

Characterizations of Recognizable Picture Series*

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Abstract

We investigate power series on pictures which map pictures to elements of a semiring and provide an extension of two-dimensional languages to a quantitative setting. We will assign weights to devices, ranging from tiling systems and domino systems to picture automata. We prove that, for commutative semirings, the behaviors of weighted picture automata are precisely alphabetic projections of series defined by rational operations and also coincide with the families of projections of tile-local and projections of hv-local series. Furthermore, we establish properties of unambiguous picture languages by characterizations in terms of unambiguous rational operations with injective projections and devices such as unambiguous (quadruplic) picture automata or unambiguous domino systems.

Keywords: picture series, two-dimensional languages, unambiguity, automata.

1 Introduction

In the literature, a variety of formal models to recognize or generate two-dimensional objects, called pictures, have been proposed [3, 12, 14, 19, 27, 29] and further properties of string languages have been formulated for two dimensions ([4, 5, 6, 17, 20, 21]). This research was motivated by problems arising from the area of image processing and pattern recognition [9, 24], and also plays a role in frameworks concerning cellular automata and other models of parallel computing [16, 28]. Restivo and Giammarresi defined the family REC of *recognizable picture languages* (cf. [10, 12]). This family is very robust and has been characterized by many different devices, generalizing well-known properties of regular word languages. Several authors obtained equivalences to the family of recognizable picture languages describing it in terms of types of automata, projections of local sets, rational operations with projections or monadic second order logic [4, 11, 13, 14, 19]. It is the goal of this paper to generalize these equivalences to a quantitative setting. Furthermore, we establish properties of unambiguous picture languages by characterizations in terms of unambiguous rational operations with injective projections and devices such as unambiguous (quadruplic) picture automata or unambiguous domino systems.

We will investigate weighted picture automata (WPA) and their behaviors, cf. [22]. The interesting model of weighted (quadruplic) automata was introduced by Bozapalidis and Grammatikopoulou [4]. These are automata operating in a natural way (the unweighted version of a WPA characterizes precisely recognizable picture languages) on pictures and whose transitions carry weights; the weights are taken as elements from a

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given semiring. The behavior or the computation of a WPA is a function which maps picture over a finite alphabet to elements of the semiring. We call such functions *picture series*. Weighted picture devices can be used to model several application-examples, e.g. the intensity of light of a picture (interpreting the alphabet as different levels of gray) or the amplitude of a monochrome subpicture of a colored picture.

Bozapalidis and Grammatikopoulou showed that picture series computed by WPA are closed under certain operations and projections on series. This was our starting point for raising the question whether the converse holds, i.e. whether the family of recognizable series and the family of projections of rational picture series coincide. The aim of this paper is to prove this equivalence for any alphabet and any commutative semiring. We will characterize the family of picture series recognized by WPA also by using tiling and domino systems, and thus obtain a robust definition of a class of recognizable picture series. Further characterizations, e.g. by weighted two-dimensional on-line tessellation automata (W2OTA) and in terms of a weighted monadic second order logic are contained in [23]. These results extend the main findings of [12, 13] to the weighted case; we get the results for languages by restricting the semiring to the Boolean semiring.

In the proofs one has to be careful when arguing in an automaton which might have several successful paths for an input picture. If necessary one has to consider or construct unambiguous picture automata in order not to count weights twice. Also, we will use ideas of constructions in [12], but involving weights and using the model of (quadrupolic) picture automata instead of 2-dimensional on-line tessellation automata (2OTA).

We will also examine new properties of unambiguous picture languages. The notion of unambiguity for picture languages as injective projections of local languages was briefly introduced in [10]. There, Giammarresi and Restivo also posed the conjecture that unambiguous languages are properly included in the family of recognizable languages. This conjecture was solved very recently in [1], where the authors showed that unlike to words, there exist recognizable picture languages that are inherently ambiguous, i.e. not computable by unambiguous 2OTA; moreover, the problem whether a tiling system is unambiguous is undecidable. We considered unambiguous picture automata and unambiguous 2OTA. In [22, 23] we showed that unambiguous picture languages are closed under injective projections and disjoint union; furthermore, the families of languages computed by unambiguous 2OTA and unambiguous tiling systems coincide (this result was independently derived in [1]). Here we will show further properties and devices to obtain an equivalence theorem for unambiguous picture languages. More precisely, we will characterize injective projections of local languages as injective projections of unambiguous rational languages, unambiguous domino recognizable languages and also as behaviors of unambiguous (quadrupolic) picture automata.

The paper is organized as follows. In Section 2, we give examples of pictures with weights, i.e. picture series and recall concepts of two-dimensional languages. In Section 3 we introduce the definitions of picture series, rational operations on them and the concept of a weighted picture automaton computing a recognizable picture series. We briefly recall the necessary notation and background for formal power series and weighted finite automata. Section 4 gives the main theorem on the coincidence of recognizable series with projections of rational series for commutative semirings. Then, in Section 5 we study unambiguous picture languages and present an equivalence theorem. Finally, in Section 6 we compare new models of weighted tile systems and weighted domino systems as extensions of local and hv-local picture languages with the family of recognizable series; the different devices defining picture series are composed in Section 7.

2 Pictures and Examples for Pictures with Weights

We recall notions and results of two-dimensional languages, required for this paper. For more details see [12, 14, 19].

Let $\mathbb{N} = \{0, 1, \dots\}$ and Σ be a finite alphabet. A *picture* over Σ is a non-empty rectangular array of elements of Σ^1 . A *picture language* is a set of pictures. The set of all pictures over Σ is denoted by Σ^{++} . Let $p \in \Sigma^{++}$. We write $p(i, j)$ or $p_{i,j}$ for the component of p at position (i, j) . Furthermore, we let $l_v(p)$ be the number of rows and $l_h(p)$ be the number of columns of p (v stands for vertical, h for horizontal). The pair $(l_v(p), l_h(p))$ is the *size* of p . The set $\Sigma^{m \times n}$ comprises all pictures with size (m, n) .

Next, we give examples of functions $S : \Sigma^{++} \rightarrow \mathbb{R} \cup \{\infty\}$ and $T : \Sigma^{++} \rightarrow \mathbb{N}$.

Example 2.1. Let $D \subset [0, 1]$ be a finite set of discrete values and let $L \subseteq D^{++}$ be a recognizable picture language. Consider the function $S : D^{++} \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$S(p) = \begin{cases} \sum_{i,j} p_{i,j} & p \in L, \\ \infty & \text{otherwise.} \end{cases}$$

One could interpret the values in D as different levels of gray [7]. Then, for each recognizable picture $p \in L$, the series S provides the total value $S(p)$ of light of p .

Example 2.2. Let C be a finite set of colors and consider $T : C^{++} \rightarrow \mathbb{N}$, defined by $T(p) = \max\{l_v(q) \cdot l_h(q) \mid q \text{ is a monochrome subpicture of } p\}$, ($p \in C^{++}$). Then $T(p)$ gives the largest size of a monochrome rectangle, contained in p .

Functions S from Σ^{++} into $\mathbb{R} \cup \{\infty\}$ or, more generally, a semiring K will be called *picture series* and abstractly defined in the next section. We will also give tools to describe the functions S and T of the above examples as the behaviors of weighted picture automata over certain semirings.

In order to consider rational operations on picture series we need two different, partial concatenations for pictures: the *column concatenation* $p \oplus q$ juxtaposes two pictures next to each other provided they have the same height, i.e. for $p \in \Sigma^{m \times k}$, $q \in \Sigma^{m \times l}$:

$$r := p \oplus q \in \Sigma^{m \times (k+l)}, \quad r(i, j) = \begin{cases} p(i, j) & j \leq k \\ q(i, j - k) & j > k. \end{cases}$$

The *row concatenation* $p \ominus q$ of two pictures p and q are defined similarly for pictures having identical width. These definitions can be extended to languages as usual and can then also be iterated, that is to say (similar for row concatenation), for $k \geq 1$, $L^{\oplus 1} := L$, $L^{\oplus k+1} := L^{\oplus k} \oplus L$ and $L^{\oplus +} := \bigcup_{k \geq 1} L^{\oplus k}$. We get operations on languages $\oplus, \ominus : \mathcal{P}(\Sigma^{++}) \rightarrow \mathcal{P}(\Sigma^{++})$ (where $\mathcal{P}(\Sigma^{++})$ denotes the set of subsets of Σ^{++}), referred to as *column closure* and *row closure*.

For any two alphabets Σ and Γ , a mapping $\pi : \Gamma \rightarrow \Sigma$ is called (*alphabetic*) *projection*. It can be lifted pointwise to pictures and picture languages as usual. If not otherwise indicated, we do not distinguish between a word w and the picture having only row (or only column) w . We fix an alphabet Σ .

In the literature, there are many equivalent devices defining or recognizing picture languages in terms of projections of local languages (tiling systems) and rational expressions [10, 11, 12], domino systems [19], two-dimensional on-line tessellation automata

¹We assume a picture to be non-empty for technical simplicity, as in [3, 14, 19].

(2OTA) [14, 15], monadic second-order (MSO) logic [13] or recently quadrupolic picture automata [4]. These devices characterize *recognizable* picture languages, collected in the class $\text{Rec}(\Sigma^{++})$.

3 Picture Series and Weighted Automata

A *semiring* $(K, +, \cdot, 0, 1)$ is a structure K such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $x \cdot 0 = 0 = 0 \cdot x$ for all elements $x \in K$. In case the multiplication is commutative, K is called *commutative*. Examples of semirings useful to model problems in operation research and carrying quantitative properties for many devices include e.g. the *Boolean* semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$, the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$, the *tropical (or min-plus)* semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$, the *arctical (or max-plus)* semiring $\text{Arc} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, the language-semiring $(\mathcal{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*)$ and $([0, 1], \max, \cdot, 0, 1)$ (to capture probabilities).

Subsequently, K will always denote a commutative semiring. Let Σ and Γ be alphabets. Now we define picture series and define some notions for them quite similarly as it is done in the theory of formal power series on words (see the end of this section).

A *picture series* is a mapping $S : \Sigma^{++} \rightarrow K$. We let $K\langle\langle\Sigma^{++}\rangle\rangle$ comprise all picture series. We write (S, p) for $S(p)$, then a series S often is written as a formal sum $S = \sum_{p \in \Sigma^{++}} (S, p) \cdot p$. The set $\text{supp}(S) = \{p \in \Sigma^{++} \mid (S, p) \neq 0\}$ is the *support* of S . Series having finite support are called *polynomials* and form the set $K\langle\Sigma^{++}\rangle$. We now define the *rational* operations $\oplus, \odot, \otimes, \ominus : (K\langle\langle\Sigma^{++}\rangle\rangle)^2 \rightarrow K\langle\langle\Sigma^{++}\rangle\rangle$ referred to as *sum*, *Hadamard product*, *horizontal multiplication* and *vertical multiplication*, respectively, and also $\oplus^+, \ominus^+ : K\langle\langle\Sigma^{++}\rangle\rangle \rightarrow K\langle\langle\Sigma^{++}\rangle\rangle$, the *horizontal star* and the *vertical star*, as follows. Fix $S, T \in K\langle\langle\Sigma^{++}\rangle\rangle$ and $p \in \Sigma^{++}$. Then we set

$$\begin{aligned} (S \oplus T, p) &:= (S, p) + (T, p) \text{ and } (S \odot T, p) := (S, p) \cdot (T, p) \\ (S \otimes T, p) &:= \sum_{p_1 \otimes p_2 = p} (S, p_1) \cdot (T, p_2) \text{ and } (S \ominus T, p) := \sum_{p_1 \ominus p_2 = p} (S, p_1) \cdot (T, p_2) \\ (S^{\oplus^+}, p) &:= \sum_{\substack{p_1 \oplus \dots \oplus p_n = p \\ n \geq 1}} (S, p_1) \cdot \dots \cdot (S, p_n) \\ (S^{\ominus^+}, p) &:= \sum_{\substack{p_1 \ominus \dots \ominus p_n = p \\ n \geq 1}} (S, p_1) \cdot \dots \cdot (S, p_n). \end{aligned}$$

The star operations are not partial since every picture is nonempty. We define the (pointwise) *scalar multiplications* with elements of the semiring, i.e. for $k \in K$, we put $(k \cdot S, p) = k \cdot (S, p)$. It defines a series in $K\langle\langle\Sigma^{++}\rangle\rangle$, as usual. For a language $L \subseteq \Sigma^{++}$, the *characteristic series* $\mathbb{1}_L : \Sigma^{++} \rightarrow K$ is defined by $(\mathbb{1}_L, p) = 1$ if $p \in L$, and $(\mathbb{1}_L, p) = 0$ otherwise. Note that $k \cdot S = (k \cdot \mathbb{1}_{\Sigma^{++}}) \odot S$.

Definition 3.1. A picture series $S \in K\langle\langle\Gamma^{++}\rangle\rangle$ is called *rational* if it is obtained from a finite set of polynomials by finitely many applications of the rational operations $\oplus, \odot, \otimes, \ominus, \oplus^+$ and \ominus^+ .

The family of rational series over a semiring K and an alphabet Γ will be denoted by $K^{\text{rat}}\langle\langle\Gamma^{++}\rangle\rangle$. Now, extending projections for languages to series, for $\pi : \Gamma \rightarrow \Sigma$ and $S' \in K\langle\langle\Gamma^{++}\rangle\rangle$, we set $(\pi(S'), p) := \sum_{\pi(p')=p} (S', p')$ for each $p \in \Sigma^{++}$. It defines a series $\pi(S') \in K\langle\langle\Sigma^{++}\rangle\rangle$ which we call the *projection* of S' by π . We say S is a *projection* of a

rational series if there exists an alphabet Γ , a series $S' \in K^{rat}\langle\langle\Gamma^{++}\rangle\rangle$ and a projection $\pi : \Gamma \rightarrow \Sigma$ with $S = \pi(S')$. We denote the family of series over Σ that are projections of rational series by $K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle$. Next we define weighted picture automata.

Definition 3.2 ([4]). A *weighted (quadrupole) picture automaton (WPA)* is a 6-tuple $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ consisting of a finite set Q of states, a finite set of rules $R \subseteq \Sigma \times K \times Q^4$, as well as four *poles of acceptance* $F_w, F_n, F_e, F_s \subseteq Q$.

Given $r = (a, k, q_w, q_n, q_e, q_s) \in R$, we denote by $\text{label}(r)$ its (input) label a , by $\text{weight}(r)$ its weight k and corresponding to the four poles $\sigma_w(r) := q_w, \sigma_n(r) := q_n, \sigma_e(r) := q_e, \sigma_s(r) := q_s$. We extend the functions label and weight to pictures by setting for $c = (c_{i,j}) \in R^{++}$: $\text{label}(c)(i, j) := \text{label}(c_{i,j})$ and $\text{weight}(c) = \prod_{i,j} \text{weight}(c_{i,j})$. We call $\text{label}(c)$ the *label* and $\text{weight}(c)$ the *weight* of c .

A *run* (or *computation*) in \mathfrak{A} is an element in R^{++} satisfying natural compatibility properties, more precisely, for $c = (c_{i,j}) \in R^{m \times n}$ we have

$$\forall i \leq m-1, j \leq n : \sigma_s(c_{i,j}) = \sigma_n(c_{i+1,j}), \forall i \leq m, j \leq n-1 : \sigma_e(c_{i,j}) = \sigma_w(c_{i,j+1}).$$

A run c is *successful* if it has its (outer) pole-states in the respective poles of acceptance, that is to say:

$$\forall i \leq m, j \leq n : \sigma_w(c_{i,1}) \in F_w, \sigma_n(c_{1,j}) \in F_n, \sigma_e(c_{i,n}) \in F_e, \sigma_s(c_{m,j}) \in F_s. \quad (1)$$

For a successful run c with $\text{label}(c) = p$ we will shortly write $c \in F_w \underset{F_s}{p} F_e$ for (1).

We define a picture series $\|\mathfrak{A}\| : \Sigma^{++} \rightarrow K$, as follows. If p has no successful run in \mathfrak{A} , $\|\mathfrak{A}\|$ sends p to 0. Otherwise, we let

$$(\|\mathfrak{A}\|, p) = \sum_{c \in F_w \underset{F_s}{p} F_e} \text{weight}(c).$$

That is, the *weight* of a picture p is the sum of the weights of all successful runs with label p . We call $\|\mathfrak{A}\|$ the *behavior* of \mathfrak{A} and also say that \mathfrak{A} *computes* (or *recognizes*) $\|\mathfrak{A}\|$.

The family of picture series computed by weighted picture automata over Σ will be denoted by $K^{rec}\langle\langle\Sigma^{++}\rangle\rangle$, elements of which are referred to as *recognizable series*.

Considering the unweighted case of Definition 3.2, where $R \subseteq \Sigma \times Q^4$, we get the definition of a *(quadrupole) picture automaton (PA)*, firstly introduced in [4]. Devices of PA over an alphabet Σ in a natural way define picture languages and were shown to compute precisely the family of recognizable picture languages ($\text{Rec}(\Sigma^{++})$) [4]. For $K = \mathbb{B}$, the correspondence $L \mapsto \mathbb{1}_L$ gives a natural bijection between languages over Σ and series in $\mathbb{B}\langle\langle\Sigma^{++}\rangle\rangle$. Results done in the next section are also true for PA.

Let us consider again Examples 2.1 and 2.2. By simulating an unweighted picture automaton \mathfrak{A} recognizing L and assigning the weight $p_{i,j}$ to every rule with label $p_{i,j}$ in \mathfrak{A} , we get a WPA \mathfrak{A}' over tropical semiring. One can prove that \mathfrak{A}' computes the function S , i.e. $S \in \mathbb{T}^{rec}\langle\langle D^{++}\rangle\rangle$. Also, there is an automaton over the max-plus semiring Arc computing T . Here, for a picture p , the automaton provides one successful path for every different monochrome subpicture of p . Since we get the behavior by adding the weights for successful runs reading p , in Arc , the maximal size is extracted.

We consider the case of words. We fix K and Σ . A formal power series is a mapping $S : \Sigma^* \rightarrow K$. The support of S is $\text{supp}(S) = \{w \in \Sigma^* \mid S(w) \neq 0\}$. A polynomial is a

series with finite support. We let $K\langle\langle\Sigma^*\rangle\rangle$ comprise all formal power series over Σ . Now let $S, T \in K\langle\langle\Sigma^*\rangle\rangle$. The sum $S + T$, the Hadamard product and the (Cauchy) product $S \cdot T$ are defined for $w \in \Sigma^*$ by

$$(S+T, w) = (S, w) + (T, w); (S \odot T, w) = (S, w) \cdot (T, w); (S \cdot T, w) = \sum_{w=w_1 w_2} (S, w_1) \cdot (T, w_2).$$

The powers $S^n (n \geq 0)$ of S are defined in a natural way, as usual. The series S is proper, if $(S, \varepsilon) = 0$. In this case, we set $S^* = \sum_{n \geq 0} S^n$, the star of S .

The class of *rational* (string) series (denoted by $K^{rat}\langle\langle\Sigma^*\rangle\rangle$) can be constructed from polynomials by applying the operations $+$, \cdot and $*$, where the star $*$ is restricted to proper series.

A *weighted finite automaton* (WFA) is a tuple $T = (Q, E, I, F)$, where Q is a finite set of states, E is a finite subset of $Q \times \Sigma \times K \times Q$, and $I, F : Q \rightarrow K$. We call the tuples in E transitions and I resp. F the *initial* (resp. *final*) weight function.

Let $n \geq 1$. A path π of length n is a sequence

$$(q_0, a_0, k_0, q_1) (q_1, a_1, k_1, q_2) \dots (q_{n-1}, a_{n-1}, k_{n-1}, q_n)$$

of transitions in E . The word $a_0 \dots a_{n-1}$ is called the *label* of π . We say that π starts at q_0 and ends at q_n . Then π is *successful* if $I(q_0), F(q_n) \neq 0$. We define $\text{weight}(\pi) := k_0 \cdot k_1 \cdot \dots \cdot k_{n-1}$, the *weight* of π . We assume that for every $q \in Q$ there is a path of length 0 which starts and ends at q , is labeled with ε and weighted with 1. For every $p, q \in Q$ and every $w \in \Sigma^*$, we denote by $p \xrightarrow{w} q$ the set of all paths with label w which start at p and end at q . The WFA T computes a formal power series $\|T\| : \Sigma^* \rightarrow K$, defined for every $w \in \Sigma^*$, by

$$\|T\|(w) = \sum_{p \in I, q \in F, \pi \in p \xrightarrow{w} q} I(p) \cdot \text{weight}(\pi) \cdot F(q).$$

We call $\|T\|$ the *behavior* of T . A formal power series $S \in K\langle\langle\Sigma^*\rangle\rangle$ is *recognizable* if S is the behavior of some WFA. We let $K^{rec}\langle\langle\Sigma^*\rangle\rangle$ comprise all recognizable series over K and Σ .

Schützenberger's theorem states the following equivalence between recognizable and rational formal power series.

Theorem 3.3 (Schützenberger [26]). *A formal power series is rational if and only if it is the behavior of some weighted finite automaton.*

For further details to notions and basic results in the theory of formal power series on words, as well as to Schützenberger's theorem, we refer [2, 8, 18, 25, 26].

4 A Kleene-Schützenberger Theorem for Picture Series

For the rest of the paper, let Σ and Γ be alphabets and K a commutative semiring.

4.1 Projections of Rational Series are Recognizable

The aim of this subsection is to show that projections of rational picture series are behaviors of weighted picture automata. We will give the initialization of a structural induction and use the results in [4, Section 4] to obtain the closure of the family of recognizable picture series under rational operations and projections. Clearly, the *monomials*, i.e. series with supports as singletons, are recognizable:

Lemma 4.1. *Let $p \in \Sigma^{++}$ and $k \in K$. Then $k \cdot \mathbb{1}_{\{p\}}$, $k \cdot \mathbb{1}_{\Sigma^{++}} \in K^{\text{rec}}\langle\langle\Sigma^{++}\rangle\rangle$.*

Proof. Let $p \in \Sigma^{m \times n}$ and $k \in K$. Then, $\mathfrak{A} = (\{0, \dots, \max\{m, n\}\}, R, \{0\}, \{0\}, \{n\}, \{m\})$ defined by

$$R = \{(p_{i+1, j+1}, c, j, i, j+1, i+1) \mid 0 \leq i \leq m, 0 \leq j \leq n\}$$

such that $c = k$ if $(i, j) = (0, 0)$ and $c = 1$ otherwise, computes $\|\mathfrak{A}\| = k \cdot \mathbb{1}_{\{p\}}$. Similar one could define an automaton for $k \cdot \mathbb{1}_{\Sigma^{++}}$. \square

Lemma 4.2 ([4]). *$K^{\text{rec}}\langle\langle\Sigma^{++}\rangle\rangle$ is closed under operations $\oplus, \odot, \ominus, \ominus^+, \oplus^+$ and \ominus^+ .*

Note, that using both lemmas, $K^{\text{rec}}\langle\langle\Sigma^{++}\rangle\rangle$ is also closed under scalar multiplication, since for $k \in K$ and $S \in K^{\text{rec}}\langle\langle\Sigma^{++}\rangle\rangle$, we get $k \cdot S = S \odot (k \cdot \mathbb{1}_{\Sigma^{++}})$.

Lemma 4.3 ([4]). *Let $\pi : \Gamma \rightarrow \Sigma$ and $T \in K^{\text{rec}}\langle\langle\Gamma^{++}\rangle\rangle$. Then $\pi(T) \in K^{\text{rec}}\langle\langle\Sigma^{++}\rangle\rangle$.*

Now the following theorem is immediate by Lemmas 4.1, 4.2 and 4.3.

Theorem 4.4 ([4]). *$K^{\text{Prat}}\langle\langle\Sigma^{++}\rangle\rangle \subseteq K^{\text{rec}}\langle\langle\Sigma^{++}\rangle\rangle$.*

4.2 Recognizable Series are Projections of Rational Series

The idea for the proof of the other direction of a Kleene-Schützenberger Theorem for picture series is to convert the automaton into some “deterministic” device of a certain type via a projection. The behavior of this deterministic device will then be proved as rational by using Schützenberger’s Theorem for recognizable and rational formal power series on words (Theorem 3.3).

Definition 4.5. A weighted picture automaton is called *rule deterministic* if for every input label a of the alphabet there is at most one rule with label a .

There is a natural correspondence between formal power series reading words and picture series reading only rows or only columns. We can consider a picture having only one row (resp. one column) also as word over Σ , and later on we will not distinguish between the notations of these two cases.

Lemma 4.6. *Let $S : \Sigma^* \rightarrow K$ be a rational formal power series over words, i.e. $S \in K^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$. There exist $S_h, S_v \in K^{\text{rat}}\langle\langle\Sigma^{++}\rangle\rangle$ such that for all $p \in \Sigma^{++}$, we have*

$$S_h(p) = \begin{cases} S(p) & p \in \Sigma^{1 \times n} (n \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}, \quad S_v(p) = \begin{cases} S(p) & p \in \Sigma^{n \times 1} (n \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the class of rational (string) series is closed under the Hadamard product [8], the series $S \odot \mathbb{1}_{\Sigma^* \setminus \{\varepsilon\}}$ is rational. We can naturally embed the polynomials of $K^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$ into $K\langle\langle\Sigma^{++}\rangle\rangle$ having their supports in $\Sigma^{1 \times \mathbb{N}}$ (resp. $\Sigma^{\mathbb{N} \times 1}$); the operations $+, \cdot, *$ are simulated by $\oplus, \ominus(\ominus), \oplus^+(\ominus^+)$. \square

Proposition 4.7. *Let $S \in K^{\text{rec}}\langle\langle\Gamma^{++}\rangle\rangle$ be a series computed by a rule deterministic WPA. Then S is rational.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be a rule deterministic WPA computing S . We group the proof into 3 steps and show that \mathfrak{A} computes a rational picture series. For $a \in \Gamma$, we set $r(a) = (a, k, q_1, q_2, q_3, q_4)$ if $(a, k, q_1, q_2, q_3, q_4) \in R$.

Step 1 We use the horizontal direction of the rules in R to define a WFA $A_h =$

(Q, E_h, I_h, F_h) over words, as follows. Let $E_h \subseteq Q \times \Gamma \times K \times Q$ be the set of transitions, defined by

$$(q_1, a, k, q_3) \in E_h \Leftrightarrow \exists r = (a, k, q_1, q_2, q_3, q_4) \in R,$$

and put

$$I_h(q) = \begin{cases} 1 & q \in F_w, \\ 0 & \text{otherwise,} \end{cases} \quad F_h(q) = \begin{cases} 1 & q \in F_e, \\ 0 & \text{otherwise} \end{cases}$$

as initial and final weight functions.

Then A_h is a WFA having successful computations for all words corresponding to rows which have a run in \mathfrak{A} leading from F_w to F_e . For such a row $w = a_1 a_2 \cdots a_n (a_i \in \Gamma)$, since \mathfrak{A} is rule deterministic, we have

$$(\|A_h\|, w) = 1 \cdot \left(\prod_{1 \leq i \leq n} \text{weight}(r(a_i)) \right) \cdot 1.$$

The series $\|A_h\|$ maps all other words in Γ^* to 0.

Using Theorem 3.3 and Lemma 4.6 we conclude that there exists a rational picture series S_h such that for all $p \in \Gamma^{1 \times \mathbb{N}}$ we have $(S_h, p) = (\|A_h\|, p)$, and S_h maps elements not in $\Gamma^{1 \times \mathbb{N}}$ to 0.

Step 2 Similarly, we use the vertical direction of rules in R for the definition of transitions of a WFA $A_v = (Q, E_v, I_v, F_v)$ over the Boolean semiring where $E_v \subseteq Q \times \Gamma \times \{0, 1\} \times Q$ is the set of transitions, defined by

$$(q_2, a, 1, q_4) \in E_v \Leftrightarrow \exists r = (a, k, q_1, q_2, q_3, q_4) \in R,$$

and $I_v(q) = \begin{cases} 1 & q \in F_n, \\ 0 & \text{otherwise,} \end{cases} \quad F_v(q) = \begin{cases} 1 & q \in F_s, \\ 0 & \text{otherwise} \end{cases}$ as weight functions.

Then A_v is an automaton having successful computations for all words corresponding to columns which have a run in \mathfrak{A} leading from F_n to F_s . Such a column $w = a_1 a_2 \cdots a_m (a_i \in \Gamma)$, again, since \mathfrak{A} is rule deterministic, satisfies $(\|A_v\|, w) = 1$. All other words are mapped to 0 by $\|A_v\|$.

Now, as before, Theorem 3.3 and Lemma 4.6 provide a rational picture series S_v over \mathbb{B} such that for all $p \in \Gamma^{\mathbb{N} \times 1}$: $(S_v, p) = (\|A_v\|, p)$.

Step 3 **(C) Claim:** $\forall x \in \Gamma^{++} : (\|\mathfrak{A}\|, x) = (S_h^{\ominus+}, x) \cdot (S_v^{\oplus+}, x).$

For pictures $x = (x_{i,j}) (1 \leq i \leq m, 1 \leq j \leq n)$ where every row has a successful run in A_h , the picture series $S_h^{\ominus+}$ is a rational series that maps x to the product of the weights of the composed rules for pixels of x in \mathfrak{A} . The series $S_h^{\ominus+}$ maps all other pictures to 0. We get

$$(S_h^{\ominus+}, x) = \prod_{i \leq m, j \leq n} \text{weight}(r(x_{i,j})). \quad (2)$$

Analogously, for a pictures $y = (y_{i,j})$ where every column has a successful run in A_v we get

$$(S_v^{\oplus+}, y) = 1. \quad (3)$$

$S_v^{\oplus+}$ maps all other pictures to 0..

Now, to prove (C), let $x = (x_{i,j}) \in \Gamma^{++} (1 \leq i \leq m, 1 \leq j \leq n)$. We distinguish between three cases. First, assume $x \in \Gamma^{m \times n}$ such that there exists an $i \in \{1, \dots, m\}$ and $(x_{i,1}x_{i,2} \cdots x_{i,n}) \in \Gamma^{1 \times n}$ has no run in \mathfrak{A} satisfying $\sigma_w(r(x_{i,1})) \in F_w, \sigma_e(r(x_{i,n})) \in F_e$. With the definition of $\|\mathfrak{A}\|$ and (2) we conclude $(\|\mathfrak{A}\|, x) = 0 = (S_h^{\ominus+}, x)$, hence: (C).

Now, let $x \in \Gamma^{m \times n}$ such that there exists an $j \in \{1, \dots, n\}$ with $(x_{1,j}x_{2,j} \cdots x_{m,j})^T \in \Gamma^{m \times 1}$ having no run in \mathfrak{A} satisfying $\sigma_n(r(x_{1,j})) \in F_n, \sigma_s(r(x_{m,j})) \in F_s$. Then using (3), we get $(\|\mathfrak{A}\|, x) = 0 = (S_v^{\oplus+}, x)$, hence (C).

For the remaining case, again, let $x \in \Gamma^{m \times n}$. For every row in x , there exists a unique computation leading in \mathfrak{A} from F_w to F_e , that is, for all $1 \leq i \leq m$ and all $1 \leq j \leq (n-1)$:

$$\sigma_e(r(x_{i,j})) = \sigma_w(r(x_{i,j+1})), \sigma_w(r(x_{i,1})) \in F_w, \sigma_e(r(x_{i,n})) \in F_e. \quad (4)$$

On the other hand, for every column in x there exists a unique computation in \mathfrak{A} having the northern state in F_n and the southern state in F_s , i.e., for all $1 \leq i \leq m-1$ and all $1 \leq j \leq n$:

$$\sigma_s(r(x_{i,j})) = \sigma_n(r(x_{i+1,j})), \sigma_n(r(x_{1,j})) \in F_n, \sigma_s(r(x_{m,j})) \in F_s. \quad (5)$$

With (4) and (5), $c := (r(x_{i,j}))_{i,j}$ forms a successful computation for x in \mathfrak{A} . Since \mathfrak{A} is rule deterministic, there is at most one computation for x . We obtain

$$\begin{aligned} (\|\mathfrak{A}\|, x) &= \prod_{i \leq m, j \leq n} \text{weight}(r(x_{i,j})) \stackrel{(2)}{=} (S_h^{\ominus+}, x) \cdot 1 \\ &\stackrel{(3)}{=} (S_h^{\ominus+}, x) \cdot (S_v^{\oplus+}, x). \end{aligned}$$

Therefore, claim (C) holds and thus (using Lemma 4.2)

$$\|\mathfrak{A}\| = S_h^{\ominus+} \odot S_v^{\oplus+} \in K^{rat} \langle \langle \Gamma^{++} \rangle \rangle.$$

□

Next we show that every recognizable series is the projection of a series computed by a rule deterministic automaton. The idea is to encode the rules of the given automaton into the new alphabet. Then we will prove that this encoding can be reversed by a projection.

Proposition 4.8. *Let \mathfrak{A} be a WPA over Σ . There exists a rule deterministic WPA \mathfrak{A}' over an alphabet Γ and a projection $\pi : \Gamma \rightarrow \Sigma$ satisfying $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be a WPA over Σ and K . We put $\Gamma := R$ and define a rule deterministic WPA over Γ as $\mathfrak{A}' = (Q, R', F_w, F_n, F_e, F_s)$ with

$$R' := \left\{ ((a, k, q_1, q_2, q_3, q_4), k, q_1, q_2, q_3, q_4) \mid (a, k, q_1, q_2, q_3, q_4) \in R \right\}.$$

For every input label $(a, k, q_1, q_2, q_3, q_4) \in \Gamma$ there is at most one rule with label $(a, k, q_1, q_2, q_3, q_4)$ in \mathfrak{A}' . We define a projection $\pi : \Gamma \rightarrow \Sigma$ by mapping pixels $(a, k, q_1, q_2, q_3, q_4)$ to a . We claim that $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$ (*).

Let $x \in \Sigma^{m \times n}$. If there was no successful run of x in \mathfrak{A} then there is no picture in Γ^{++} with a successful run in \mathfrak{A}' which is mapped to x by π , so (*) holds. For the other case,

let $\{c_1, c_2, \dots, c_s\} \subseteq R^{++}$ be the set of successful runs for x in \mathfrak{A} . These runs belong to successful runs $\{c'_1, c'_2, \dots, c'_s\} \subseteq R'^{++}$ in \mathfrak{A}' such that

$$\forall 1 \leq i \leq s : \pi(l(c'_i)) = x, \quad \sum_{1 \leq i \leq s} \text{weight}(c_i) = \sum_{1 \leq i \leq s} \text{weight}(c'_i).$$

Since there cannot be other successful runs in \mathfrak{A}' mapped by the projection π to x , we conclude (*):

$$(\|\mathfrak{A}\|, x) = \sum_{1 \leq i \leq s} \text{weight}(c_i) = \sum_{\pi(x')=x} (\|\mathfrak{A}'\|, x') = (\pi(\|\mathfrak{A}'\|), x).$$

□

Theorem 4.9. $K^{rec}\langle\langle\Sigma^{++}\rangle\rangle \subseteq K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle$.

Proof. Immediate by Propositions 4.7 and 4.8. □

As a consequence of Theorems 4.4 and 4.9, we obtain the following Kleene-Schützenberger-like theorem for picture series:

Theorem 4.10. *Let K be a commutative semiring and Σ an alphabet. Then*

$$K^{rec}\langle\langle\Sigma^{++}\rangle\rangle = K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle.$$

Remark 4.11. With above theorem, the theorem on the coincidence of projections of rational picture languages and recognizable picture languages ([12]) follows by considering the Boolean semiring and noting that a language $L \subseteq \Sigma^{++}$ is recognizable if and only if $\mathbb{1}_L \in \mathbb{B}\langle\langle\Sigma^{++}\rangle\rangle$ is recognizable [4]; and a language $L \subseteq \Sigma^{++}$ is a projection of a rational language if and only if $\mathbb{1}_L \in \mathbb{B}\langle\langle\Sigma^{++}\rangle\rangle$ is a projection of a rational series (finite languages correspond to polynomials and rational operations on languages are simulated by the respective operation of the corresponding characteristic series).

Now, having this relation, as in the case of picture languages ([12]), for the definition of the class of rational (resp. recognizable) picture series, the operations and projections used are necessary. For instance, defining $L = \{x \in \{a\}^{++} \mid l_v(x) = l_h(x)\}$, using the relationship between languages and characteristic series over \mathbb{B} , the series $\mathbb{1}_L$ clearly is recognizable over \mathbb{B} , but not in $\mathbb{B}^{rat}\langle\langle\Sigma^{++}\rangle\rangle$.

5 Unambiguous Picture Languages

As mentioned in the introduction, we will now establish properties of unambiguous picture languages. The notion of ambiguity for picture languages in the context of tiling systems was briefly introduced in [10]. The authors defined the class $\text{UPLoc}(\Sigma^{++})$ (in [10] this class was denoted by UREC) of picture languages that are injective projections of local languages and posed the conjecture that $\text{UPLoc}(\Sigma^{++})$ is properly included in the family of recognizable picture languages. Very recently, in [1] it was shown that this conjecture is true. The authors proved that there are recognizable picture languages that are inherently ambiguous. Furthermore, they showed that it is undecidable whether a tiling system is unambiguous. In [23], in order to prove the equivalence of the class of picture series defined by sentences of a weighted MSO logic on pictures with the family $K^{rec}\langle\langle\Sigma^{++}, W2OTA\rangle\rangle$ (which coincides with $K^{rec}\langle\langle\Sigma^{++}\rangle\rangle$), it was also shown that the

family $\text{UPLoc}(\Sigma^{++})$ coincides with the class of languages that are definable by unambiguous 2-dimensional on-line tessellation automata (this result was independently derived in [1]).

It is the aim of this section to provide further properties and equivalent devices for an equivalence theorem for unambiguous picture languages. More precisely, we will characterize injective projections of local languages also as injective projections of unambiguously rational languages and unambiguous domino recognizable languages. In some of the proofs in Section 6, we will construct unambiguous picture automata in order to be able to define the right assignment of the weights.

Let Σ and Γ be alphabets, $L \subseteq \Gamma^{++}$. We call a projection $\pi : \Gamma \rightarrow \Sigma$, *injective* on L if $\pi : L \rightarrow \Sigma^{++}$ is an injective mapping. For a picture p , we denote by \hat{p} the picture that results from p by surrounding it with the (new) boundary symbol $\#$. If p has size (m, n) then \hat{p} has size $(m + 2, n + 2)$.

As is known, the closure of the set of finite picture languages over Σ under the *rational* operations $\cup, \cap, \oplus, \ominus, \oplus^+, \ominus^+$ coincides with $\text{Rec}(\Sigma^{++})$ ([12]). Now, as in the classical theory of formal languages, we will consider the *unambiguously rational* subsets of Σ^{++} and restrict the rational operations in such a way that only unambiguously rational subsets can be obtained. Let $A, B \subseteq \Sigma^{++}$. The union $A \cup B$ can be formed only if $A \cap B = \emptyset$, the product $A \oplus B$ can be formed only if $a_1 \oplus b_1 = a_2 \oplus b_2$ with $a_1, a_2 \in A, b_1, b_2 \in B$ implies $a_1 = a_2, b_1 = b_2$. In order to form A^{\oplus^+} we require that each $A^{\oplus^n}, n > 1$ can be formed and further that the resulting sets $A, A^{\oplus^2}, \dots, A^{\oplus^n}, \dots$ are disjoint. In an analogical way we restrict the operations \ominus and \ominus^+ . We do not restrict intersection. The six operations $\cap, \cup, \oplus, \ominus, \oplus^+, \ominus^+$ just described, with the restrictions concerning their applicability, will be called *unambiguously rational* operations. Furthermore, we only consider injective projections as operations on languages.

Definition 5.1. The class $\text{URat}(\Sigma^{++})$ of *unambiguously rational* languages is the smallest class \mathfrak{C} of subsets of Σ^{++} such that:

- all singletons are in \mathfrak{C}
- $\emptyset \in \mathfrak{C}$
- \mathfrak{C} is closed under unambiguously rational operations.

Also, the class $\text{UPRat}(\Sigma^{++})$ is smallest among families of subsets of Σ^{++} which contain $\text{URat}(\Sigma^{++})$ and are closed under injective projections. (*Unambiguously rational picture expressions* are defined as usual.

For words, the following result holds (observe also again the facts for series on words at the end of Section 3), where an unambiguously rational word language is defined analogously to Definition 5.1 using union, product and star for (word) languages.

Proposition 5.2 ([8]). *Every rational string language is unambiguously rational.*

Tiles are pictures of size $(2, 2)$ and *dominoes* have size $(1, 2)$ or $(2, 1)$. For a picture p , we denote by $T(p)$ (resp. $D(p)$) the set of all sub-tiles (resp. sub-dominoes) of p . A language $L \subseteq \Gamma^{++}$ is *local* (resp. *hv-local*) if there exists a set Θ of tiles (resp. dominoes) over $\Gamma \cup \{\#\}$, such that $L = \{p \in \Gamma^{++} \mid T(\hat{p}) \subseteq \Theta\}$ (resp. $L = \{p \in \Gamma^{++} \mid D(\hat{p}) \subseteq \Theta\}$). Then (Γ, Θ) *characterizes* L . We write $L = \mathcal{L}(\Theta)$.

Lemma 5.3. *Every hv-local language is unambiguously rational. This inclusion is strict.*

Proof. Let $K \subseteq \Gamma^{++}$ be hv-local. Following the proof in [12, Theorem 8.4] there exist rational string languages $S_h, S_v \subseteq \Gamma^+$ with rational (string) expressions (Proposition 5.2) α_h (α_v resp.) denoting S_h (S_v resp.) such that a picture $p \in \Gamma^{++}$ belongs to K if and only if the strings corresponding to the rows and to the columns of p belong to S_h and to S_v respectively (this fact often is written like $K = S_h \oplus S_v$). With Proposition 5.2 we can assume α_h, α_v to be unambiguously rational. Now, as in the cited proof, let β_h (β_v resp.) be the rational (picture) expression obtained replacing in α_h (α_v resp.) the concatenation with \oplus (\ominus resp.) and the $*$ operation with \oplus^+ (\ominus^+ resp.). Then $\beta_h, \beta_v \in \text{URat}(\Gamma^{++})$ and

$$K = (\beta_h)^{\ominus^+} \cap (\beta_v)^{\oplus^+}. \quad (6)$$

Now, we have $K \in \text{URat}(\Gamma^{++})$, since the operations \ominus^+ and \oplus^+ in (6) are unambiguous.

Also, for $\Sigma = \{0\}$, the language $L = \{p \in \{0\}^{++} \mid l_h(p) = 2\} \subset \Sigma^{++}$ is unambiguously rational but not hv-local. To see $L \in \text{URat}(\Sigma^{++})$, denote by E the unambiguously rational (word) expression for the language $\{0^n \mid n \geq 1\}$. With the proof of Lemma 4.6 we get an unambiguously rational expression \bar{E} for the picture language $\{\{0\}^{\oplus^+}\}$. Then $\bar{E} \oplus \bar{E}$ is an unambiguously rational expression for L . \square

Definition 5.4. A language $L \subseteq \Sigma^{++}$ is *unambiguous tiling (resp. domino) recognizable* if there exists an alphabet Γ , a local (hv-local) language $L' \subseteq \Gamma^{++}$, characterized by (Γ, Θ) , and a projection $\pi : \Gamma \rightarrow \Sigma$ such that π is injective on L' and $\pi(L') = L$. The *tiling (resp. domino) system* $(\Sigma, \Gamma, \Theta, \pi)$ is called an *unambiguous representation* for L .

For an alphabet Σ , the family of languages that are unambiguous tiling (domino resp.) recognizable, we denote by $\text{UPLoc}(\Sigma^{++})$ ($\text{UPDom}(\Sigma^{++})$ resp.).

Since a finite language is local, we conclude a finite language as unambiguous tiling recognizable using the identity projection to get an unambiguous representation. The closure of $\text{UPLoc}(\Sigma^{++})$ under unambiguously rational operations is proved by the following lemma.

Lemma 5.5. *$\text{UPLoc}(\Sigma^{++})$ is closed under unambiguous column/row concatenation, unambiguous column/row closure, disjoint union, intersection and injective projections.*

Proof. Let $L_1, L_2 \subseteq \Sigma^{++}$ be unambiguous tiling recognizable and $(\Sigma, \Gamma_i, \Theta_i, \pi_i)$, ($i = 1, 2$) the respective unambiguous representations, i.e. π_i is injective on $\mathcal{L}(\Theta_i)$, $i = 1, 2$. Assume \oplus as unambiguous for L_1, L_2 . We define $(\Sigma, \Gamma_1 \cup \Gamma_2, \Theta, \pi)$ for $L_1 \oplus L_2$ as in [12, Theorem 7.2] with

$$\mathcal{L}(\Theta) = \mathcal{L}(\Theta_1) \oplus \mathcal{L}(\Theta_2), \quad \forall a \in \Gamma, \pi(a) = \begin{cases} \pi_1(a) & a \in \Gamma_1 \\ \pi_2(a) & a \in \Gamma_2. \end{cases} \quad (7)$$

We have to show, $\pi : \Gamma \rightarrow \Sigma$ is injective on $\mathcal{L}(\Theta)$. Clearly, for $p \in L_1 \oplus L_2$, since this operation \oplus is unambiguous, there exist unique $p_i \in L_i$ satisfying $p_1 \oplus p_2 = p$. Using the prerequisites, there are unique $p'_i \in \mathcal{L}(\Theta_i)$ such that $\pi_i(p'_i) = p_i$, hence a unique $p' \in \mathcal{L}(\Theta)$ such that $\pi(p') = p$ (7).

The case of the unambiguous column closure of L_1 can be reduced to the construction above by considering two distinct unambiguous representations for L_1 and iterating the tiles. In an analogous way, one proves the closeness under unambiguous row concatenation and closure.

For the disjoint union, we follow the construction in [12, Theorem 7.4]. One can prove that the constructed tiling system for $L := L_1 \cup L_2$ is an unambiguous representation for L provided to act on unambiguous representations for L_1 and L_2 and disjoint union.

Similarly for the intersection, as in [12, Theorem 7.4] we construct a tiling system $(\Sigma, \Gamma, \Theta, \pi)$ for $L = L_1 \cap L_2$. Assuming π_i as injective on $\mathcal{L}(\Theta_i)$, one concludes π as being injective on $\mathcal{L}(\Theta)$, hence $(\Sigma, \Gamma, \Theta, \pi)$ is an unambiguous system for L .

Let Γ and Δ be two alphabets and $(\Gamma, \Delta, \Theta, \psi)$ an unambiguous tiling system for $L \subseteq \Gamma^{++}$. We consider $\pi : \Gamma \rightarrow \Sigma$ as an injective projection on L . The tiling system $\tau := (\Sigma, \Delta, \Theta, \psi \circ \pi)$ computes $\pi(L)$. Let $p \in \pi(L)$. Since π is injective on L , there exists a unique $p' \in L$ with $\pi(p') = p$, hence a unique $p'' \in \mathcal{L}(\Theta)$ with $\psi(p'') = p'$ (ψ is injective on $\mathcal{L}(\Theta)$). It follows that $\psi \circ \pi$ is injective on $\mathcal{L}(\Theta)$ and therefore τ is an unambiguous representation for $\pi(L)$. \square

Lemma 5.6. $\text{UPLoc}(\Sigma^{++}) \subseteq \text{UPDom}(\Sigma^{++})$

Proof. It suffices to show that every local language L is unambiguous domino recognizable, since, simulating precisely the proof of Lemma 5.5, the class $\text{UPDom}(\Sigma^{++})$ is closed under injective projections.

In [12, Lemma 7.1] it is shown, that every local set is the projection of an hv-local set by constructing dominoes of tiles. One can prove that the given projection in the cited proof is injective on the specified hv-local set characterized by these dominoes of tiles, hence the presented domino system is an unambiguous representation for L . \square

We want to give the notion of unambiguity as a property of automata. As noted before, in [23] it is shown that the class $\text{UPLoc}(\Sigma^{++})$ coincides with the family of languages that are computable by unambiguous 2OTA. Now, in this paper our automata-theoretic model are quadrupole picture automata, so we have to reformulate and slightly reprove this result.

We define a (possibly weighted) picture automaton \mathfrak{A} as *unambiguous* if for every input picture there exists at most one successful run in \mathfrak{A} . The family of languages over an alphabet Σ that are computable by unambiguous PA will be denoted by $\text{URec}(\Sigma^{++})$. With common constructions, we get:

Lemma 5.7. *Let $L \in \text{URec}(\Gamma^{++})$ and let $\pi : \Gamma \rightarrow \Sigma$ be injective on L . There exists an unambiguous picture automaton computing $\pi(L)$.*

Proof. We transform a rule $r = (a, q_w, q_n, q_e, q_s)$ in an unambiguous PA for L to a corresponding transition $r' := (\pi(a), q_w, q_n, q_e, q_s)$ in a PA for $\pi(L)$. The resulting PA is unambiguous. \square

Lemma 5.8. $\text{URec}(\Sigma^{++}) = \text{UPLoc}(\Sigma^{++})$.

Proof. For the inclusion from right to left, it suffices to construct an unambiguous PA for any local picture language (Lemma 5.5). For this we follow ideas in [12, Lemma 8.2], but we have to formulate the constructed sets in order to describe PA instead of 2OTA.

Let $L \subseteq \Sigma^{++}$ be local, characterized by $(\Sigma, \Theta), \Theta \subseteq (\Sigma \cup \{\#\})^{2 \times 2}$. We define $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ as a PA over Σ computing L by putting $Q = (\Sigma \cup \{\#\})^{2 \times 2}$ and

$$\bullet F_w = \left\{ \begin{array}{|c|} \hline \# & a \\ \hline \# & b \\ \hline \end{array} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}, F_n = \left\{ \begin{array}{|c|} \hline \# & \# \\ \hline a & b \\ \hline \end{array} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}$$

$$\bullet F_e = \left\{ \begin{bmatrix} a \# \\ b \# \end{bmatrix} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}, F_s = \left\{ \begin{bmatrix} a \ b \\ \# \ # \end{bmatrix} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}$$

We set $R = R_{ulc} \cup R_{ue} \cup R_{le} \cup R_m \subseteq \Sigma \times Q^4$ (where ulc, ue, le, m stand for “upper left corner”, “upper edge”, “left edge”, “middle”, respectively) with $(a, b, c, d, f, g, h, x, y, t, z \in \Sigma \cup \{\#\})$:

$$\begin{aligned} \bullet R_{ulc} &= \left\{ e = \left(a, \begin{bmatrix} \# \ a \\ \# \ c \end{bmatrix}, \begin{bmatrix} a \ b \\ c \ d \end{bmatrix}, \begin{bmatrix} a \ b \\ c \ d \end{bmatrix}, \begin{bmatrix} \# \ # \\ a \ b \end{bmatrix} \right) \mid a \in \Sigma \right\} \\ \bullet R_{ue} &= \left\{ e = \left(b, \begin{bmatrix} a \ b \\ h \ d \end{bmatrix}, \begin{bmatrix} b \ c \\ d \ f \end{bmatrix}, \begin{bmatrix} b \ c \\ d \ f \end{bmatrix}, \begin{bmatrix} \# \ # \\ b \ c \end{bmatrix} \right) \mid a, b \in \Sigma \right\} \\ \bullet R_{le} &= \left\{ e = \left(c, \begin{bmatrix} \# \ c \\ \# \ g \end{bmatrix}, \begin{bmatrix} c \ d \\ g \ h \end{bmatrix}, \begin{bmatrix} c \ d \\ g \ h \end{bmatrix}, \begin{bmatrix} a \ b \\ c \ d \end{bmatrix} \right) \mid a, c \in \Sigma \right\} \\ \bullet R_m &= \left\{ e = \left(a, \begin{bmatrix} z \ a \\ t \ c \end{bmatrix}, \begin{bmatrix} a \ b \\ c \ d \end{bmatrix}, \begin{bmatrix} a \ b \\ c \ d \end{bmatrix}, \begin{bmatrix} x \ y \\ a \ b \end{bmatrix} \right) \mid a, x, z \in \Sigma \right\} \end{aligned}$$

Then \mathfrak{A} computes L and is unambiguous.

For the inclusion from left to right, let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be a PA for L (it suffices to assume \mathfrak{A} as rule deterministic by Proposition 4.8 and Lemma 5.7). We use the notations of the proof of Proposition 4.7 and succeeding Definition 3.2. Let $a, b, c, d \in \Sigma$. If the occurring rules exist, we define (Σ, Θ) characterizing L as follows:

$$\begin{aligned} \begin{bmatrix} \# \ # \\ \# \ a \end{bmatrix} &\in \Theta, \text{ if } \sigma_w(r(a)) \in F_w, \sigma_n(r(a)) \in F_n \\ \begin{bmatrix} \# \ # \\ a \ # \end{bmatrix} &\in \Theta, \text{ if } \sigma_e(r(a)) \in F_e, \sigma_n(r(a)) \in F_n \\ \begin{bmatrix} \# \ a \\ \# \ # \end{bmatrix} &\in \Theta, \text{ if } \sigma_w(r(a)) \in F_w, \sigma_s(r(a)) \in F_s \\ \begin{bmatrix} a \ # \\ \# \ # \end{bmatrix} &\in \Theta, \text{ if } \sigma_e(r(a)) \in F_e, \sigma_s(r(a)) \in F_s \\ \begin{bmatrix} \# \ # \\ a \ b \end{bmatrix} &\in \Theta, \text{ if } \sigma_n(r(a)) \in F_n, \sigma_n(r(b)) \in F_n, \sigma_e(r(a)) = \sigma_w(r(b)) \\ \begin{bmatrix} a \ b \\ \# \ # \end{bmatrix} &\in \Theta, \text{ if } \sigma_s(r(a)) \in F_s, \sigma_s(r(b)) \in F_s, \sigma_e(r(a)) = \sigma_w(r(b)) \\ \begin{bmatrix} \# \ a \\ \# \ b \end{bmatrix} &\in \Theta, \text{ if } \sigma_w(r(a)) \in F_w, \sigma_w(r(b)) \in F_w, \sigma_s(r(a)) = \sigma_n(r(b)) \\ \begin{bmatrix} a \ # \\ b \ # \end{bmatrix} &\in \Theta, \text{ if } \sigma_e(r(a)) \in F_e, \sigma_e(r(b)) \in F_e, \sigma_s(r(a)) = \sigma_n(r(b)) \\ \begin{bmatrix} a \ b \\ c \ d \end{bmatrix} &\in \Theta, \text{ if } \sigma_e(r(a)) = \sigma_w(r(b)), \sigma_e(r(c)) = \sigma_w(r(d)), \sigma_s(r(a)) = \sigma_n(r(c)), \\ &\sigma_s(r(b)) = \sigma_n(r(d)). \end{aligned}$$

Then, (Σ, Θ) characterizes L , hence L is unambiguous tiling recognizable. \square

We get the following characterization result for picture languages computed by unambiguous (quadrapolic) picture automata:

Proposition 5.9. $\text{URec}(\Sigma^{++}) = \text{UPRat}(\Sigma^{++}) = \text{UPLoc}(\Sigma^{++}) = \text{UPDom}(\Sigma^{++})$.

Now, we call all languages in this class *unambiguous*.

Remark 5.10. In Subsection 4.2 we also defined rule deterministic weighted picture automata. If we apply the proof of Proposition 4.8 to an (unweighted) PA, the constructed projection clearly is injective, hence unambiguous PA define injective projections of languages that are computable by rule deterministic PA. On the other hand, clearly, rule deterministic PA are unambiguous and therefore, injective projections of languages computed by rule deterministic 2OTA are unambiguous. One interesting problem would be to find a preferably “small” class of languages such that injective projections of this class coincide with URec. For recognizable languages one could formulate a similar problem using arbitrary projections.

Example 5.11. Let $\Sigma = \{0, 1\}$ and T_1 the language containing pictures $p \in \Sigma^{(n+1) \times n}$ with the property that there exists an i ($1 \leq i \leq n$) such that the i -th row equals the $(n+1)$ th row. Then T_1 is computable by a 2OTA but not a deterministic one [14]. In fact, T_1 is an unambiguous picture language. Briefly, the proof proceeds as follows. Clockwise rotation on languages preserves unambiguity, hence it suffices to prove that $T'_1 := ((T_1)^R)^R$ is unambiguous. One even can show that T'_1 is computable by a deterministic 2OTA.

Clearly, the proof of this example also shows that the class of languages computable by deterministic 2OTA is not closed under rotation. With the proof of Proposition 4.8 and the arguments of Remark 5.10, it follows that rule deterministic and deterministic languages are not closed under injective projections.

Example 5.12. We consider the language $T_2 \subset \{0, 1\}^{++}$ such that T_2 contains all pictures with the property that there exists two rows which are identical. In [1], the authors proved that T_2 is inherently ambiguous, i.e. T_2 is recognizable but not unambiguous.

Example 5.13. Let $\Sigma = \{0, 1\}$ and consider the language T_3 of all squares over Σ of odd side-length where the central position carries a 1. Then T_3 is an unambiguous language. Indeed, let $T = (\Sigma, \Gamma, \Theta, \pi)$ a tiling system, defined as follows:

- $\Gamma = \{1, a_i, b_i, c_i, d_i, e_i \mid i = 0, 1\}$
- $\pi : \Sigma \rightarrow \Gamma; 1 \mapsto 1, l_1 \mapsto i (l = a, b, c, d, e; i = 0, 1)$
- Θ is the set of sub-tiles of the following picture *ex*

$$ex = \begin{array}{|c|} \hline \begin{array}{cccccccc} \# & \# & \# & \# & \# & \# & \# & \# \\ \# & \mathbf{a} & d & d & d & d & d & \# \\ \# & c & \mathbf{a} & d & d & d & d & \# \\ \# & c & c & \mathbf{a} & d & d & d & \# \\ \# & c & c & c & \mathbf{1} & d & d & \# \\ \# & c & c & \mathbf{b} & e & \mathbf{a} & d & \# \\ \# & c & \mathbf{b} & e & e & e & \mathbf{a} & \# \\ \# & \mathbf{b} & e & e & e & e & e & \mathbf{a} \\ \# & \# & \# & \# & \# & \# & \# & \# \end{array} \\ \hline \end{array}$$

where $l = l_0, l_1$ for $l = a, b, c, d, e$.

Then the local language $\mathcal{L}(\Theta)$ contains all squares over Γ that “look” like ex and have more than one line. One can prove that $\pi(\mathcal{L}(\Theta)) = T_3 \setminus \{\boxed{1}\}$. Now, since π is injective on $\mathcal{L}(\Theta)$, using Lemma 5.5, we conclude $T_3 \in \text{UPLoc}(\Sigma^{++})$, thus T_3 is unambiguous.

6 Tile-local and Hv-local Series

Local sets play an important role in the theory of recognizable string languages. Several authors generalized this notion to picture languages [12, 19] (see also the previous section). We will now assign weights to these local and hv-local picture devices using tiles or dominoes. This yields, via a projection, a very simple local definition and characterization of recognizable picture series.

For an alphabet Σ and a picture $p \in \Sigma^{m \times n}$, we will consider sub-tiles (sub-dominoes) at certain positions of \hat{p} . For tiles, we define

$$\forall 1 \leq i \leq m+1 \quad \forall 1 \leq j \leq n+1 : \quad t(\hat{p}_{i,j}) := \boxed{\begin{array}{cc} \hat{p}_{i,j} & \hat{p}_{i,j+1} \\ \hat{p}_{i+1,j} & \hat{p}_{i+1,j+1} \end{array}}.$$

Also, we consider the sub-dominoes in horizontal or vertical direction distinguished by their positions in \hat{p} :

$$\forall 1 \leq i \leq m+2 \quad \forall 1 \leq j \leq n+1 : \quad d^h(\hat{p}_{i,j}) := \boxed{\hat{p}_{i,j} \hat{p}_{i,j+1}}$$

$$\forall 1 \leq i \leq m+1 \quad \forall 1 \leq j \leq n+2 : \quad d^v(\hat{p}_{i,j}) := \boxed{\begin{array}{c} \hat{p}_{i,j} \\ \hat{p}_{i+1,j} \end{array}}.$$

We give the following definitions.

Definition 6.1. We call $\mathcal{T} = (\Sigma, T)$, where $T : (\Sigma \cup \{\#\})^{2 \times 2} \rightarrow K$ is a function mapping tiles over Σ to K , a (*weighted*) *tile-system*. It computes the picture series $\|\mathcal{T}\| : \Sigma^{++} \rightarrow K$, defined by

$$\forall p \in \Sigma^{++} : \|\mathcal{T}\|(p) := \prod_{\substack{1 \leq i \leq l_v(p)+1 \\ 1 \leq j \leq l_h(p)+1}} T(t(\hat{p}_{i,j})).$$

We call $S : \Sigma^{++} \rightarrow K$ *tile-local* if there exists a tile-system \mathcal{T} satisfying $\|\mathcal{T}\| = S$.

Similarly for dominoes we have:

Definition 6.2. A pair $\mathcal{D} = (\Sigma, D)$, where $D : (\Sigma \cup \{\#\})^{2 \times 1, 1 \times 2} \rightarrow K$ maps dominoes over Σ to K , is a (*weighted*) *domino-system*. It computes the series $\|\mathcal{D}\| : \Sigma^{++} \rightarrow K$, defined by

$$\forall p \in \Sigma^{++} : \|\mathcal{D}\|(p) := \prod_{\substack{1 \leq i \leq l_v(p)+2 \\ 1 \leq j \leq l_h(p)+1}} D(d^h(\hat{p}_{i,j})) \cdot \prod_{\substack{1 \leq i \leq l_v(p)+1 \\ 1 \leq j \leq l_h(p)+2}} D(d^v(\hat{p}_{i,j})).$$

A picture series $S : \Sigma^{++} \rightarrow K$ is called *hv-local* if there exists a domino-system \mathcal{D} satisfying $\|\mathcal{D}\| = S$. We denote the families of tile-local and hv-local series by $K^{loc}\langle\langle \Sigma^{++} \rangle\rangle$ and $K^{hv}\langle\langle \Sigma^{++} \rangle\rangle$, respectively. We call the functions T (resp. D) *tile* (resp. *domino*)-*function*. For a picture p , tile-systems (domino-systems) then compute the product of these functions ranging over the (*canonical*) *tile* (resp. *domino*)-*covering* of \hat{p} . As usual, one defines

projections of tile-local and hv-local series. We denote the families of series that are projections of tile-local (resp. hv-local) series by $K^{Ploc}\langle\langle\Sigma^{++}\rangle\rangle$ (resp. $K^{Phv}\langle\langle\Sigma^{++}\rangle\rangle$).

The following proposition holds, indicating that the given devices generalize the notion of local and hv-local picture languages.

Proposition 6.3. *A picture language $L \subseteq \Gamma^{++}$ is local (hv-local resp.) if and only if its characteristic series $\mathbb{1}_L \in \mathbb{B}\langle\langle\Gamma^{++}\rangle\rangle$ is tile-local (hv-local resp.).*

Proof. If (Γ, Θ) characterizes L , we define a weighted tile-system (Γ, T) by setting, for $t \in (\Gamma \cup \{\#\})^{2 \times 2}$: $T(t) = 1$, if $t \in \Theta$ and $T(t) = 0$, otherwise. Then, for all $p \in \Gamma^{++}$, we have: $\|(\Gamma, \Theta)\|(p) = 1 \iff p \in L$. Similarly, we define a weighted domino-system in the domino-case. \square

We will show that series computed by WPA are presentable as projections of hv-local series. For this, we define a domino-system in such a way that for a picture p the domino function (taken over the canonical domino-covering of \hat{p}) coincides with the weight of the unique computation (in case it exists) for p in a rule deterministic automaton.

Proposition 6.4. *We have $K^{rec}\langle\langle\Sigma^{++}\rangle\rangle \subseteq K^{Phv}\langle\langle\Sigma^{++}\rangle\rangle$.*

Proof. We restrict ourselves to rule deterministic automata, using a projection (Proposition 4.8). Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be rule deterministic, computing $\|\mathfrak{A}\| = S$. We may use the notations of the proof of Proposition 4.7 and succeeding Definition 3.2. For $a, b \in \Sigma$, in case the occurring rules exist, we define a domino-function $D : (\Sigma \cup \{\#\})^{2 \times 1, 1 \times 2} \rightarrow K$ as follows:

$$\begin{array}{l|l}
 \boxed{\# \#} \mapsto 1 & \boxed{\frac{\#}{a}} \mapsto 1, \text{ if } \sigma_n(r(a)) \in F_n \\
 \boxed{\# a} \mapsto \text{weight}(r(a)), \text{ if } \sigma_w(r(a)) \in F_w & \boxed{\frac{a}{\#}} \mapsto 1, \text{ if } \sigma_s(r(a)) \in F_s \\
 \boxed{a \#} \mapsto 1, \text{ if } \sigma_e(r(a)) \in F_e & \boxed{\frac{a}{b}} \mapsto 1, \text{ if } \sigma_s(r(a)) = \sigma_n(r(b)) \\
 \boxed{a b} \mapsto \text{weight}(r(b)), \text{ if } \sigma_e(r(a)) = \sigma_w(r(b)) & \boxed{\frac{\#}{\#}} \mapsto 1.
 \end{array}$$

D maps all other dominoes to 0. Then $\mathcal{D} := (\Sigma, D)$ is a domino-system. For a picture p with (unique) successful computation $c \in R^{++}$ in \mathfrak{A} , the product of values of D (taken over the canonical domino-covering of \hat{p}) coincides with $\text{weight}(c)$. On the other hand, if p has no successful computation in \mathfrak{A} then, clearly the definition of D gives $\|\mathcal{D}\|(p) = 0$. Thus $\|\mathcal{D}\| = S$. \square

Every hv-local language is local [12, 19]. The analogous result for picture series provides the following proposition. In the proof here we have to define the tile-function using the weights of the given domino-function such that respective products of the canonical coverings for a picture coincide.

Proposition 6.5. *Every hv-local series is tile-local.*

Proof. Let $S : \Gamma^{++} \rightarrow K$ be hv-local, computed by $\mathcal{D} = (\Gamma, D)$. Define $\mathcal{T} = (\Gamma, T)$ as a tile-system computing S such that $T = T_{ulc} \cup T_{ue} \cup T_{le} \cup T_m : (\Gamma \cup \{\#\})^{2 \times 2} \rightarrow K$ denotes the tile-function (where ulc, ue, le, m stand for ‘‘upper left corner’’, ‘‘upper edge’’, ‘‘left edge’’, ‘‘middle’’, respectively). For $a \in \Gamma$ and $b, c, d \in \Gamma \cup \{\#\}$, we put

- $T_{ulc} \left(\begin{array}{|c|} \hline \# \# \\ \hline \# a \\ \hline \end{array} \right) = D \left(\begin{array}{|c|} \hline \# \# \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline \# a \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline \# \\ \hline \# \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline \# \\ \hline a \\ \hline \end{array} \right)$
- $T_{ue} \left(\begin{array}{|c|} \hline \# \# \\ \hline a b \\ \hline \end{array} \right) = D \left(\begin{array}{|c|} \hline \# \# \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline a b \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline \# \\ \hline b \\ \hline \end{array} \right)$
- $T_{le} \left(\begin{array}{|c|} \hline \# a \\ \hline \# b \\ \hline \end{array} \right) = D \left(\begin{array}{|c|} \hline \# \\ \hline \# \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline \# b \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \right)$
- $T_m \left(\begin{array}{|c|} \hline a b \\ \hline c d \\ \hline \end{array} \right) = D \left(\begin{array}{|c|} \hline c d \\ \hline \end{array} \right) \cdot D \left(\begin{array}{|c|} \hline b \\ \hline d \\ \hline \end{array} \right)$

The values of D over a domino covering of a picture are distributed with T over the tile covering. For $p \in \Gamma^{++}$ we get $\|\mathcal{T}\|(p) = \|\mathcal{D}\|(p) = (S, p)$. \square

In fact, to finish our argument for an equivalence theorem, we will show that projections of these tile-local series are recognizable. Since the image of a picture is composed by the weights of the contained tiles, the idea is to encode the tiles into the states of the rules similar to a construction in [12]. But there, the authors considered the model of a 2-dimensional on-line tessellation automata. Here we will derive a WPA that simulates the constructed underlying on-line tessellation automaton by defining rules that identify their southern and eastern poles. Also, since we now have weights we will construct an unambiguous automaton in order not to add outputs over several runs reading identical pictures.

Proposition 6.6. $K^{Ploc} \langle \langle \Sigma^{++} \rangle \rangle \subseteq K^{rec} \langle \langle \Sigma^{++} \rangle \rangle$.

Proof. It suffices to prove the result for a tile-local series (Lemma 4.3). Let $S : \Sigma^{++} \rightarrow K$ be tile local, computed by $\mathcal{T} = (\Sigma, T)$ with tile-function $T : (\Sigma \cup \{\#\})^{2 \times 2} \rightarrow K$. We define $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ as a WPA over Σ computing S by putting $Q = (\Sigma \cup \{\#\})^{2 \times 2}$ and

- $F_w = \left\{ \begin{array}{|c|} \hline \# a \\ \hline \# b \\ \hline \end{array} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}$, $F_n = \left\{ \begin{array}{|c|} \hline \# \# \\ \hline a b \\ \hline \end{array} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}$
- $F_e = \left\{ \begin{array}{|c|} \hline a \# \\ \hline b \# \\ \hline \end{array} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}$, $F_s = \left\{ \begin{array}{|c|} \hline a b \\ \hline \# \# \\ \hline \end{array} \mid a \in \Sigma, b \in \Sigma \cup \{\#\} \right\}$
- $R = R_{ulc} \cup R_{ue} \cup R_{le} \cup R_m \subseteq \Sigma \times K \times Q^4$ (where ulc, ue, le, m stand for “upper left corner”, “upper edge”, “left edge”, “middle”, respectively) with $(a, b, c, d, f, g, h, x, y, t, z \in \Sigma \cup \{\#\})$:
- $R_{ulc} = \left\{ e = \left(a, w(e), \begin{array}{|c|} \hline \# a \\ \hline \# c \\ \hline \end{array}, \begin{array}{|c|} \hline a b \\ \hline c d \\ \hline \end{array}, \begin{array}{|c|} \hline a b \\ \hline c d \\ \hline \end{array}, \begin{array}{|c|} \hline \# \# \\ \hline a b \\ \hline \end{array} \right) \mid a \in \Sigma \right\}$
and $w(e) = T \left(\begin{array}{|c|} \hline \# \# \\ \hline \# a \\ \hline \end{array} \right) \cdot T \left(\begin{array}{|c|} \hline \# \# \\ \hline a b \\ \hline \end{array} \right) \cdot T \left(\begin{array}{|c|} \hline \# a \\ \hline \# c \\ \hline \end{array} \right) \cdot T \left(\begin{array}{|c|} \hline a b \\ \hline c d \\ \hline \end{array} \right)$
- $R_{ue} = \left\{ e = \left(b, w(e), \begin{array}{|c|} \hline a b \\ \hline h d \\ \hline \end{array}, \begin{array}{|c|} \hline b c \\ \hline d f \\ \hline \end{array}, \begin{array}{|c|} \hline b c \\ \hline d f \\ \hline \end{array}, \begin{array}{|c|} \hline \# \# \\ \hline b c \\ \hline \end{array} \right) \mid a, b \in \Sigma \right\}$
and $w(e) = T \left(\begin{array}{|c|} \hline \# \# \\ \hline b c \\ \hline \end{array} \right) \cdot T \left(\begin{array}{|c|} \hline b c \\ \hline d f \\ \hline \end{array} \right)$

- $R_{le} = \left\{ e = \left(c, w(e), \begin{bmatrix} \# & c \\ \# & g \end{bmatrix}, \begin{bmatrix} c & d \\ g & h \end{bmatrix}, \begin{bmatrix} c & d \\ g & h \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \mid a, c \in \Sigma \right\}$
and $w(e) = T\left(\begin{bmatrix} \# & c \\ \# & g \end{bmatrix}\right) \cdot T\left(\begin{bmatrix} c & d \\ g & h \end{bmatrix}\right)$
- $R_m = \left\{ e = \left(a, w(e), \begin{bmatrix} z & a \\ i & c \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ a & b \end{bmatrix} \right) \mid a, x, z \in \Sigma \right\}$
and $w(e) = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

To prove $\|\mathfrak{A}\| = S$, we observe the following. Given a picture $p \in \Sigma^{++}$ with successful computation $c \in R^{++}$ in \mathfrak{A} , for $\text{weight}(c)$, every tile of the canonical covering of \hat{p} occurs exactly once in the multiplication. On the other hand, the tiles of an arbitrary picture p are encoded in Q . The given construction with its accepting condition defines an unambiguous weighted picture automaton which has a unique successful run for every element in Σ^{++} . Hence for $p \in \Sigma^{++}$ we have

$$\|\mathfrak{A}\|(p) = \prod_{\substack{1 \leq i \leq l_v(p)+1 \\ 1 \leq j \leq l_h(p)+1}} T(t(\hat{p}_{i,j})) = \|\mathcal{T}\|(p) = (S, p).$$

□

Now, we can prove a result originally stated by S. Bozapalidis (private communication):

Theorem 6.7. $K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle = K^{\text{P}hv}\langle\langle \Sigma^{++} \rangle\rangle = K^{\text{P}loc}\langle\langle \Sigma^{++} \rangle\rangle$.

Proof. Immediate by Propositions 6.4, 6.5 and 6.6

□

There is also a direct proof for the inclusion from the first to the third class, similar to the construction for the inclusion from left to right in Lemma 5.8.

7 Comparing all Families

We introduced different devices to characterize recognizable picture series. The theorem below shows that the definition of a recognizable picture series is very robust.

Theorem 7.1. *Let Σ be an alphabet, K a commutative semiring and $S : \Sigma^{++} \rightarrow K$ a picture series. The following assertions are equivalent.*

1. S is the behavior of a weighted picture automaton.
2. S is the projection of a rational picture series.
3. S is the projection of a tile-local series.
4. S is the projection of an hv -local series.

The proof immediately follows from Theorems 4.10 and 6.7. It extends the main equivalences for characterizing recognizable picture languages to the weighted case of picture series over arbitrary commutative semirings. In [23], we established a notion of a weighted MSO logics over pictures and we proved that the class of picture series defined by sentences of the weighted logics coincides with the family $K^{\text{rec}}\langle\langle \Sigma^{++} \rangle\rangle$. We generalized the results of [12, 13] to the quantitative setting of series, the respective results for languages follow by considering the Boolean semiring and using Remark 4.11 and Proposition 6.3.

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