picture series defined by sentences of the weighted logics coincides with the family of picture series that are computable by W2OTA. Moreover, behaviours of W2OTA coincide precisely with the recognizable picture series characterized in [18].

1 Introduction

In the literature, a variety of formal models to recognize or generate two-dimensional arrays of symbols, called pictures, have been proposed [2, 11, 13, 15]. This research was motivated by problems arising from the area of image processing and pattern recognition [8, 19], and also plays a role in frameworks concerning cellular automata and other models of parallel computing. Different authors obtained an equivalence theorem for picture languages describing languages in terms of types of automata, sets of tiles, rational operations or existential monadic second-order (MSO) logic [10, 12, 13, 15]. New notions of weighted recognizability for picture languages defined by weighted picture automata and picture series were introduced in [3]. The weights are taken from some commutative semiring. In [18], we showed that the family of behaviours of such weighted picture automata coincides with the class of projections of certain rational picture series and can be characterized also by using tiling and domino systems. These equivalent weighted picture devices can be used to model several application examples.

Recently, Droste and Gastin [4] introduced the framework of a weighted logic over words and characterized recognizable formal power series, computed by weighted finite automata, as semantics of monadic second-order sentences within their logic. Here, we will establish a weighted MSO logic for pictures. The semantics of a weighted sentence will be a picture series that maps pictures over the underlying alphabet to elements of a commutative semiring. We also introduce weighted 2-dimensional online tessellation automata (W2OTA). This model extends the known notion of 2-dimensional online tessellation automata (2OTA) [13] for picture languages and is equivalent to the concept

---

* Supported by the GK 446/3 of the German Research Foundation.

of recognizability in [18, 3]. Our main result proves that for an alphabet and any commutative semiring the family of picture series computable by W2OTA coincides with the family of series that are definable by weighted monadic second-order sentences.

For the syntax, we basically follow classical logic. But additionally, similar to [4], we also let elements of the semiring be atomic formulas, hence are able to formulate quantitative properties of picture languages: imagine for instance the number of a's occurring in a picture. The other atomic formulas will have a semantics with values in $\{0, 1\}$. Problems arise when negation is applied, because it is not clear how to define the semantics of a negated formula. Therefore, we apply negation only to atomic formulas. Universal quantification does not preserve recognizability. Hence, as in [4], we disallow universal set quantification, but here we restrict universal first-order (FO) quantification in a new way to particular formulas, since not every recognizable picture language is determinizable and the proof in [4] does not work for two dimensions.

Crucial for proving the main theorem is to show, that the universal FO quantification of a formula, with restricted semantics, defines a recognizable series. Unlike to the word-case, we build a formula instead of constructing a certain (unweighted) automaton. This formula defines a picture language which is computable by a 2OTA that is unambiguous. Also, we use successor relations instead of built-in relations $\leq_v$ and $\leq_h$, since there are $(\leq_v, \leq_h)$-definable picture languages that are not recognizable [16]. Using successor relations, in contrast to words, not every (unweighted) FO-formulas can be made unambiguous.

Considering (unweighted) logic, our restriction of the formulas is not an essential restriction, since every (unweighted) existential MSO-formula is equivalent (in the sense of defining identical languages) to a formula in which negation is only applied to atomic formulas, and since every recognizable picture language is definable by a restricted formula. We obtain the corresponding classical equivalence when restricting to the Boolean semiring. The main result of the paper indicates that the notion of weighted recognizability for picture series is robust, since it can be characterized in terms of a logic and different automata-theoretic devices and generalizes the common frameworks for picture languages.

2 Pictures and EMSO-Logic

We recall notions and results of two-dimensional languages and MSO-logic over pictures. We assume the reader is familiar with principles in MSO logic and the equivalence theorem for picture languages [11–13, 21].

Let $\mathbb{N} = \{0, 1, \ldots\}$ and Σ and Γ be finite alphabets. A *picture* over Σ is a non-empty rectangular array of elements in Σ.[1] A *picture language* is a set of pictures. The set of all pictures over Σ is denoted by Σ^{++}. Let $p \in \Sigma^{++}$. We write $p(i, j)$ or $p_{i,j}$ for the component of p at position (i, j) and let $l_v(p)$ $(l_h(p))$ be the number of rows (columns) of p (v stands for vertical, h for horizontal). The pair $(l_v(p), l_h(p))$ is the *size* of p. The set $\Sigma^{m \times n}$ comprises all pictures with size (m, n). The domain of p is $\mathrm{Dom}(p) = \{1, \ldots, l_v(p)\} \times \{1, \ldots, l_h(p)\}$. A mapping $\pi : \Gamma \to \Sigma$ is called *projection*. It can be extended pointwise to pictures and languages as usual.

---

[1] We assume a picture to be non-empty for technical simplicity, as in [2, 13, 15].

We fix an alphabet Σ. In the literature, there are many equivalent models defining or recognizing picture languages [9–11, 13, 15]. These devices define *recognizable* picture languages and form the class $\mathrm{Rec}(\Sigma^{++})$. The set $\mathrm{MSO}(\Sigma^{++})$ of MSO-formulas over Σ is defined recursively by

$$\begin{aligned}\varphi ::= P_a(x) \mid xS_vy \mid xS_hy \mid x \in X \mid x = y \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \\ \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi\end{aligned}$$

where $a \in \Sigma$, x, y are FO variables and X is a second-order variable. A picture p is represented by the relational structure $(\mathrm{Dom}(p), S_v, S_h, (R_a)_{a\in\Sigma})$ where $R_a = \{(i,j) \in \mathrm{Dom}(p) \mid p(i,j) = a\}$, $(a \in \Sigma)$ and S_v, S_h are the two successor relations of both directions: $(i,j)S_v(i+1,j), (i,j)S_h(i,j+1)$. Formulas containing no set quantification are collected in $\mathrm{FO}(\Sigma^{++})$. We denote by $\mathrm{EMSO}(\Sigma^{++})$ the set of formulas of the form $\exists X_1, \ldots \exists X_n.\psi$ such that $\psi \in \mathrm{FO}(\Sigma^{++})$. Languages definable by formulas in $Z \subseteq \mathrm{MSO}(\Sigma^{++})$ form the set $\mathcal{L}(Z)$.

For a finite set $\mathcal{V}$ of variables, a $(\mathcal{V}, p)$-assignment σ maps FO variables in $\mathcal{V}$ to elements of $\mathrm{Dom}(p)$ and second-order variables in $\mathcal{V}$ to subsets of $\mathrm{Dom}(p)$. If x is a FO variable and $(i,j) \in \mathrm{Dom}(p)$ then $\sigma[x \to (i,j)]$ coincides with σ on $\mathcal{V} \setminus \{x\}$ and assigns (i,j) to x (similarly $\sigma[X \to I]$ for $I \subseteq \mathrm{Dom}(p)$). We encode (p,σ) where σ is a $(\mathcal{V}, p)$-assignment as a picture over $\Sigma_\mathcal{V} = \Sigma \times \{0,1\}^\mathcal{V}$. Conversely, an element in ${\Sigma_\mathcal{V}}^{++}$ is a pair (p,σ) where p is the projection over Σ and σ is the projection over $\{0,1\}^\mathcal{V}$. Then σ represents a *valid* assignment over $\mathcal{V}$ if for each FO variable $x \in \mathcal{V}$, the projection of σ to the x-coordinate contains exactly one 1. In this case, we identify σ with the $(\mathcal{V}, p)$-assignment. Let $N_\mathcal{V} \subseteq \Sigma_\mathcal{V}^{++}$ comprise $\{(p,\sigma) \mid \sigma \text{ is valid}\}$. Clearly, $N_\mathcal{V}$ is a recognizable picture language. We write $\mathrm{Free}(\varphi)$ for the set of all free variables in φ and $N_\varphi = N_{\mathrm{Free}(\varphi)}$. If $\mathcal{V}$ contains $\mathrm{Free}(\varphi)$, the definition that (p,σ) *satisfies* φ, i.e. $(p,\sigma) \models \varphi$ is as usual and we let $\mathcal{L}_\mathcal{V}(\varphi) = \{(p,\sigma) \in N_\mathcal{V} \mid (p,\sigma) \models \varphi\}$. We say that the formula φ *defines* the picture language $\mathcal{L}_{\mathrm{Free}(\varphi)}(\varphi) =: \mathcal{L}(\varphi)$.

Proposition 2.1 ([12]). *A language is* EMSO*-definable iff it is recognizable.*

The aim of this paper is to generalize this result to a quantitative setting. For this, we will define weighted 2-dimensional online tessellation automata (W2OTA). The weights are taken from a commutative semiring.

3 Weighted Automata over Pictures

A *semiring* $(K, +, \cdot, 0, 1)$ is a structure K such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $x \cdot 0 = 0 = 0 \cdot x$ for all $x \in K$. If multiplication is commutative, K is called *commutative*. Examples of semirings useful to model problems in operations research and carrying quantitative properties for many devices include e.g. the *Boolean* semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$, the natural numbers $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$, the *tropical* semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$, the *arctical (or max-plus)* semiring $\mathrm{Arc} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, the language-semiring $(\mathscr{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*)$ and $([0,1], \max, \cdot, 0, 1)$ (to capture probabilities).

We will now assign weights to pictures. This provides a generalization of the theory of picture languages to formal power series over pictures, cf. [18] and [1, 7, 14, 20].

Examples are given below. Subsequently, K will always denote a commutative semiring and Σ, Δ, Γ are alphabets.

A *picture series* is a mapping $S : \Sigma^{++} \to K$. We let $K\langle\langle \Sigma^{++} \rangle\rangle$ comprise all picture series. We write (S, p) for $S(p)$, then a series S often is written as a formal sum $S = \sum_{p \in \Sigma^{++}} (S, p) \cdot p$. The set $\text{supp}(S) = \{p \in \Sigma^{++} \mid (S, p) \neq 0\}$ is the *support* of S. For a language $L \subseteq \Sigma^{++}$, the *characteristic series* $\mathbb{1}_L : \Sigma^{++} \to K$ is defined by $(\mathbb{1}_L, p) = 1$ if $p \in L$, and $(\mathbb{1}_L, p) = 0$ otherwise. For $K = \mathbb{B}$, the mapping $L \mapsto \mathbb{1}_L$ gives a natural bijection between languages over Σ and series in $\mathbb{B}\langle\langle \Sigma^{++} \rangle\rangle$.

Definition 3.1. *A weighted 2-dimensional online tessellation automaton over Σ is a tuple $\mathfrak{A} = (\Sigma, Q, I, F, E)$, consisting of a finite set Q of states, sets of initial and final states $I, F \subseteq Q$, respectively, and a finite set of transitions $E \subseteq Q \times Q \times \Sigma \times K \times Q$.*

For $r = (q_v, q_h, a, k, q) \in E$, we set $\sigma_v(r) = q_v, \sigma_h(r) = q_h, \sigma(r) = q, \text{label}(r) = a, weight(r) = k$, and, extending this to pictures, get a function $\text{label} : E^{++} \to \Sigma^{++}$. A *run* (or *computation*) in $\mathfrak{A}$ is an element in $E^{m \times n}$ satisfying natural compatibility properties, more precisely, for $c = (c_{i,j}) \in E^{m \times n}$ we have

$$\forall\, 1 \leq i \leq m, 1 \leq j \leq n : \ \sigma_v(c_{i,j}) = \sigma(c_{i-1,j}), \sigma_h(c_{i,j}) = \sigma(c_{i,j-1}).$$

We put $weight(c) = \prod_{i,j} weight(c_{i,j})$. A run c in $\mathfrak{A}$ is *successful* if for all $1 \leq i \leq m$ and $1 \leq j \leq n$, we have $\sigma_v(c_{1,j}), \sigma_h(c_{i,1}) \in I$ and $\sigma(c_{m,n}) \in F$. The set of all successful runs labelled with p is denoted by $I \overset{p}{\rightsquigarrow} F$. The automaton $\mathfrak{A}$ *computes* (or *recognizes*) the picture series $\|\mathfrak{A}\| : \Sigma^{++} \to K$, defined for a picture $p \in \Sigma^{++}$, as $(\|\mathfrak{A}\|, p) = \sum_{c \in I \overset{p}{\rightsquigarrow} F} weight(c)$. We call $\|\mathfrak{A}\|$ the *behaviour* of $\mathfrak{A}$ and write $K^{rec}\langle\langle \Sigma^{++} \rangle\rangle$ for the family of series that are computable by W2OTA over Σ.

Considering the unweighted case, instead of E, one could also define a *transition function* $\delta : Q \times Q \times \Sigma \to 2^Q$. If $|I| = 1$ and $\delta : Q \times Q \times \Sigma \to Q$, we call $\mathfrak{A}$ *deterministic*. W2OTA generalize in a straightforward way the automata-theoretic recognizability of 2OTA for picture languages.

For motivation, we now give two examples of functions $S : \Sigma^{++} \to \mathbb{R} \cup \{\infty\}$ and $T : \Sigma^{++} \to \mathbb{N}$.

Example 3.2. Let $D \subset [0, 1]$ be a finite set of discrete values and let $L \subseteq D^{++}$ be recognizable. Consider $S : D^{++} \to \mathbb{R} \cup \{\infty\}$, mapping p to $S(p) = \sum_{i,j} p_{i,j}$ if $p \in L$ and to ∞ otherwise. One could interpret the values in D as different levels of gray [6]. Then, for each picture $p \in L$, the series S provides the total value $S(p)$ of light of p.

Example 3.3. Let C be a finite set of colors and consider $T : C^{++} \to \mathbb{N}$, defined by $(T, p) = \max\{l_v(q) \cdot l_h(q) \mid q \text{ is a monochrome subpicture of } p\}, (p \in C^{++})$. Then $T(p)$ gives the largest size of a monochrome rectangle, contained in p.

One can prove that the functions S and T are computable by W2OTA, more precisely $S \in \mathbb{T}^{rec}\langle\langle D^{++} \rangle\rangle$ and $T \in \text{Arc}^{rec}\langle\langle C^{++} \rangle\rangle$.

We define *rational* operations $\oplus$ and $\odot$, referred to as *sum* and *Hadamard product*, and also *scalar multiplications*, in the following way. For $S, T \in K\langle\langle \Sigma^{++} \rangle\rangle, k \in K$ and $p \in \Sigma^{++}$, we set $(S \oplus T, p) := (S, p) + (T, p)$, $(S \odot T, p) := (S, p) \cdot (T, p)$

and $(k \cdot S, p) := k \cdot (S, p)$. Extending projections and inverse projections to series, for $\pi : \Gamma \to \Sigma$, $R \in K\langle\langle \Gamma^{++} \rangle\rangle$ and $q \in \Gamma^{++}$, we set $(\pi(R), p) := \sum_{\pi(p')=p} (R, p')$ and $\big(\pi^{-1}(S), q\big) := \big(S, \pi(q)\big)$. Then , $\pi(R) \in K\langle\langle \Sigma^{++} \rangle\rangle$ and $\pi^{-1}(S) \in K\langle\langle \Gamma^{++} \rangle\rangle$.

Now, similar to common constructions and using ideas in [3], we can prove

Proposition 3.4. *Recognizable picture series are closed under* $\odot, \oplus, \cdot$, *projections and inverse projections. For languages, inverse projections preserve deterministic devices. If* L *is deterministically recognizable then* $\mathbb{1}_L$ *is recognizable.*

4 Weighted Logics

In this section we introduce the syntax and semantics of the weighted MSO-logic on pictures. We fix K and Σ. For $a \in \Sigma$, P_a denotes a unary predicate symbol. Formulas of the *weighted MSO-logic* are defined recursively as follows:

$$\varphi ::= k \mid P_a(x) \mid \neg P_a(x) \mid xS_v y \mid \neg(xS_v y) \mid xS_h y \mid \neg(xS_h y) \mid x \in X \mid \neg(x \in X) \\ \mid x = y \mid \neg(x = y) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi$$

where $k \in K, a \in \Sigma$ and x, y (resp. X) are first (resp. second)-order variables. The class $\mathrm{MSO}(K, \Sigma)$ comprises all such weighted MSO-formulas φ. The formulas $k, P_a(x), xS_v y, xS_h y$ and $x = y$ are referred to as *atomic formulas*. Subsequently, we will also consider the class $\mathrm{FO}(K, \Sigma) \subset \mathrm{MSO}(K, \Sigma)$ of all formulas having no set quantification. Clearly, formulas in $\mathrm{MSO}(K, \Sigma)$, containing no fragment of the form k, may also be regarded as unweighted formula defining a language $\mathcal{L}(\varphi)$. Now, similar to [4] we give the semantics of weighted MSO-formulas φ.

Definition 4.1. *Let* $\varphi \in \mathrm{MSO}(K, \Sigma)$ *and* $\mathcal{V}$ *be a finite set of variables containing* $\mathrm{Free}(\varphi)$. *The* semantics *of* φ *is a series* $[\![\varphi]\!]_{\mathcal{V}} : \Sigma_{\mathcal{V}}^{++} \to K$. *Let* $(p, \sigma) \in \Sigma_{\mathcal{V}}^{++}$. *If* σ *is not a valid* $\mathcal{V}$*-assignmen, then we set* $[\![\varphi]\!]_{\mathcal{V}}(p, \sigma) = 0$. *Otherwise, we define* $[\![\varphi]\!]_{\mathcal{V}}(p, \sigma) \in K$ *inductively as:*

$$[\![k]\!]_{\mathcal{V}}(p, \sigma) = k \qquad [\![P_a(x)]\!]_{\mathcal{V}}(p, \sigma) = \begin{cases} 1 & \textit{if } p(\sigma(x)) = a \\ 0 & \textit{otherwise} \end{cases}$$

$$[\![xS_v y]\!]_{\mathcal{V}}(p, \sigma) = \begin{cases} 1 & \textit{if } \sigma(x) S_v \sigma(y) \\ 0 & \textit{otherwise} \end{cases} \qquad [\![xS_h y]\!]_{\mathcal{V}}(p, \sigma) = \begin{cases} 1 & \textit{if } \sigma(x) S_h \sigma(y) \\ 0 & \textit{otherwise} \end{cases}$$

$$[\![x \in X]\!]_{\mathcal{V}}(p, \sigma) = \begin{cases} 1 & \textit{if } \sigma(x) \in \sigma(X) \\ 0 & \textit{otherwise} \end{cases} \qquad [\![x = y]\!]_{\mathcal{V}}(p, \sigma) = \begin{cases} 1 & \textit{if } \sigma(x) = \sigma(y) \\ 0 & \textit{otherwise} \end{cases}$$

$$[\![\neg\varphi]\!]_{\mathcal{V}}(p, \sigma) = \begin{cases} 1 & \textit{if } [\![\varphi]\!]_{\mathcal{V}}(p, \sigma) = 0 \\ 0 & \textit{if } [\![\varphi]\!]_{\mathcal{V}}(p, \sigma) = 1 \end{cases} \begin{matrix} \textit{if } \varphi \textit{ is of the form } P_a(x),\ x = y, \\ (xS_v y),\ (xS_h y) \textit{ or } (x \in X) \end{matrix}$$

$$[\![\varphi \vee \psi]\!]_{\mathcal{V}}(p, \sigma) = [\![\varphi]\!]_{\mathcal{V}}(p, \sigma) + [\![\psi]\!]_{\mathcal{V}}(p, \sigma)$$

$$[\![\varphi \wedge \psi]\!]_{\mathcal{V}}(p, \sigma) = [\![\varphi]\!]_{\mathcal{V}}(p, \sigma) \cdot [\![\psi]\!]_{\mathcal{V}}(p, \sigma)$$

$$[\![\exists x.\varphi]\!]_{\mathcal{V}}(p, \sigma) = \sum_{(i,j) \in \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}(p, \sigma[x \to (i, j)])$$

$$\begin{aligned}
[\![\exists X.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \sum_{I\subseteq \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(p,\sigma[X\to I]) \\
[\![\forall x.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \prod_{(i,j)\in \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(p,\sigma[x\to (i,j)]) \\
[\![\forall X.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \prod_{I\subseteq \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(p,\sigma[X\to I]).
\end{aligned}$$

We write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$. In case φ is a sentence, then $[\![\varphi]\!] \in K\langle\!\langle \Sigma^{++}\rangle\!\rangle$. For $Z \subseteq \mathrm{MSO}(K,\Sigma)$, we call a series $S : \Sigma^{++} \to K$ *Z-definable* if there exists a sentence $\varphi \in Z$ satisfying $[\![\varphi]\!] = S$.

Example 4.2. Consider the formula $\varphi = \exists x.P_a(x) \in \mathrm{MSO}(\mathbb{N},\{a,b,c\})$. Then $[\![\varphi]\!]$ is the series that computes for a picture $p \in \{a,b,c\}^{++}$ the number of occurrences of the letter a in p. Also, consider Example 3.2 of Section 3 again. For $L = D^{++}$, the formula $\psi = \forall x.\big(\bigvee_{d\in D}(P_d(x)\wedge d)\big) \in \mathrm{MSO}(\mathbb{T},D)$ satisfies $[\![\psi]\!] = S$.

For different sets of variables $\mathcal{V}$, we show that our semantics are consistent:

Proposition 4.3. *Let $\varphi \in \mathrm{MSO}(K,\Sigma)$, $\mathcal{V}$ be finite containing $\mathrm{Free}(\varphi)$ and $(p,\sigma) \in N_{\mathcal{V}}$. Then $[\![\varphi]\!]_{\mathcal{V}}(p,\sigma) = [\![\varphi]\!](p,\sigma_{|\,\mathrm{Free}(\varphi)})$, and $[\![\varphi]\!]$ is recognizable iff $[\![\varphi]\!]_{\mathcal{V}}$ is recognizable.*

Proof. The first claim is proved by induction. Let $[\![\varphi]\!] \in K^{\mathrm{rec}}\langle\!\langle \Sigma_\varphi^{++}\rangle\!\rangle$ and $\pi : \Sigma_{\mathcal{V}} \to \Sigma_\varphi$ the projection. Then, $[\![\varphi]\!]_{\mathcal{V}} = (\pi^{-1}[\![\varphi]\!]) \odot \mathbb{1}_{N_{\mathcal{V}}} \in K^{\mathrm{rec}}\langle\!\langle \Sigma_{\mathcal{V}}^{++}\rangle\!\rangle$ by Proposition 3.4. Now, let $[\![\varphi]\!]_{\mathcal{V}} \in K^{\mathrm{rec}}\langle\!\langle \Sigma_{\mathcal{V}}^{++}\rangle\!\rangle$ and $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) be the set of first (resp. second)-order variables in $\mathcal{V}$. Then, $N^{\mathrm{norm}} = \Big\{(p,\sigma) \in N_{\mathcal{V}} \big| {\scriptstyle \forall x\in V_1\setminus\mathrm{Free}(\varphi):\sigma(x)=(1,1), \atop \forall X\in V_2\setminus\mathrm{Free}(\varphi):\sigma(X)=\{(1,1)\}}\Big\}$ is deterministically recognizable. For $(p,\sigma) \in N_\varphi$, π maps exactly one element $(p,\sigma^{\mathrm{norm}}) \in N^{\mathrm{norm}}$ on (p,σ). With the above and Proposition 3.4, we conclude

$$\big(\pi([\![\varphi]\!]_{\mathcal{V}} \odot \mathbb{1}_{N^{\mathrm{norm}}}),(p,\sigma)\big) = \sum_{\substack{\pi(p,\sigma')=(p,\sigma)\\ (p,\sigma')\in N^{\mathrm{norm}}}} [\![\varphi]\!]_{\mathcal{V}}(p,\sigma') = [\![\varphi]\!]_{\mathcal{V}}(p,\sigma^{\mathrm{norm}}) = [\![\varphi]\!](p,\sigma).\square$$

For words, examples show that unrestricted application of universal first-order quantification does not preserve recognizability [4, Ex. 3.3, 3.4]. These settings are contained in our context of the weighted MSO logic and series over pictures.

Definition 4.4. *A picture series $S : \Sigma^{++} \to K$ is a* first-order step function *(FO step function), if $S = \bigoplus_{i=1}^{n} k_i \cdot \mathbb{1}_{L_i}$ for some $n \in \mathbb{N}, k_i \in K$ and languages $L_i \in \mathcal{L}(\mathrm{FO}(\Sigma^{++}))$ ($i = 1,\ldots,n$) that are definable by* FO *formulas.*

We will call $\varphi \in \mathrm{MSO}(K,\Sigma)$ *restricted*, if φ contains no universal set quantification of the form $\forall X.\psi$, and whenever φ contains a universal quantification $\forall x.\psi$, then $[\![\psi]\!]$ is a FO step function. We let $\mathrm{RMSO}(K,\Sigma)$ comprise all restricted formulas of $\mathrm{MSO}(K,\Sigma)$. Furthermore, let $\mathrm{REMSO}(K,\Sigma)$ contain all restricted *existential* MSO-formulas φ, i.e. φ is of the form $\varphi = \exists X_1,\ldots,X_n.\psi$ such that $\psi \in \mathrm{FO}(K,\Sigma) \cap \mathrm{RMSO}(K,\Sigma)$. The families $K^{\mathrm{rmso}}\langle\!\langle \Sigma^{++}\rangle\!\rangle$ (resp. $K^{\mathrm{remso}}\langle\!\langle \Sigma^{++}\rangle\!\rangle$) contain all picture series $S \in K\langle\!\langle \Sigma^{++}\rangle\!\rangle$ which are definable by some sentence in $\mathrm{RMSO}(K,\Sigma)$ (resp. in

$\mathrm{REMSO}(K, \Sigma)$). The following equivalence theorem states that for an alphabet Σ and any commutative semiring K, the family of recognizable picture series coincides with the families of series defined in terms of weighted RMSO resp. REMSO logic.

Theorem 4.5. *We have* $K^{\mathrm{rec}}\langle\langle \Sigma^{++} \rangle\rangle = K^{\mathrm{rmso}}\langle\langle \Sigma^{++} \rangle\rangle = K^{\mathrm{remso}}\langle\langle \Sigma^{++} \rangle\rangle$.

In parts of our proofs, we follow ideas of [4]. The crucial difference concerns the universal FO quantification. For pictures, not every recognizable language is determinizable, but this is one important property within the proofs of [4]. Here we consider a restriction of this quantification to formulas having a semantics which is a FO step function. But still the proof of the word-case does not work due to the two dimensions of a run in an automaton. We therefore rather build a formula instead of constructing a certain (unweighted) automaton. For the disposition of weights, the key property will be that this unweighted formula defines a language which is computable by a 2OTA that is unambiguous. Also, observe that, in the Theorem 4.5, going from RMSO to REMSO is not at all clear, since unlike to the situation of words [4, Lemma 5.2], in the framework of pictures using successor relations, instead of $\leq_v$ and $\leq_h$, not every (unweighted) FO-formula can be made unambiguous. However, we have to handle successor relations, since there are $(\leq_v, \leq_h)$-definable picture languages that are not recognizable.

5 Unambiguous Picture Languages

We call a possibly weighted 2OTA $\mathfrak{A}$ *unambiguous* if for any input picture there exists at most one successful run in $\mathfrak{A}$. Simulating the proof of Proposition 3.4, if L is unambiguously computable, then $\mathbb{1}_L \in K^{\mathrm{rec}}\langle\langle \Sigma^{++} \rangle\rangle$. For $L \subseteq \Gamma^{++}$ and a projection $\pi : \Gamma \to \Sigma$, we call π *injective* on L if $\pi : L \to \Sigma^{++}$ is an injective mapping. For $p \in \Sigma^{++}$, $\hat{p}$ denotes the picture that results from p by surrounding it with the (new) boundary symbol $\#$. If p has size (m, n) then $\hat{p}$ has size $(m{+}2, n{+}2)$. *Tiles* are pictures of size $(2, 2)$. We denote by $T(p)$ the set of all sub-tiles of p. A language $L \subseteq \Gamma^{++}$ is *local* if there exists a set Θ of tiles over $\Gamma \cup \{\#\}$, such that $L = \{p \in \Gamma^{++} \mid T(\hat{p}) \subseteq \Theta\}$. Then (Γ, Θ) *characterizes* L. We write $L = \mathcal{L}(\Theta)$. In [9], the authors briefly mention the notion of ambiguity for picture languages in the context of tiling systems (TS). We define $L \subseteq \Sigma^{++}$ as *unambiguously tiling recognizable* if there exists a local language $L' \subseteq \Gamma^{++}$, characterized by (Γ, Θ), and a projection $\pi : \Gamma \to \Sigma$ such that π is injective on L' and $\pi(L') = L$. In this case, we call $(\Sigma, \Gamma, \Theta, \pi)$ an *unambiguous* TS *computing* L. If the projection is not necessarily injective, we obtain the known definition of a TS. Unambiguously tiling recognizable languages over Σ are collected in $\mathrm{UPLoc}(\Sigma^{++})$.

Lemma 5.1. $\mathrm{UPLoc}(\Sigma^{++})$ *is closed under injective projections and disjoint union. A language L is recognizable by an unambiguous 2OTA if and only if it is computable by an unambiguous tiling system.*

Proof. Let Σ, Γ, Δ be alphabets and $(\Gamma, \Delta, \Theta, \psi)$ unambiguous for $L \subseteq \Gamma^{++}$. If $\pi : \Gamma \to \Sigma$ is injective on L, then $\tau := (\Sigma, \Delta, \Theta, \psi \circ \pi)$ is unambiguous for $\pi(L)$. Let $L_1, L_2 \in \mathrm{UPLoc}\, \Sigma^{++}, L_1 \cap L_2 = \emptyset$. We follow the construction in [11, Theorem 7.4]. The given TS for $L := L_1 \cup L_2$ is unambiguous since the union is disjoint. For the second claim, the TS, constructed in [11, Lemma 8.1] and also the automaton constructed in [11, Lemma 8.2] are unambiguous. □

We call languages in $\mathrm{UPLoc}(\Sigma^{++})$ *unambiguous*. We obtain further equivalences by injective projections of unambiguous rational operations and unambiguously domino recognizable series, cf. [17]. We now show, that FO definable picture languages are unambiguous. This will be crucial for proving Lemma 6.3 below. The idea for the course of the proof is to follow constructions in [12]. But, we now need unambiguous picture languages, hence we have to construct injective projections and disjoint unions.

Proposition 5.2. *If L be $\mathrm{FO}(\Sigma^{++})$-definable. Then L is unambiguous.*

Proof. For $d, t \geq 1$, (d,t)-locally threshold testable (LTT) picture languages can be characterized by subsquares of dimension $\leq d$, where the occurrences are counted up to a threshold t. A language is FO-definable iff it is LTT and every LTT language L is recognizable [12]. We show that LTT languages are unambiguous. Let $L \subseteq \Sigma^{++}$ be LTT for (d,t). As in Lemma 3.7 [12], we partition L into a union of strictly LTT languages (where strictly means, only squares of dimension d are considered). This union is easily proved as disjoint. Strictly LTT languages are projections d-local languages (for d-locality we use a set $\Theta^{(d)}$ of $(d \times d)$-tiles instead of (2×2)-tiles for local sets)[12, Lemma 3.9]. For a given (d,t)-strictly LTT language L', in the construction, one performs a scanning of $p' \in L'$ using certain d-squares and counts occurrences up to t. For acceptance, one compares these computed values with the tuples characterizing L'. It defines a d-local language L'' and a projection π satisfying $\pi(L'') = L'$. We can modify the d-tiles (and hence L'') by strengthening their border-conditions in such a way that for every $p' \in L'$ there exists one uniquely determined $p'' \in L''$ with $\pi(p'') = p'$. Hence, the modified projection then is injective on L''.

It remains to show that every d-local language M is unambiguous. For this, let Δ be arbitrary, $d \geq 3$ and M characterized by $(\Delta, \Theta^{(d)})$. We can assume $M \subseteq \Delta^{m \times n}$ such that $m, n \geq d-2$ ([12, Lemma 3.10]). We prove that M is an injective projection of a local set, that is, M is computable by an unambiguous tiling system. We define $T = (\Delta, \Gamma, \Theta, \pi)$ as

- $\overline{\Theta^{(d)}} := \left\{ \boxed{\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}} \in (\Delta \,\dot\cup\, \{\#\} \,\dot\cup\, \{+\})^{d \times d} \mid B = C = D \equiv +\,,\ \exists\ \boxed{\begin{array}{c|c} \mathrm{A1} & \mathrm{A2} \\ \hline \mathrm{A3} & \mathrm{A} \end{array}} \in \Theta^{(d)} \right\}$
- $\Gamma := \overline{\Theta^{(d)}} \setminus \{p \mid p_{1,1} = \#\}$; border symbols: $\{p \in \overline{\Theta^{(d)}} \mid p_{1,1} = \#\}$
- $\Theta := \left\{ \boxed{\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}} \in \Gamma^{2\times 2} | A = \boxed{\begin{array}{c|c} \mathrm{a} & \mathrm{N} \\ \hline \mathrm{W} & \mathrm{Q} \end{array}}, B = \boxed{\begin{array}{c|c} \mathrm{N} & \mathrm{b} \\ \hline \mathrm{Q} & \mathrm{E} \end{array}}, C = \boxed{\begin{array}{c|c} \mathrm{W} & \mathrm{Q} \\ \hline \mathrm{c} & \mathrm{S} \end{array}}, D = \boxed{\begin{array}{c|c} \mathrm{Q} & \mathrm{E} \\ \hline \mathrm{S} & \mathrm{d} \end{array}} \right\}$
  where $Q \in (\Delta \,\dot\cup\, \{\#\} \,\dot\cup\, \{+\})^{(d-1)\times(d-1)}$ and W, S, E, N, a, b, c, d accordant.

We set $\pi : \Gamma \to \Delta,\ p \mapsto p_{1,1}$ and show $\pi(\mathcal{L}(\Theta)) = M$. Let $p \in M$. We extend p to $\bar{p} \in (\Delta \,\dot\cup\, \{\#\} \,\dot\cup\, \{+\})^{(m+d-1)\times(n+d-1)}$ and define $p' \in \Gamma^{m \times n}$, as

$$\bar{p}(i,j) = \begin{cases} p(i,j) & i \leq m, j \leq n \\ \# & \begin{array}{l} i = m+1, j \leq n+1 \\ or\ i \leq m+1, j = n+1 \end{array} \\ + & otherwise. \end{cases}, \quad p'(i,j) = \boxed{\begin{array}{lcr} \bar{p}(i,j) & & \bar{p}(i,j+d-1) \\ & \cdots & \\ \bar{p}(i+d-1,j) & & \bar{p}(i+d-1,j+d-1) \end{array}}.$$

Then, $p' \in \mathcal{L}(\Theta)$ and $\pi(p') = p$. Now let $p' \in \mathcal{L}(\Theta)$ and q be a $(d \times d)$-subpicture of $\hat{p}'$. It suffices to show $\pi(q) \in \Theta^{(d)}$. With the construction of $\overline{\Theta^{(d)}}$ we have $q_{1,1} \in \Theta^{(d)}$. But, $q_{1,1} = \pi(q)$. By the structure of the d-tiles in Θ, one can show that T is unambiguous, i.e., π is injective on $\mathcal{L}(\Theta)$. We constructed unambiguous languages and disjoint unions. With Lemma 5.1, L is unambiguous. □

6 Definable Picture Series are Recognizable

The aim of this section is to show that semantics of sentences in $\mathrm{RMSO}(K, \Sigma)$ are recognizable series. We prove this implication by structural induction on the formulas in $\mathrm{RMSO}(K, \Sigma)$.

Lemma 6.1. *Let $\mathcal{V}$ be a set of variables. Then the set $N_{\mathcal{V}}$ is FO-definable. The class $\mathcal{L}(\mathrm{FO}(\Sigma^{++}))$ is closed under boolean operations.*

Lemma 6.2. *Let $\varphi, \psi \in \mathrm{MSO}(K, \Sigma)$. Then the following holds.*

(a) If φ is atomic or the negation of an atomic formula, then φ is recognizable.
(b) If $[\![\varphi]\!]$ and $[\![\psi]\!]$ are recognizable, then $[\![\varphi \vee \psi]\!]$ and $[\![\varphi \wedge \psi]\!]$ are recognizable.
(c) If $[\![\varphi]\!]$ is recognizable, then $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are recognizable.

Proof. (a) We construct W2OTA using Proposition 3.4. The other proofs are similar to the word-case ([4, Lemma 4.1]) and use Propositions 4.3 and 3.4. □

The next Lemma shows that for FO step functions the application of the universal first-order quantification provides a recognizable semantics. We use ideas of [4, Lemma 4.2], but these did not completely work in this setting.

Lemma 6.3. *Let $\varphi \in \mathrm{MSO}(K, \Sigma)$ such that $[\![\varphi]\!]$ is a first-order step function. Then $[\![\forall x.\varphi]\!]$ is a recognizable picture series.*

Proof. As prerequisite, let $\mathcal{W} = \mathrm{Free}(\varphi)$, $\mathcal{V} = \mathrm{Free}(\forall x.\varphi) = \mathcal{W} \setminus \{x\}$ and assume $[\![\varphi]\!] = \bigoplus_{l=1,\ldots,n} k_l \cdot \mathbb{1}_{L_l}$ with $n \in \mathbb{N}$, $k_l \in K$ and $L_l \in \mathcal{L}(\mathrm{FO}(\Sigma_{\mathcal{W}}^{++}))$ $(l = 1, \ldots, n)$ such that the languages L_l form a partition (use Lemma 6.1). Assume $x \in \mathcal{W}$.

The definition of the semantics of the universal FO quantification of a formula maps a picture p to the product over all positions in p of certain values in K. In our setting, the factors are the elements k_l corresponding to the supports of $[\![\varphi]\!]$. We mark positions of p by their respective index l of k_l. Let $\widetilde{\Sigma} = \Sigma \times \{1, \ldots, n\}$. A picture in $(\widetilde{\Sigma}_{\mathcal{V}})^{++}$ will be written as (p, ν, σ) where $(p, \sigma) \in \Sigma_{\mathcal{V}}^{++}$ and $\nu \in \{1, \ldots, n\}^{++}$ is interpreted as a mapping from $\mathrm{Dom}(p)$ to $\{1, \ldots, n\}$. Let $\widetilde{L}$ be the set of $(p, \nu, \sigma) \in (\widetilde{\Sigma}_{\mathcal{V}})^{++}$ such that $\nu(i,j) = l \iff (p, \sigma[x \to (i,j)]) \in L_l$ for all $(i,j) \in \mathrm{Dom}(p)$ and $l \in \{1, \ldots, n\}$. We prove $\widetilde{L} \in \mathrm{FO}(\widetilde{\Sigma}_{\mathcal{V}}^{++})$ by presenting a formula. Let $1 \leq l \leq n$ and φ_l be an FO-sentence over ${\Sigma_{\mathcal{W}}}^{++}$ for L_l. We define $\widetilde{\varphi}_l \in \mathrm{FO}((\widetilde{\Sigma_{\mathcal{W}}})^{++})$ as φ_l where all occurrences of $P_{(a,r)}(y)$ (here, $a \in \Sigma, r \in \{0,1\}^{\mathcal{W}}$) are replaced by $\bigvee_{1 \leq l \leq n} P_{(a,r,l)}(y)$. Then, for $(p, \tau, \nu) \in \widetilde{\Sigma_{\mathcal{W}}}^{++}$, we conclude $(p, \tau, \nu) \in \mathcal{L}(\widetilde{\varphi}_l)$ iff $(p, \tau) \in \mathcal{L}(\varphi_l)$. Additionally, we define $\widetilde{\varphi}_l{}'$ as $\widetilde{\varphi}_l$, modified as follows. Occurrences of $P_{(a,r,l)}(y)$ satisfying $r(x) = 1$ become $P_{(a,r',l)}(y) \wedge (x = y)$ and occurrences of $P_{(a,r,l)}(y)$ with $r(x) = 0$ become $P_{(a,r',l)}(y) \wedge \neg(x = y)$, where r' is the restriction of r to $\mathcal{W} \setminus \{x\}$. Then, $\widetilde{\varphi}_l{}'$ is an FO-formula over the alphabet $\widetilde{\Sigma}_{\mathcal{V}}$ with $\mathrm{Free}(\widetilde{\varphi}_l{}') = \{x\}$

satisfying for all $(p,\tau,\nu) \in N_{\widetilde{\varphi_l}'}$, that $(p,\tau',\nu) \in \mathcal{L}(\widetilde{\varphi_l}')$ if and only if $(p,\tau,\nu) \in \mathcal{L}(\widetilde{\varphi}_l)$. Now, set $\widetilde{\varphi} := \forall x. \bigwedge_{1\leq l\leq n} \left[(\nu(x) = l) \Leftrightarrow \widetilde{\varphi_l}'\right]$ where $\nu(x) = l$ and $\Leftrightarrow$ are standard abbreviations. Now, $\widetilde{\varphi}$ is an FO-sentence over $\widetilde{\Sigma}_\mathcal{V}$. We show $\mathcal{L}(\widetilde{\varphi}) = \widetilde{L}$.

Let $(q,\nu) \in (\widetilde{\Sigma}_\mathcal{V})^{++}$ (here, $q \in \Sigma_\mathcal{V}^{++}$). Then $(q,\nu) \models \widetilde{\varphi}$ iff for all $(i,j) \in \mathrm{Dom}(q)$ and all $1 \leq l \leq n$, $(q,\nu,[x \to (i,j)]) \in \mathcal{L}\big((\nu(x) = l) \Leftrightarrow \widetilde{\varphi_l}'\big)$, where $[x \to (i,j)]$ denotes the assignment defined on $\{x\}$ mapping x to (i,j). Now, $(q,\nu,[x \to (i,j)]) \in \mathcal{L}(\nu(x) = l)$ iff $\nu(i,j) = l$ and $(q,\nu,[x \to (i,j)]) \in \mathcal{L}(\widetilde{\varphi_l}')$ iff $(q,\nu,\sigma[x \to (i,j)]) \in \mathcal{L}(\widetilde{\varphi}_l)$ iff $(q,[x \to (i,j)]) \in \mathcal{L}(\varphi_l)$ iff $(q,[x \to (i,j)]) \in L_l$. Hence the constructed formula $\widetilde{\varphi}$ defines $\widetilde{L}$.

Now, using Proposition 5.2 and Lemma 5.1, there exists an unambiguous 2OTA $\widetilde{A} = \big(\widetilde{\Sigma}_\mathcal{V}, Q, I, F, E\big)$ computing $\widetilde{L}$. We obtain a W2OTA $\widetilde{\mathfrak{A}} = \big(\widetilde{\Sigma}_\mathcal{V}, Q, I, F, \bar{E}\big)$ disposing weights along $\widetilde{A}$ as: $\big(p,q,(a,l,s),r\big) \in E$ iff $\big(p,q,(a,l,s),k_l,r\big) \in \bar{E}$, where $[\![\varphi]\!] = \bigoplus_{l=1,\ldots,n} k_l \cdot \mathbb{1}_{L_l}$. Then $\widetilde{\mathfrak{A}}$ is unambiguous. Similar to [4, Lemma 4.2], using Proposition 3.4 and Lemma 6.2, for the projection $\pi : \widetilde{\Sigma_\mathcal{V}} \to \Sigma_\mathcal{V}$, one proves for $(p,\sigma) \in \Sigma_\mathcal{V}^{++}$: $\big(\pi(\|\widetilde{\mathfrak{A}}\|),(p,\sigma)\big) = [\![\forall x.\varphi]\!](p,\sigma)$, hence $[\![\forall x.\varphi]\!]$ is recognizable. The case $x \notin \mathcal{W}$ is reduced to above. $\square$

Theorem 6.4. *We have* $K^{\mathrm{rmso}}\langle\!\langle \Sigma^{++} \rangle\!\rangle \subseteq K^{\mathrm{rec}}\langle\!\langle \Sigma^{++} \rangle\!\rangle$.

7 Recognizable Picture Series are Definable

We want to show that recognizable series are REMSO-definable. Similar to [4, 5], for a W2OTA $\mathfrak{A}$ we construct a weighted EMSO-sentence γ such that $\|\mathfrak{A}\| = [\![\gamma]\!]$. It then remains to prove that γ is restricted. We also need the notion of unambiguous formulas. We note that, unlike to the word-cases, we use successor relations, here not every (unweighted) FO-formula can be made unambiguous. The class of *unambiguous* formulas in $\mathrm{FO}(K,\Sigma)$ is defined inductively as follows: All atomic formulas and their negations are unambiguous. If φ, ψ are unambiguous, then $\varphi \wedge \psi, \forall x.\varphi$ are unambiguous. If φ, ψ are unambiguous and $\mathrm{supp}([\![\varphi]\!]) \cap \mathrm{supp}([\![\psi]\!]) = \emptyset$, then $\varphi \vee \psi$ is unambiguous. Let $\mathcal{V} = \mathrm{Free}(\varphi)$. If φ is unambiguous and for any $(p,\sigma) \in \Sigma_\mathcal{V}^{++}$ there is at most one element $(i,j) \in \mathrm{Dom}(p)$ such that $[\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(p,\sigma[x \to (i,j)]) \neq 0$, then $\exists x.\varphi$ is unambiguous.

By $\text{qf-MSO}^-(K,\Sigma)$, we denote formulas in $\mathrm{MSO}(K,\Sigma)$ having no quantification and no subformula of the form k. To make such formulas unambiguous we perform a syntactic transformations $(.)^+$ and $(.)^-$ in a simultaneous induction such that, for $\varphi, \psi \in \text{qf-MSO}^-(K,\Sigma)$, we have $\mathcal{L}(\varphi^+) = \mathcal{L}(\varphi)$ and $\mathcal{L}(\varphi^-) = \Sigma_{\mathrm{Free}(\varphi)}^{++} \setminus \mathcal{L}(\varphi)$. Now, similar to [4, Prop. 5.1], we get:

Lemma 7.1. *Let* $\varphi \in \mathrm{FO}(K,\Sigma)$ *be unambiguous. Then* $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$. *For* $\psi \in \text{qf-MSO}^-(K,\Sigma)$, *the formula* ψ^+ *is unambiguous.*

Notions like $\min_v(x), \max_h(z)$ or $\mathrm{part}(X_1,\ldots,X_l)$ abbreviate common formulas. We set (analog init_W):

$$\mathrm{init}_\mathrm{N} := \forall x.\Big(\Big[\min_v(x) \wedge \Big(\bigvee_{q_h^x, q^x \in Q, q_v^x \in I, a \in \Sigma} x \in X_{(q_v^x, q_h^x, a, q^x)}\Big)^+\Big] \vee \exists s.(s S_v x)\Big).$$

For intuition, the formulas $\mathrm{init}_{\mathrm{N}}$ (resp. $\mathrm{init}_{\mathrm{W}}$) simulate accepting conditions of the automaton for the first row (resp. first column) of an input picture.

Theorem 7.2. *We have* $K^{\mathrm{rec}}\langle\!\langle \Sigma^{++}\rangle\!\rangle \subseteq K^{\mathrm{remso}}\langle\!\langle \Sigma^{++}\rangle\!\rangle$.

Proof (sketch). Let $\mathfrak{A} = (\Sigma, Q, I, F, E)$ be a W2OTA. For $a \in \Sigma, q_v, q_h, q \in Q$, we set $\mu_{(q_v,q_h,q)}(a) = \sum_{(q_v,q_h,a,k,q)\in E} k$. Let $\mathcal{V} = \big\{X_{(q_v,q_h,a,q)} \mid (q_v, q_h, a, q) \in Q^2 \times \Sigma \times Q\big\}$ the set of set variables, $(X_1, \ldots, X_l)$ an enumeration of $\mathcal{V}$. We set

$$\alpha(X_1, \ldots, X_l) := \mathrm{part}(X_1, \ldots, X_l) \wedge \bigwedge_{q_v,q_h,a,q} \forall x.\Bigg((x \in X_{(q_v,q_h,a,q)}) \to P_a(x)\Big)$$
$$\wedge\ \ \forall x \forall z.\big((x S_v z) \to \bigvee_{q_v^x,q_h^x,q^x,q_h^z,q^z \in Q; a,b\in\Sigma} (x \in X_{(q_v^x,q_h^x,a,q^x)}) \wedge (z \in X_{(q^x,q_h^z,b,q^z)})\Bigg)^+$$
$$\wedge\ \ \forall y \forall z.\Bigg((y S_h z) \to \bigvee_{q_v^y,q_h^y,q^y,q_v^z,q^z \in Q; c,b\in\Sigma} (y \in X_{(q_v^y,q_h^y,c,q^y)}) \wedge (z \in X_{(q_v^z,q^y,b,q^z)})\Bigg)^+ .$$

The formula α qualifies unweighted runs in $\mathfrak{A}$. Now, let $\beta(X_1, \ldots, X_l) :=$

$$\alpha \wedge \bigwedge_{q_v,q_h,a,q} \forall x.\Big(x \in X_{(q_v,q_h,a,q)}) \to \mu_{(q_v,q_h,q)}(a)\Big) \wedge \mathrm{init}_{\mathrm{N}}\ \wedge \mathrm{init}_{\mathrm{W}}$$
$$\wedge\, \exists z.\Bigg(\max{}_v(z) \wedge \max{}_h(z) \wedge \bigvee_{q_v^z,q_h^z \in Q, q^z\in F, b\in\Sigma} (z \in X_{(q_v^z,q_h^z,b,q^z)}\Bigg).$$

Here, β simulates the distribution of weights along transitions and successful runs. Let $\gamma := \exists X_1 \cdots \exists X_l.\beta(X_1, \ldots, X_l)$ and $p \in \Sigma^{++}$. Then $[\![\gamma]\!](p) = (\|\mathfrak{A}\|, p)$, hence $\|\mathfrak{A}\| = [\![\gamma]\!]$. Furthermore, using Lemma 7.1 and remarks above, one can show that the specified formula γ lies in $\mathrm{REMSO}(K, \Sigma)$. ☐

8 Conclusion

In [18] we assigned weights to tiling systems, domino systems or weighted (quadrapolic) picture automata and proved for an alphabet Σ and any commutative semiring K the coincidence of the corresponding series with the projections of series defined by rational operations. In fact, one can prove that this very class coincides with $K^{\mathrm{rec}}\langle\!\langle \Sigma^{++}\rangle\!\rangle$ [17]. With Theorem 4.5, this implies that the notion of weighted recognizability presented here is robust and extends the main result of [11] to the weighted case. Furthermore, in [3] it is shown, that a picture language is recognizable if and only if it is the support of a recognizable series with coefficients in $\mathbb{B}$. Hence, we obtain the classical equivalence in [12] by restricting to $\mathbb{B}$.

Acknowledgements. I would like to thank Manfred Droste and Dietrich Kuske for their helpful discussions and comments, as well as the unknown referees whose remarks resulted in improvements of this paper.

References

1. J. Berstel and C. Reutenauer. *Rational Series and Their Languages*, volume 12 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, 1984.
2. M. Blum and C. Hewitt. Automata on a 2-dimensional tape. In *IEEE Symposium on Switching and Automata Theory*, pages 155–160. 1967.
3. S. Bozapalidis and A. Grammatikopoulou. Recognizable picture series. In *special volume on Weighted Automata, presented at WATA 2004, Dresden*, Journal of Automata, Languages and Combinatorics. 2005, in print.
4. M. Droste and P. Gastin. Weighted automata and weighted logics. In *32th ICALP, Proceedings*, volume 3580 of *Lecture Notes in Computer Science*, pages 513–525. Springer-Verlag, 2005.
5. M. Droste and H. Vogler. Weighted tree automata and weighted logics. Submitted, 2005.
6. D. Dubois, E. Hüllermeier, and H. Prade. A systematic approach to the assessment of fuzzy association rules. Technical report, Philipps-Universität Marburg, 2004.
7. S. Eilenberg. *Automata, Languages, and Machines,* Vol. A. Academic Press, New York, 1974.
8. K. S. Fu. *Syntactic Methods in Pattern Recognition*. Academic Press, New York, 1974.
9. D. Giammarresi and A. Restivo. Recognizable picture languages. *International Journal Pattern Recognition and Artificial Intelligence*, 6(2-3):31–46, 1992. Issue on *Parallel Image Processing*. M. Nivat, A. Saoudi, and P.S.P. Wangs (Eds.).
10. D. Giammarresi and A. Restivo. Two-dimensional finite state recognizability. *Fundam. Inform.*, 25(3):399–422, 1996.
11. D. Giammarresi and A. Restivo. Two-dimensional languages. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages, Vol. 3, Beyond Words*, pages 215–267. Springer-Verlag, Berlin, 1997.
12. D. Giammarresi, A. Restivo, S. Seibert, and W. Thomas. Monadic second-order logic over rectangular pictures and recognizability by tiling systems. *Information and Computation*, 125:32–45, 1996.
13. K. Inoue and A. Nakamura. Some properties of two-dimensional on-line tessellation acceptors. *Information Sciences*, 13:95–121, 1977.
14. W. Kuich and A. Salomaa. *Semirings, Automata, Languages*, volume 6 of *EATCS Monographs in Theoretical Computer Science*. Springer-Verlag, Berlin, 1986.
15. M. Latteux and D. Simplot. Recognizable picture languages and domino tiling. *Theoretical Computer Science*, 178:275–283, 1997.
16. O. Matz. On piecewise testable, starfree, and recognizable picture languages. In P. van Emde Boas et al., editors, *FoSSaCS*, volume 1378 of *Lecture Notes in Computer Science*, pages 203–210. Springer-Verlag, Berlin, 1998.
17. I. Mäurer. Characterizations of weighted picture series. In preparation, 2005.
18. I. Mäurer. Recognizable and rational picture series. In *Conference on Algebraic Informatics*, pages 141–155. Aristotle University of Thessaloniki Press, 2005.
19. M. Minski and S. Papert. *Perceptron*. M.I.T. Press, Cambridge, Mass., 1969.
20. A. Salomaa and M. Soittola. *Automata-Theoretic Aspects of Formal Power Series*. Texts and Monographs on Computer Science. Springer-Verlag, 1978.
21. W. Thomas. On logics, tilings, and automata. In *18th ICALP, Proceedings*, volume 510 of *Lecture Notes in Computer Science*, pages 441–453. Springer-Verlag, 1991.