Weighted Picture Automata and Weighted Logics*

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Abstract

We investigate formal power series on pictures. These are functions that map pictures to elements of a semiring and provide an extension of two-dimensional languages to a quantitative setting. We establish a notion of a weighted MSO logics over pictures. The semantics of a weighted formula will be a picture series. We introduce weighted 2-dimensional on-line tessellation automata (W2OTA) and prove that for commutative semirings, the class of picture series defined by sentences of the weighted logics coincides with the family of picture series that are computable by W2OTA. Moreover, we show that the family of behaviors of W2OTA coincide precisely with the class of picture series characterized by weighted (quadrapolic) picture automata and consequently, the notion of weighted recognizability presented here is robust. However, the weighted structures can not be used to get better decidability properties than in the language case. For every commutative semiring, it is undecidable whether a given MSO formula has restricted structure or whether the semantics of a formula has empty support.

Keywords: picture series, weighted logics, two-dimensional languages, weighted picture automata.

1 Introduction

In the literature, a variety of formal models to recognize or generate two-dimensional arrays of symbols, called pictures, have been proposed [3, 15, 17, 21, 31, 33] and various properties of string languages have been formulated for two dimensions ([4, 5, 6, 19, 23, 24]). This research was motivated by problems arising from the area of image processing and pattern recognition [12, 29], and also plays a role in frameworks concerning cellular automata and other models of parallel computing [22, 32]. Restivo and Giammarresi defined the family REC of *recognizable picture languages* (cf. [13, 15]). This family is very robust and has been characterized by many different devices, generalizing well-known properties of regular word languages. The work of several authors can combined to an equivalence theorem for picture languages describing recognizable languages in terms of types of automata, sets of tiles, rational operations or existential monadic second-order (MSO) logic [4, 14, 16, 17, 21].

Notions of weighted recognizability for picture languages defined by weighted picture automata (WPA) were introduced in [4]. The weights are taken from some commutative

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semiring. The behavior of a weighted picture automaton is a picture series mapping pictures over an alphabet to some semiring. In [25, 26], we showed that the family of behaviors of WPA coincides with the class of projections of certain rational picture series and can be characterized also by using tiling and domino systems. These equivalent weighted picture devices can be used to model several application examples, e.g. the intensity of light of a picture (interpreting the alphabet as different levels of gray) or the amplitude of a monochrome subpicture of a colored picture.

Recently, Droste and Gastin [7] introduced the framework of a weighted logics over words and characterized recognizable formal power series, computed by weighted finite automata, as semantics of monadic second-order sentences within their logic. Meanwhile, the idea of a weighted logic was also applied to devices recognizing more general structures such as weighted tree automata and tree series, weighted automata on infinite words or even traces [8, 9, 28]. In this paper we will establish a weighted MSO logic for pictures. The semantics of a weighted formula will be a picture series over some commutative semiring. We will introduce weighted 2-dimensional on-line tessellation automata (W2OTA). This model extends the known notion of 2-dimensional on-line tessellation automata (2OTA) [17] for picture languages. Our main result proves that for an alphabet and any commutative semiring the family of picture series computable by W2OTA coincides with the family of series that are definable by weighted monadic second-order sentences. This generalizes the main result of [16] to the weighted case. Furthermore, we will show that W2OTA are equivalent to the concept of recognizability in [4, 25, 26] where weighted (quadrapolic) picture automata are considered. More precisely, a picture series is recognizable by some weighted 2-dimensional on-line tessellation automaton if and only if it is the behavior of a weighted picture automaton. The main theorem and this coincidence are proved by a circular argument.

For the syntax, we basically follow classical logic. But additionally, similar to [7], we also let elements of the semiring be atomic formulas, hence are able to formulate quantitative properties of picture languages: imagine for instance the number of a's occurring in a picture. The other atomic formulas will have a semantics with values in $\{0, 1\}$. Problems arise when negation is applied, because it is not clear how to define the semantics of a negated formula. Therefore, we apply negation only to atomic formulas. Universal quantification in general does not preserve recognizability. Hence, as in [7], we disallow universal set quantification, but here we restrict universal first-order (FO) quantification in a new way to particular formulas, since not every recognizable picture language is determinizable and the proof in [7] does not work for two dimensions due to counter examples.

Crucial for proving the main theorem is to show, that the universal FO quantification of a formula, with restricted semantics, defines a recognizable series. Unlike the word-case, we build a formula instead of constructing a certain (unweighted) automaton. This formula defines a picture language which is computable by a 2OTA that is unambiguous. Also, we will use successor relations instead of built-in relations $\leq_v$ and $\leq_h$, since there are $(\leq_v, \leq_h)$-definable picture languages that are not recognizable [24].

Considering (unweighted) logic, our constraints of the formulas is not an essential restriction, since every (unweighted) existential MSO-formula is equivalent (in the sense of defining identical languages) to a formula in which negation is only applied to atomic formulas, and since every recognizable picture language is definable by a restricted formula. We obtain the corresponding classical equivalence when restricting to the Boolean semiring. The main result of the paper indicates that the notion of weighted recognizability for picture series is robust, since it can be characterized in terms of a logic and different automata-theoretic devices and generalizes the common frameworks for picture languages.

Concering decidability, it is well-known that classical decision problems for languages as emptyness or finiteness are undecidable for picture languages. Since picture languages can be viewed as series weighted in the Boolean semiring, the question arises whether other weight structures (for instance fields) could lead to better decidability properties. Here we show that for any commutative semiring this is not the case. More precisely, for every commutative semiring, it is undecidable whether a given MSO formula has restricted structure or whether the semantics of a formula has empty support. Also, for arbitrary commutative semirings we can not decide whether two W2OTA are equivalent.

The organization of the paper is as follows. In Section 2, we briefly present the classical equivalence theorem and recall basic concepts of the theory of two-dimensional languages. Section 3 provides the required definitions and results of picture series and introduces the new concept of a weighted 2-dimensional on-line tessellation automaton. In Section 4 we define the syntax and semantics of our weighted MSO-logic on pictures, and show basic properties. Next, in Section 5, unambiguous picture languages are examined. Sections 6,7 and 8 correspond to the three inclusions of the circular argument containing the main results. In Section 9 we present the indicated undecidability results.

2 Pictures and EMSO-Logic

We recall notions and results of two-dimensional languages and MSO-logic over pictures. For more details see [15, 16, 33].

Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of natural numbers. Let Σ and Γ be finite alphabets. A *picture* over Σ is a non-empty rectangular array of elements in Σ.[1] A *picture language* is a set of pictures. The set of all pictures over Σ is denoted by Σ^{++}. Let $p \in \Sigma^{++}$. We write $p(i, j)$ or $p_{i,j}$ for the component of p at position (i, j) and let $l_v(p)$ (resp. $l_h(p)$) be the number of rows (resp. columns) of p (v stands for vertical, h for horizontal). The pair $(l_v(p), l_h(p))$ is the *size* of p. The set $\Sigma^{m\times n}$ comprises all pictures with size (m, n). The domain of p is the set $\mathrm{Dom}(p) = \{1, \ldots, l_v(p)\} \times \{1, \ldots, l_h(p)\}$. A mapping $\pi : \Gamma \to \Sigma$ is called *projection*. It can be extended pointwise to pictures and picture languages as usual.

We fix an alphabet Σ. In the literature, there are many equivalent models defining or recognizing picture languages in terms of projections of local languages (tiling systems) and rational expressions [13, 14, 15], domino systems [21], two-dimensional on-line tessellation automata (2OTA) [17, 18], monadic second-order (MSO) logic [16] or recently quadrapolic picture automata [4]. These devices characterize *recognizable* picture languages, collected in the class $\mathrm{Rec}(\Sigma^{++})$.

The set $\mathrm{MSO}(\Sigma^{++})$ of MSO-formulas over Σ is defined recursively by

$$\begin{aligned}\varphi \quad ::= \quad & P_a(x) \mid xS_vy \mid xS_hy \mid x = y \mid x \in X \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \\ & \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi\end{aligned}$$

where $a \in \Sigma$, x, y are FO variables and X is a second-order variable. A picture p is represented by the relational structure $(\mathrm{Dom}(p), S_v, S_h, (R_a)_{a\in\Sigma})$ where $R_a = \{(i, j) \in$

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[1] We assume a picture to be non-empty for technical simplicity, as in [3, 17, 21].

$\mathrm{Dom}(p) \mid p(i,j) = a\}$, $(a \in \Sigma)$ and S_v, S_h are the two successor relations of both directions, defined by

$$(i,j)S_v(i+1,j), (i,j)S_h(i,j+1),$$

in case the occuring positions are elements of the domain of p. Formulas containing no set quantification (but possibly including atomic formulas of the form $(x \in X)$) are collected in $\mathrm{FO}(\Sigma^{++})$ and called first-order formulas. We denote by $\mathrm{EMSO}(\Sigma^{++})$ the set of formulas of the form $\exists X_1, \dots \exists X_n.\psi$ such that ψ contains only first-order quantifiers, i.e. $\psi \in \mathrm{FO}(\Sigma^{++})$.

For a finite set $\mathcal{V}$ of first-order or second-order variables, a $(\mathcal{V}, p)$-assignment σ maps FO variables in $\mathcal{V}$ to elements of $\mathrm{Dom}(p)$ and second-order variables in $\mathcal{V}$ to subsets of $\mathrm{Dom}(p)$. If x is a FO variable and $(i,j) \in \mathrm{Dom}(p)$ then $\sigma[x \to (i,j)]$ coincides with σ on $\mathcal{V} \setminus \{x\}$ and assigns (i,j) to x (similarly $\sigma[X \to I]$ for $I \subseteq \mathrm{Dom}(p)$). We encode (p, σ) where σ is a $(\mathcal{V}, p)$-assignment as a picture over $\Sigma_{\mathcal{V}} = \Sigma \times \{0,1\}^{\mathcal{V}}$. Conversely, an element in ${\Sigma_{\mathcal{V}}}^{++}$ is a pair (p, σ) where p is the projection over Σ and σ is the projection over $\{0,1\}^{\mathcal{V}}$. Then σ represents a *valid* assignment over $\mathcal{V}$ if for each FO variable $x \in \mathcal{V}$, the projection of σ to the x-coordinate contains exactly one 1. In this case, we identify σ with the $(\mathcal{V}, p)$-assignment. Let $N_{\mathcal{V}} \subseteq \Sigma_{\mathcal{V}}^{++}$ comprise $\{(p, \sigma) \mid \sigma \text{ is valid}\}$. Clearly, $N_{\mathcal{V}}$ is a recognizable picture language. We write $\mathrm{Free}(\varphi)$ for the set of all free variables in φ and $N_\varphi = N_{\mathrm{Free}(\varphi)}$. If $\mathcal{V}$ contains $\mathrm{Free}(\varphi)$, the definition that (p, σ) *satisfies* φ, i.e. $(p, \sigma) \models \varphi$ is as usual and we let

$$\mathcal{L}_{\mathcal{V}}(\varphi) = \{(p, \sigma) \in N_{\mathcal{V}} \mid (p, \sigma) \models \varphi\}.$$

We say that the formula φ *defines* the picture language $\mathcal{L}_{\mathrm{Free}(\varphi)}(\varphi) =: \mathcal{L}(\varphi)$. The language $\mathcal{L}_{\mathcal{V}}(\varphi)$ is recognizable over $\Sigma_{\mathcal{V}}$. Let $Z \subseteq \mathrm{MSO}(\Sigma^{++})$ a set of MSO formulas. We set

$$\mathcal{L}(Z) := \{\mathcal{L}(\varphi) \mid \varphi \in Z\}.$$

A picture language $L \subseteq \Sigma^{++}$ is *FO (resp. EMSO)-definable* if there exists a sentence $\varphi \in \mathrm{FO}(\Sigma^{++})$ (resp. $\varphi \in \mathrm{EMSO}(\Sigma^{++})$) such that $\mathcal{L}(\varphi) = L$.

Remark 2.1. Note, that by introducing a predicate over an extented alphabet for every subformula of the form $(x \in X)$, we can transform a formula $\varphi \in \mathrm{FO}(\Sigma^{++})$ with $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) as the set of first-order (resp. second-order) variables, into a formula $\varphi' \in \mathrm{FO}((\Sigma_{\mathcal{V}_1})^{++})$ having no second-order variable anymore and satisfying $\mathcal{L}(\varphi) = \mathcal{L}(\varphi')$. Essentially using the same argument, we can also show that $\mathcal{L}(\varphi)$ is FO-definable.

The equivalence between recognizable picture languages and EMSO-definable picture languages can be formulated as follows.

Theorem 2.2 ([16]). *A language $L \subseteq \Sigma^{++}$ is definable by a sentence $\varphi \in \mathrm{EMSO}(\Sigma^{++})$ if and only if L is recognizable.*

The aim of this paper is to generalize this result to a quantitative setting. For this, we will define two types of weighted automata devices on pictures: weighted 2-dimensional on-line tessellation automata (W2OTA) and weighted picture automata (WPA). The weights are taken from a commutative semiring.

3 Weighted Automata over Pictures

A *semiring* $(K, +, \cdot, 0, 1)$ is a structure K such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $x \cdot 0 = 0 = 0 \cdot x$ for all elements $x \in K$. If multiplication is commutative, K is called *commutative.* Examples of semirings useful to model problems in operations research and carrying quantitative properties for many devices include e.g. the *Boolean* semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$, the natural numbers $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$, the *tropical* semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$, the *arctical (or max-plus)* semiring $\mathrm{Arc} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, the language-semiring $(\mathscr{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*)$ and $([0,1], \max, \cdot, 0, 1)$ (to capture probabilities).

Subsequently, K will always denote a commutative semiring. Let Σ, Δ, Γ be alphabets. We will now assign weights to pictures. This provides a generalization of the theory of picture languages to formal power series over pictures, cf. [4, 25, 26, 27]. Next we define some notions for picture series quite similarly as it is done in the theory of formal power series on words [2, 11, 20, 30].

A *picture series* is a mapping $S : \Sigma^{++} \to K$. We let $K\langle\langle\Sigma^{++}\rangle\rangle$ comprise all picture series. We write (S, p) for $S(p)$, then a picture series S often is written as a formal sum $S = \sum_{p \in \Sigma^{++}} (S, p) \cdot p$. The set $\mathrm{supp}(S) = \{p \in \Sigma^{++} \mid (S, p) \neq 0\}$ is the *support* of S. For a language $L \subseteq \Sigma^{++}$, the *characteristic series* $\mathbb{1}_L : \Sigma^{++} \to K$ is defined by setting $(\mathbb{1}_L, p) = 1$ if $p \in L$, and $(\mathbb{1}_L, p) = 0$ otherwise. For $K = \mathbb{B}$, the mapping $L \mapsto \mathbb{1}_L$ gives a natural bijection between languages over Σ and series in $\mathbb{B}\langle\langle\Sigma^{++}\rangle\rangle$.

We define *rational* operations $\oplus$ and $\odot$, referred to as *sum* and *Hadamard product*, respectively, and also $\cdot : K \times K\langle\langle\Sigma^{++}\rangle\rangle \to K\langle\langle\Sigma^{++}\rangle\rangle$, the *scalar multiplications* with elements of the semiring, in the following way. For two series $S, T \in K\langle\langle\Sigma^{++}\rangle\rangle, k \in K$ and $p \in \Sigma^{++}$, we set

$$(S \oplus T, p) := (S, p) + (T, p), \ (S \odot T, p) := (S, p) \cdot (T, p) \text{ and } (k \cdot S, p) := k \cdot (S, p).$$

Note that $k \cdot S = (k \cdot \mathbb{1}_{\Sigma^{++}}) \odot S$. Now, defining projections and inverse projections for series, given additionally $\pi : \Gamma \to \Sigma$, $R \in K\langle\langle\Gamma^{++}\rangle\rangle$ and $q \in \Gamma^{++}$, we put

$$(\pi(R), p) := \sum_{\pi(p')=p} (R, p') \text{ and } \big(\pi^{-1}(S), q\big) := \big(S, \pi(q)\big).$$

We will call the series, $\pi(R) \in K\langle\langle\Sigma^{++}\rangle\rangle$ *projection* of R by π and $\pi^{-1}(S) \in K\langle\langle\Gamma^{++}\rangle\rangle$ *inverse projection* of S by π, respectively. In the boolean case we get for languages $L \subseteq \Sigma^{++} : \pi^{-1}(L) = \{p \in \Gamma^{++} \mid \pi(p) \in L\}$. There are further rational operations on picture series like horizontal/vertical multiplication and horizontal/vertical star. The closure of the class of series having finite support (polynomials) under rational operations and projections defines the family of projections of rational picture series which coincides with the family of series that are behaviors of weighted picture automta (WPA), cf. [25, 26]. We will prove in this paper the equivalence of weighted 2-dimensional on-line tessellation automata and WPA.

We now present the detailed definition of a weighted 2-dimensional on-line tessellation automaton. It generalizes in a straightforward way the automata-theoretic definition of recognizablility for picture languges in termes of 2-dimensional on-line tessellation automata.

Definition 3.1. A *weighted 2-dimensional on-line tessellation automaton (W2OTA)* over Σ is a tuple $\mathfrak{A} = (Q, E, I, F)$, consisting of a finite set Q of states, a finite set of transitions $E \subseteq Q \times Q \times \Sigma \times K \times Q$ and sets of initial and final states $I, F \subseteq Q$, respectively.

For a transition $e = (q_h, q_v, a, k, q) \in E$, we set $\sigma_h(e) = q_h, \sigma_v(e) = q_v$ and $\sigma(e) = q$. We denote by label(e) its label a and by weight(e) its weight k. We extend these both functions to pictures by setting, for $c = (c_{i,j}) \in E^{m\times n}$:

$$\text{label}(c)(i,j) := \text{label}(c_{i,j}), \quad \text{weight}(c) = \prod_{i,j} \text{weight}(c_{i,j}).$$

It defines functions label : $E^{++} \to \Sigma^{++}$ and weight : $E^{++} \to K$. We call label(c) the *label* and weight(c) the *weight* of c. A *run* (or *computation*) in $\mathfrak{A}$ is an element in $E^{m\times n}$ satisfying natural compatibility properties, more precisely, for $c = (c_{i,j}) \in E^{m\times n}$ we have

$$\forall\ 1 \leq i \leq m, 1 \leq j \leq n : \ \sigma_v(c_{i,j}) = \sigma(c_{i-1,j}), \sigma_h(c_{i,j}) = \sigma(c_{i,j-1}).$$

A run $c \in E^{m\times n}$ is *successful* if for all $1 \leq i \leq m$ and $1 \leq j \leq n$, we have $\sigma_v(c_{1,j}), \sigma_h(c_{i,1}) \in I$ and $\sigma(c_{m,n}) \in F$. The set of all successful runs labelled with a picture p is denoted by $I \overset{p}{\rightsquigarrow} F$.

We define a picture series $\|\mathfrak{A}\|$ as follows. If $p \in \Sigma^{++}$ has no successful run in $\mathfrak{A}$, $\|\mathfrak{A}\|$ sends p to 0. Otherwise, we define

$$(\|\mathfrak{A}\|, p) = \sum_{c \in I \overset{p}{\rightsquigarrow} F} \text{weight}(c).$$

Intuitively, the weight of a picture p is the sum of the weights of all successful runs in $\mathfrak{A}$ that read p. We call $\|\mathfrak{A}\|$ the *behavior* of $\mathfrak{A}$ and say that the automaton $\mathfrak{A}$ *computes* (or *recognizes*) the picture series $\|\mathfrak{A}\| : \Sigma^{++} \to K$. We write $K^{rec}\langle\langle \Sigma^{++}, W2OTA\rangle\rangle$ for the family of series that are computable by W2OTA over Σ, elements of which are referred to as *W2OTA-recognizable* series.

Considering above Definition 3.1, where K equals $\mathbb{B}$, we get precisely the definition of a *2-dimensional on-line tessellation automaton (2OTA)*. Here, instead of E, one could also define a *transition function* $\delta : Q \times Q \times \Sigma \to 2^Q$. If $|I| = 1$ and $\delta : Q \times Q \times \Sigma \to Q$, we call $\mathfrak{A}$ *deterministic*. For an alphabet Σ, devices of 2OTA over Σ define picture languages and were shown to compute precisely the family Rec(Σ^{++}) of recognizable picture languages [15].

For motivation, we now give two examples of picture series $S : \Sigma^{++} \to \mathbb{R} \cup \{\infty\}$ and $T : \Sigma^{++} \to \mathbb{N}$.

Example 3.2. Let $D \subset [0, 1]$ be a finite set of discrete values and let the language $L \subseteq D^{++}$ be any language recognizable by a 2OTA. Consider the following function $S : D^{++} \to \mathbb{R} \cup \{\infty\}$, defined for $p \in D^{++}$ by

$$S(p) = \begin{cases} \sum_{i,j} p_{i,j} & p \in L \\ \infty & \text{otherwise.} \end{cases}$$

One could interpret the values in D as different levels of gray [10]. Then, for each picture $p \in L$, the function S provides the total value $S(p)$ of light of p.

Example 3.3. Let C be a finite set of colors and consider $T : C^{++} \to \mathbb{N}$, defined by $T(p) = \max\{l_v(q) \cdot l_h(q) \mid q \text{ is a monochrome subpicture of } p\}, (p \in C^{++})$. Then $T(p)$ gives the largest size of a monochrome rectangle, contained in p.

By simulating a 2OTA recognizing L and assigning weights, one can prove that the function S is computable by a W2OTA over the tropical semiring, that is $S \in \mathbb{T}^{rec}\langle\langle D^{++}, W2OTA\rangle\rangle$. Also, there is a W2OTA over the max-plus semiring Arc on the alphabet C computing T. Here, for a picture p, the automaton provides one successful path for every different monochrome subpicture of p. Since we get the behaviour by adding the weights for successful runs reading p, in Arc, the maximal size is extracted.

As illustrated before, the main result will be an extension of Theorem 2.2 to the described weighted scenery. For this, we first establish required properties of W2OTA. Similar to common constructions on automata and using ideas in [4], we have the following.

Proposition 3.4. *W2OTA-recognizable picture series are closed under* $\odot, \oplus, \cdot$*, projections and inverse projections. For languages, inverse projections preserve deterministic devices. If* L *is deterministically recognizable then* $\mathbb{1}_L$ *is recognizable.*

Proof. As usual, for $\odot$ and $\oplus$ we use the union and the direct product of automata, respectively. Let $k \in K$. The W2OTA $\mathfrak{A} = \big(\{0,1\}, E, \{0\}, \{1\}\big)$ defined by

$$E = \bigcup_{a \in \Sigma} \Big\{(0,0,a,k,1), (0,1,a,1,1), (1,0,a,1,1), (1,1,a,1,1)\Big\}$$

computes $\|\mathfrak{A}\| = k \cdot \mathbb{1}_{\Sigma^{++}}$. Now, since in general, for a series $S \in K\langle\langle\Sigma^{++}\rangle\rangle$, we have $k \cdot S = (k \cdot \mathbb{1}_{\Sigma^{++}}) \odot S$, we get the assertion for scalar multiplications. Let $\pi : \Gamma \to \Sigma$. We transform a transition $e = (q_h, q_v, a, k, q)$ in a W2OTA computing $R : \Gamma^{++} \to K$ to a corresponding transition $e' := (q_h, q_v, \pi(a), k, q)$ in a W2OTA computing $\pi(R)$. Now let $\mathfrak{A} = (Q, E, I, F)$ be a W2OTA computing $S : \Sigma^{++} \to K$ with $E \subseteq Q^2 \times \Sigma \times K \times Q$. We obtain a W2OTA $\mathfrak{A}' = (Q, E', I, F)$ on Γ for $\pi^{-1}(S)$ by putting

$$E' := \big\{(q_h, q_v, a, k, q) \mid (q_h, q_v, \pi(a), k, q) \in E\big\}.$$

This construction works also for the language case of 2OTA und then preserves deterministic devices. Let $\mathcal{B}$ be a deterministic 2OTA recognizing L. Assigning $1 \in K$ to every transition in $\mathcal{B}$, will result in a W2OTA recognizing $\mathbb{1}_L$. $\square$

We will now define the notion of a WPA, firstly introduced in [4].

Definition 3.5 ([4])**.** A *weighted (quadrapolic) picture automaton (WPA)* is a 6-tuple $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ consisting of a finite set Q of states, a finite set of rules $R \subseteq \Sigma \times K \times Q^4$, as well as four *poles of acceptance* $F_w, F_n, F_e, F_s \subseteq Q$.

Precisely as with W2OTA in Definition 3.1, for $r = (a, k, q_w, q_n, q_e, q_s) \in R$, we denote by $\text{label}(r)$ its *(input) label* a (extented then to pictures), by $\text{weight}(r)$ its weight k and corresponding to the four poles $\sigma_w(r) := q_w, \sigma_n(r) := q_n, \sigma_e(r) := q_e, \sigma_s(r) := q_s$. A *run* is an element $c = (c_{i,j}) \in R^{m \times n}$ satisfying

$$\forall i \leq m-1, j \leq n : \sigma_s(c_{i,j}) = \sigma_n(c_{i+1,j}),\ \forall i \leq m, j \leq n-1 : \sigma_e(c_{i,j}) = \sigma_w(c_{i,j+1}).$$

We put $\text{weight}(c) = \prod_{i,j} \text{weight}(c_{i,j})$ and call $\text{weight}(c)$ the *weight* of c. A run c is *successful* if it has its (outer) pole-states in the respective poles of acceptance, that is to say:

$$\forall i \leq m, j \leq n : \sigma_w(c_{i,1}) \in F_w, \sigma_n(c_{1,j}) \in F_n, \sigma_e(c_{i,n}) \in F_e, \sigma_s(c_{m,j}) \in F_s. \qquad (1)$$

For a successful run c with $\text{label}(c) = p$ we will shortly write $c \in \text{Succ}(p)$. The automaton *computes* a picture series $\|\mathfrak{A}\| : \Sigma^{++} \to K$ such that

$$(\|\mathfrak{A}\|, p) = \sum_{c \in \text{Succ}(p)} \text{weight}(c),$$

called the *behavior* of $\mathfrak{A}$. The *weight* of a picture p is the sum of the weights of all successful runs with label p.

The family of picture series computed by weighted picture automata over Σ will be denoted by $K^{rec}\langle\langle \Sigma^{++}, WPA \rangle\rangle$. We call elements of this family *WPA-recognizable.*

Again, for the unweighted case of Definition 3.5, where $R \subseteq \Sigma \times Q^4$, we get the description of a *(quadrapolic) picture automaton (PA).* In [4], the authors proved that PA characterize precisely the family of recognizable picture languages.

4 Weighted Logics

In this section we introduce the syntax and semantics of the weighted MSO-logic on pictures. It provides an extension of the unweighted MSO-logic on pictures (Section 2). We will also present basic properties of this logic. We fix a commutative semiring K and an alphabet Σ. For $a \in \Sigma$, P_a denotes a unary predicate symbol. Formulas of the *weighted MSO-logic* are defined recursively as follows:

$$\begin{aligned} \varphi \ ::= \ & k \mid P_a(x) \mid \neg P_a(x) \mid xS_vy \mid \neg(xS_vy) \mid xS_hy \mid \neg(xS_hy) \mid x = y \mid \neg(x = y) \\ & \mid x \in X \mid \neg(x \in X) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi \end{aligned}$$

where $k \in K, a \in \Sigma$ and x, y (resp. X) are first (resp. second)-order variables. The class $\text{MSO}(K, \Sigma)$ comprises all such weighted MSO-formulas φ. The formulas $k, P_a(x), xS_vy, xS_hy, x = y$ and $x \in X$ are referred to as *atomic formulas.* Observe that the negation is applied only to atomic formulas (except k), this is because it would not be clear how to define the semantics of the negation of an arbitrary formula. If we consider again the definition of the general syntax of an (unweighted) MSO-formula in Section 2, this is equivalent (meaning: defining the same languages) to a syntax where negation is only applied to atomic formulas. In this sense our weighted MSO-syntax extends the classical syntax. We can understand a given formula φ, that is weighted in $\mathbb{B}$, as a classical formula defining languages by replacing the values 0 and 1 by their equivalents $\forall x.xS_hx$ and $\forall x.\neg(xS_hx)$, respectively. On the other hand, we can understand a classical MSO-formula as the syntax of a weighted formula.

Subsequently, we will also consider the class $\text{FO}(K, \Sigma) \subset \text{MSO}(K, \Sigma)$ of all formulas containing no set quantifier. Clearly, formulas in $\text{MSO}(K, \Sigma)$, containing no fragment of the form k, may also be regarded as unweighted formulas defining a picture languages. Now, similar to [7] we give the semantics of weighted MSO-formulas φ.

Definition 4.1. Let $\varphi \in \text{MSO}(K, \Sigma)$ and $\mathcal{V}$ be a finite set of variables containing $\text{Free}(\varphi)$. The *semantics* of φ will be a series $[\![\varphi]\!]_{\mathcal{V}} : \Sigma_{\mathcal{V}}^{++} \to K$. Let $(p, \sigma) \in \Sigma_{\mathcal{V}}^{++}$. If σ is not a valid $\mathcal{V}$-assignmen, then we set $[\![\varphi]\!]_{\mathcal{V}}(p, \sigma) = 0$. Otherwise, we define $[\![\varphi]\!]_{\mathcal{V}}(p, \sigma) \in K$

inductively as:

$$
\begin{aligned}
[\![k]\!]_{\mathcal{V}}(p,\sigma) &= k & [\![P_a(x)]\!]_{\mathcal{V}}(p,\sigma) &= \begin{cases} 1 & \text{if } p(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases} \\
[\![xS_vy]\!]_{\mathcal{V}}(p,\sigma) &= \begin{cases} 1 & \text{if } \sigma(x) S_v \sigma(y) \\ 0 & \text{otherwise} \end{cases} & [\![xS_hy]\!]_{\mathcal{V}}(p,\sigma) &= \begin{cases} 1 & \text{if } \sigma(x) S_h \sigma(y) \\ 0 & \text{otherwise} \end{cases} \\
[\![x=y]\!]_{\mathcal{V}}(p,\sigma) &= \begin{cases} 1 & \text{if } \sigma(x) = \sigma(y) \\ 0 & \text{otherwise} \end{cases} & [\![x \in X]\!]_{\mathcal{V}}(p,\sigma) &= \begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
[\![\neg\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathcal{V}}(p,\sigma) = 0 \quad \text{if } \varphi \text{ is of the form } P_a(x),\, x=y, \\ 0 & \text{if } [\![\varphi]\!]_{\mathcal{V}}(p,\sigma) = 1 \quad (xS_vy),\, (xS_hy) \text{ or } (x \in X) \end{cases} \\
[\![\varphi \vee \psi]\!]_{\mathcal{V}}(p,\sigma) &= [\![\varphi]\!]_{\mathcal{V}}(p,\sigma) + [\![\psi]\!]_{\mathcal{V}}(p,\sigma) \\
[\![\varphi \wedge \psi]\!]_{\mathcal{V}}(p,\sigma) &= [\![\varphi]\!]_{\mathcal{V}}(p,\sigma) \cdot [\![\psi]\!]_{\mathcal{V}}(p,\sigma) \\
[\![\exists x.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \sum_{(i,j)\in \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(p,\sigma[x \to (i,j)]) \\
[\![\exists X.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \sum_{I \subseteq \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(p,\sigma[X \to I]) \\
[\![\forall x.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \prod_{(i,j)\in \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(p,\sigma[x \to (i,j)]) \\
[\![\forall X.\varphi]\!]_{\mathcal{V}}(p,\sigma) &= \prod_{I \subseteq \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(p,\sigma[X \to I]).
\end{aligned}
$$

We write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$. In case φ is a sentence, then $[\![\varphi]\!] \in K\langle\!\langle \Sigma^{++} \rangle\!\rangle$. For $Z \subseteq \mathrm{MSO}(K,\Sigma)$, we call a series $S : \Sigma^{++} \to K$ *Z-definable* if there exists a sentence $\varphi \in Z$ satisfying $[\![\varphi]\!] = S$.

Example 4.2. Consider the formula $\varphi = \exists x.P_a(x) \in \mathrm{MSO}(\mathbb{N},\{a,b,c\})$. Then $[\![\varphi]\!]$ is the series that computes for a picture $p \in \{a,b,c\}^{++}$ the number of occurrences of the letter a in p. Also, consider Example 3.2 of Section 3 again. For $L = D^{++}$, the formula $\psi = \forall x.\big(\bigvee_{d\in D}(P_d(x) \wedge d)\big) \in \mathrm{MSO}(\mathbb{T}, D)$ satisfies $[\![\psi]\!] = S$.

For different sets of variables $\mathcal{V}$, we show that our semantics are consistent:

Proposition 4.3. *Let $\varphi \in \mathrm{MSO}(K,\Sigma)$, $\mathcal{V}$ be finite containing $\mathrm{Free}(\varphi)$ and $(p,\sigma) \in N_{\mathcal{V}}$. Then $[\![\varphi]\!]_{\mathcal{V}}(p,\sigma) = [\![\varphi]\!](p,\sigma_{|\,\mathrm{Free}(\varphi)})$, and $[\![\varphi]\!]$ is W2OTA-recognizable if and only if $[\![\varphi]\!]_{\mathcal{V}}$ is W2OTA-recognizable.*

Proof. The first claim is an induction on φ. For the second assertion, let $[\![\varphi]\!] \in K^{\mathrm{rec}}\langle\!\langle \Sigma_{\varphi}^{++}, W2OTA \rangle\!\rangle$ and $\pi : \Sigma_{\mathcal{V}} \to \Sigma_{\varphi}$ the projection (erasing additional layers). Then, $[\![\varphi]\!]_{\mathcal{V}} = (\pi^{-1}[\![\varphi]\!]) \odot \mathbb{1}_{N_{\mathcal{V}}} \in K^{\mathrm{rec}}\langle\!\langle \Sigma_{\mathcal{V}}^{++}, W2OTA \rangle\!\rangle$ by Proposition 3.4 and the fact that the set $N_{\mathcal{V}}$ is recognized by a deterministic 2OTA. Now, let $[\![\varphi]\!]_{\mathcal{V}} \in K^{\mathrm{rec}}\langle\!\langle \Sigma_{\mathcal{V}}^{++}, W2OTA \rangle\!\rangle$ and $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) be the set of first (resp. second)-order variables in $\mathcal{V}$. Then,

$$N^{\mathrm{norm}} = \left\{ (p,\sigma) \in N_{\mathcal{V}} \,\middle|\, {\scriptstyle \forall x \in \mathcal{V}_1 \setminus \mathrm{Free}(\varphi) : \sigma(x) = (1,1), \atop \forall X \in \mathcal{V}_2 \setminus \mathrm{Free}(\varphi) : \sigma(X) = \{(1,1)\}} \right\}$$

is deterministically recognizable. For $(p,\sigma) \in N_\varphi$, π maps exactly one element $(p, \sigma^{\mathrm{norm}}) \in N^{\mathrm{norm}}$ on (p,σ). With the above and Proposition 3.4, we conclude

$$\big(\pi(\llbracket\varphi\rrbracket_{\mathcal{V}} \odot \mathbb{1}_{N^{\mathrm{norm}}}), (p,\sigma)\big) = \sum_{\substack{\pi(p,\sigma')=(p,\sigma)\\ (p,\sigma')\in N^{\mathrm{norm}}}} \llbracket\varphi\rrbracket_{\mathcal{V}}(p,\sigma') = \llbracket\varphi\rrbracket_{\mathcal{V}}(p,\sigma^{\mathrm{norm}}) = \llbracket\varphi\rrbracket(p,\sigma).$$

□

For words, examples show that unrestricted application of universal first-order quantification does not preserve recognizability [7, Ex. 3.3, 3.4]. These settings are contained in our context of the weighted MSO logic and series over pictures. For example, for the semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$, the formula picture series $\forall y \forall x.2 \in \mathrm{MSO}(\mathbb{N}, \Sigma)$ having semantics $\llbracket \forall y \forall x.2 \rrbracket \in \mathbb{N}\langle\langle \Sigma^{++} \rangle\rangle$, defined as

$$p \in \Sigma^{m\times n} : \quad p \mapsto 2^{(m\cdot n)^2},$$

is not W2OTA-recognizable. Therefore, we define the following.

Definition 4.4. A picture series $S : \Sigma^{++} \to K$ is a *first-order step function* (FO step function), if $S = \bigoplus_{i=1}^{n} k_i \cdot \mathbb{1}_{L_i}$ for some $n \in \mathbb{N}, k_i \in K$ and picture languages $L_i \subseteq \Sigma^{++}$ $(i = 1, \ldots, n)$ that are definable by first-order formulas.

We will call $\varphi \in \mathrm{MSO}(K,\Sigma)$ *restricted*, if φ contains no universal set quantification of the form $\forall X.\psi$, and whenever φ contains a universal quantification $\forall x.\psi$, then $\llbracket\psi\rrbracket$ is a FO step function. We let $\mathrm{RMSO}(K,\Sigma)$ comprise all restricted formulas of $\mathrm{MSO}(K,\Sigma)$. Furthermore, let $\mathrm{REMSO}(K,\Sigma)$ contain all restricted *existential* MSO-formulas φ, i.e. φ is of the form $\varphi = \exists X_1, \ldots, X_n.\psi$ such that $\psi \in \mathrm{FO}(K,\Sigma) \cap \mathrm{RMSO}(K,\Sigma)$. The families $K^{\mathrm{rmso}}\langle\langle\Sigma^{++}\rangle\rangle$ (resp. $K^{\mathrm{remso}}\langle\langle\Sigma^{++}\rangle\rangle$) contain all picture series $S \in K\langle\langle\Sigma^{++}\rangle\rangle$ which are definable by some sentence in $\mathrm{RMSO}(K,\Sigma)$ (resp. in $\mathrm{REMSO}(K,\Sigma)$). The following equivalence theorem states that the families of picture series that are W2OTA-recognizable resp. WPA-recognizable coincide with the families of series defined by sentences of weighted RMSO resp. REMSO logic.

Theorem 4.5. *Let Σ be an alphabet and K any commutative semiring. Then*

$$K^{\mathrm{rec}}\langle\langle\Sigma^{++}, WPA\rangle\rangle = K^{\mathrm{rec}}\langle\langle\Sigma^{++}, W2OTA\rangle\rangle = K^{\mathrm{rmso}}\langle\langle\Sigma^{++}\rangle\rangle = K^{\mathrm{remso}}\langle\langle\Sigma^{++}\rangle\rangle.$$

We prove this theorem in Sections 6 till 8 using a circular argument combining the inclusions of the respective families of picture series. In parts of our proofs, we follow ideas of [7]. The crucial difference concerns the universal FO quantification. Over pictures, not every recognizable language is determinizable, but this is one important property within the proofs of [7]. Here we consider a restriction of this quantification to formulas having a semantics which is a FO step function. But still the proof of the word-case does not work due to the two dimensions of a run in a picture automaton. We will present a counter example proving that the definition of a recognizable step function occuring in the word case would not been adequate here. We therefore rather build a formula instead of constructing a certain (unweighted) automaton. An open problem here would be to find a more direct automata-theoretic construction.

In our proof, for the disposition of weights, the key property will be that a certain constructed unweighted formula defines a language which is computable by a 2OTA that is unambiguous. Also, observe that, in Theorem 4.5, going from RMSO to REMSO is not at all clear, since in the framework of pictures using successor relations instead of $\leq_v$ and $\leq_h$, the proof in the one-dimensional case of formal power series on words, for making FO formulas unambiguous [7, Lemma 5.2] does not work in our case. However, we have to handle successor relations, since there are $(\leq_v, \leq_h)$-definable picture languages that are not recognizable.

5 Unambiguous Picture Languages

The notion of ambiguity for picture languages in the context of tiling systems was briefly introduced in [13]. The authors defined the class $\mathrm{UPLoc}(\Sigma^{++})$ (in [13] this class was denoted by UREC) of picture languages that are injective projections of local languages and posed the conjecture that $\mathrm{UPLoc}(\Sigma^{++})$ is properly included in the family of recognizable picture languages. Very recently, in [1] it was shown that this conjecture is true. The authors proved that there are recognizable picture languages that are inherently ambiguous. Furthermore, they showed that it is undecidable whether a tiling system is unambiguous. We considered unambiguous picture automata and unambiguous 2OTA in [25, 27]. We showed that unambiguous picture languages are closed under injective projections and disjoint union, as well as the coincidence of the families of languages computed by unambiguous 2OTA and unambiguous tiling systems (this result was independently derived in [1]). In [26], we introduced unambiguous rational operations on picture languages and provided further properties and devices to obtain an equivalence theorem for unambiguous picture languages. More precisely, we characterized injective projections of local languages as injective projections of unambiguous rational languages, unambiguous domino recognizable languages and also as behaviors of unambiguous (quadrapolic) picture automata.

Let Σ and Γ be alphabets. We call a possibly weighted 2OTA $\mathfrak{A}$ *unambiguous* if for any input picture there exists at most one successful run in $\mathfrak{A}$. Clearly, every deterministic 2OTA is unambiguous. Simulating the proof of Proposition 3.4, if L is computable by an unambiguous 2OTA, then $\mathbb{1}_L \in K^{\mathrm{rec}}\langle\langle \Sigma^{++}, W2OTA\rangle\rangle$.

For $L \subseteq \Gamma^{++}$, we call a projection $\pi : \Gamma \to \Sigma$ *injective* on L if $\pi : L \to \Sigma^{++}$ is an injective mapping. For a picture p, we denote by $\hat{p}$ the picture that results from p by surrounding it with the (new) boundary symbol $\#$. If p has size (m, n) then $\hat{p}$ has size $(m + 2, n + 2)$. *Tiles* are pictures of size $(2, 2)$. We denote by $T(p)$ the set of all sub-tiles of p. A language $L \subseteq \Gamma^{++}$ is *local* if there exists a set Θ of tiles over $\Gamma \cup \{\#\}$, such that $L = \{p \in \Gamma^{++} \mid T(\hat{p}) \subseteq \Theta\}$. Then (Γ, Θ) *characterizes* L. We write $L = \mathcal{L}(\Theta)$. We define $L \subseteq \Sigma^{++}$ as *unambiguous tiling recognizable* if there exists a local language $L' \subseteq \Gamma^{++}$, characterized by (Γ, Θ), and a projection $\pi : \Gamma \to \Sigma$ such that π is injective on L' and $\pi(L') = L$. In this case, we call $(\Sigma, \Gamma, \Theta, \pi)$ an *unambiguous* tiling system *computing* L. If the projection is not necessarily injective, we obtain the known definition of a tiling system (TS). Unambiguous tiling recognizable languages over Σ are collected in $\mathrm{UPLoc}(\Sigma^{++})$.

Lemma 5.1. *The class* $\mathrm{UPLoc}(\Sigma^{++})$ *is closed under injective projections and disjoint union. A language L is recognizable by an unambiguous 2OTA if and only if it is computable by an unambiguous tiling system.*

Proof. Let Σ, Γ, Δ be alphabets and $(\Gamma, \Delta, \Theta, \psi)$ unambiguous computing $L \subseteq \Gamma^{++}$. If $\pi : \Gamma \to \Sigma$ is injective on L, then $\tau := (\Sigma, \Delta, \Theta, \psi \circ \pi)$ is unambiguous for $\pi(L)$. Let $L_1, L_2 \in \mathrm{UPLoc}\,\Sigma^{++}, L_1 \cap L_2 = \emptyset$. We follow the construction in [15, Theorem 7.4]. The given TS for $L := L_1 \cup L_2$ is unambiguous since the union is disjoint. For the second claim, the TS, constructed in [15, Lemma 8.1] and also the automaton constructed in [15, Lemma 8.2] are unambiguous. $\square$

We call languages in $\mathrm{UPLoc}(\Sigma^{++})$ *unambiguous*. For the next steps we will use the notion of a locally threshold testable picture language [16]. We recall here the definition. Let Σ be an alphabet. For $p \in \Sigma^{m \times n}$ and $q \in \Sigma^{m' \times n'}$, we call p a *subblock* of q if there are $k \leq m' - m$ and $l \leq n' - n$ such that $q(i,j) = p(k+i, l+j)$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Let $d, t \geq 1$. Two pictures q_1, q_2 are *(d,t)-block-threshold-equivalent* iff for every square picture p of size $d' \times d'$ (with $d' \leq d$), the number of occurrences of p as a subblock in $\hat{q_1}$ (respectively $\hat{q_2}$) are equal or both $> t$. A picture language L is *locally threshold testable (LTT)* if there are d, t such that L is a union of (d,t)-block-threshold-equivalence classes.

For picture languages it holds the following relation:

Proposition 5.2 ([16])**.** *A language is locally threshold testable if and only if it is FO-definable.*

We will now show, that first-order-definable picture languages are unambiguous. The inclusion of the other direction does not hold, more precisely, there are unambiguous picture languages that are not FO-definable. An example would be the set of all squares over an alphabet. The next proposition will be crucial for proving Lemma 6.3 below. The idea for the course of the proof is to follow constructions in [16]. But, we now need unambiguous picture languages, hence we have to construct injective projections and disjoint unions.

Proposition 5.3. *Let L be FO-definable. Then L is unambiguous.*

Proof. Using Proposition 5.2, we have to show that LTT languages are unambiguous. Let $L \subseteq \Sigma^{++}$ be LTT for (d,t). As in Lemma 3.7 [16], we partition L into a union of strictly LTT languages (where strictly means, only squares of dimension d are considered). This union is easily proved as disjoint. Strictly LTT languages are projections d-local languages (for d-locality we use a set $\Theta^{(d)}$ of $(d \times d)$-tiles instead of (2×2)-tiles for for the definition of local sets)[16, Lemma 3.9]. For a given (d,t)-strictly LTT language L', in the construction, we perform a scanning of $p' \in L'$ using certain d-squares and counting occurrences up to threshold t. For acceptance, one compares these computed values with the tuples characterizing L'. This defines a d-local language L'' and a projection π satisfying $\pi(L'') = L'$. We can modify the set of d-tiles (and hence L'') by strengthening their border-conditions in such a way that for every $p' \in L'$ there exists one uniquely determined $p'' \in L''$ satisfying $\pi(p'') = p'$. Hence, the modified projection then is injective on L''.

It remains to show that every d-local language M is unambiguous. For this, let Δ be arbitrary, $d \geq 3$ and M characterized by $(\Delta, \Theta^{(d)})$. For every $p \in M$ we can assume $p \in \Delta^{m \times n}$ such that $m, n \geq d-2$ ([16, Lemma 3.10]). We prove that M is an injective projection of a local set, that is, M is computable by an unambiguous tiling system. We define $T = (\Delta, \Gamma, \Theta, \pi)$ as

- $\overline{\Theta^{(d)}} := \left\{ \begin{array}{|c|c|} \hline \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \\ \hline \end{array} \in (\Delta \mathbin{\dot\cup} \{\#\} \mathbin{\dot\cup} \{+\})^{d \times d} \mid B = C = D \ \equiv \ + \ , \ \exists \ \begin{array}{|c|c|} \hline \mathrm{A1} & \mathrm{A2} \\ \hline \mathrm{A3} & \mathrm{A} \\ \hline \end{array} \in \Theta^{(d)} \right\}$
- $\Gamma := \overline{\Theta^{(d)}} \setminus \{p \mid p_{1,1} = \#\}$
- border symbols: $\{p \in \overline{\Theta^{(d)}} \mid p_{1,1} = \#\}$

- $\Theta := \left\{ \boxed{\begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array}} \in \Gamma^{2\times 2} \middle| A = \boxed{\begin{array}{c|c} \text{a} & \text{N} \\ \hline \text{W} & \text{Q} \end{array}}, B = \boxed{\begin{array}{c|c} \text{N} & \text{b} \\ \hline \text{Q} & \text{E} \end{array}}, C = \boxed{\begin{array}{c|c} \text{W} & \text{Q} \\ \hline \text{c} & \text{S} \end{array}}, D = \boxed{\begin{array}{c|c} \text{Q} & \text{E} \\ \hline \text{S} & \text{d} \end{array}} \right\}$

  where $Q \in (\Delta \mathbin{\dot\cup} \{\#\} \mathbin{\dot\cup} \{+\})^{(d-1)\times(d-1)}$ and W, S, E, N, a, b, c, d accordingly.

We define a projection $\pi : \Gamma \to \Delta$ by $p \mapsto p_{1,1}$ and show $\pi(\mathcal{L}(\Theta)) = M$. For this, let $p \in M$. We extend p to a picture $\bar{p} \in (\Delta \mathbin{\dot\cup} \{\#\} \mathbin{\dot\cup} \{+\})^{(m+d-1)\times(n+d-1)}$ and then define $p' \in \Gamma^{m\times n}$, as follows

$$\bar{p}(i,j) = \begin{cases} p(i,j) & i \leq m, j \leq n \\ \# & \begin{array}{l} i = m+1, j \leq n+1 \\ \mathit{or}\ i \leq m+1, j = n+1 \end{array} \\ + & \mathit{otherwise}, \end{cases}$$

$$p'(i,j) = \boxed{\begin{array}{lcr} \bar{p}(i,j) & \cdots & \bar{p}(i,j+d-1) \\ \vdots & & \vdots \\ \bar{p}(i+d-1,j) & \cdots & \bar{p}(i+d-1,j+d-1) \end{array}}.$$

Then, $p' \in \mathcal{L}(\Theta)$ and $\pi(p') = p$. Now let $p' \in \mathcal{L}(\Theta)$ and q be a $(d \times d)$-subpicture of $\hat{p'}$. It suffices to show $\pi(q) \in \Theta^{(d)}$. With the construction of $\overline{\Theta^{(d)}}$ above we conclude $q_{1,1} \in \Theta^{(d)}$. But, $q_{1,1} = \pi(q)$. By the structure of the d-tiles in Θ, we can show that T is unambiguous, i.e., π is injective on $\mathcal{L}(\Theta)$. We constructed unambiguous languages and disjoint unions. With Lemma 5.1, L is unambiguous. ☐

6 Definable Picture Series are W2OTA-Recognizable

The aim of this section is to show that semantics of sentences in $\mathrm{RMSO}(K, \Sigma)$ are series recognized by W2OTA. We prove this implication by structural induction on the formulas in $\mathrm{RMSO}(K, \Sigma)$. Let Σ be an alphabet. The following lemma is immediate by the results in [16] and Remark 2.1.

Lemma 6.1. *Let $\mathcal{V}$ be a set of variables. Then the set $N_{\mathcal{V}} \subseteq (\Sigma \times \{0,1\}^{\mathcal{V}})^{++}$ is FO-definable. The class $\mathcal{L}(\mathrm{FO}(\Sigma^{++}))$ is closed under boolean operations.*

Lemma 6.2. *Let $\varphi, \psi \in \mathrm{MSO}(K, \Sigma)$. Then the following holds.*

(a) If φ is atomic or the negation of an atomic formula, then φ is W2OTA-recognizable.

(b) If $[\![\varphi]\!]$ and $[\![\psi]\!]$ are W2OTA-recognizable, then $[\![\varphi \vee \psi]\!]$ and $[\![\varphi \wedge \psi]\!]$ are W2OTA-recognizable.

(c) If $[\![\varphi]\!]$ is W2OTA-recognizable, then $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are W2OTA-recognizable.

Proof. (a) We construct W2OTA using Proposition 3.4. We have $[\![k]\!] = k \cdot \mathbb{1}_{\Sigma^{++}}$ and the semantics of the other atomic formulas are characteristic series. All occuring supports are deterministically recognizable picture languages. We get the assertion by Proposition 3.4. For (b), let $\mathcal{V} = \mathrm{Free}(\varphi) \cup \mathrm{Free}(\psi)$. By definition, we have $[\![\varphi \vee \psi]\!] = [\![\varphi]\!]_{\mathcal{V}} + [\![\psi]\!]_{\mathcal{V}}$ and $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!]_{\mathcal{V}} \odot [\![\psi]\!]_{\mathcal{V}}$. These series are W2OTA-recognizable using Propositions 4.3 and 3.4. In (c) we prove the set variable case. For this, let $\mathcal{V} = \mathrm{Free}(\exists X.\varphi)$ and put $\pi : \Sigma_{\mathcal{V}\cup\{X\}} \to \Sigma_{\mathcal{V}}$ the projection erasing the X-level. Let $(p, \sigma) \in N_{\mathcal{V}}$ and observe, that

σ is a valid $\mathcal{V}$-assignment iff $\sigma[X \to I]$ is a valid $\mathcal{V} \cup \{X\}$-assignment for all $I \subseteq \mathrm{Dom}(p)$. Then, for $(p, \sigma) \in N_{\mathcal{V}}$, we conclude

$$[\![\exists. X\varphi]\!](p,\sigma) = \sum_{I \subseteq \mathrm{Dom}(p)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(p, \sigma[X \to I]) = \pi([\![\varphi]\!]_{\mathcal{V}\cup\{X\}})(p,\sigma).$$

Now, $[\![\varphi]\!]_{\mathcal{V}\cup\{X\}}$ and $[\![\exists X.\varphi]\!]$ are W2OTA-recognizable with Proposition 4.3. Similarly, we prove the result for first-order variables. □

The next lemma shows that for formulas having a semantics which is a first-order step function, the application of the universal first-order quantification provides a recognizable semantics. We use ideas of the corresponding proof for words in [7, Lemma 4.2]. There, the authors construct automata for the resulting semantics, but these did not work in this setting due to the two dimensions of a run in an automaton on pictures and due to the fact that not every recognizable picture language is determinizable. In fact, at the end of this section we will present a counter example proving that the definition of a recognizable step function occuring in the word case would not been adequate here.

Lemma 6.3. *Let* $\varphi \in \mathrm{MSO}(K, \Sigma)$ *such that* $[\![\varphi]\!]$ *is a first-order step function. Then* $[\![\forall x.\varphi]\!]$ *is a W2OTA-recognizable picture series.*

Proof. As prerequisite, let $\mathcal{W} = \mathrm{Free}(\varphi)$, $\mathcal{V} = \mathrm{Free}(\forall x.\varphi) = \mathcal{W} \setminus \{x\}$ and assume $[\![\varphi]\!] = \bigoplus_{l=1,\dots,n} k_l \cdot \mathbb{1}_{L_l}$ with $n \in \mathbb{N}$, $k_l \in K$ and L_l definable by FO sentences ($l = 1, \dots, n$) such that the languages L_l form a partition of N_φ (use Remark 2.1 and Lemma 6.1). Assume $x \in \mathcal{W}$.

The definition of the semantics of the universal first-order quantification of a formula maps a picture p to the product over all positions in p of certain values in K. In our setting here, the factors are the elements k_l corresponding to the supports of $[\![\varphi]\!]$. We mark positions of p by their respective index l of k_l. Let $\widetilde{\Sigma} = \Sigma \times \{1, \dots, n\}$. A picture in $(\widetilde{\Sigma}_{\mathcal{V}})^{++}$ will be written as a tuple (p, ν, σ) where $(p, \sigma) \in \Sigma_{\mathcal{V}}^{++}$ and $\nu \in \{1, \dots, n\}^{++}$ is interpreted as a mapping from $\mathrm{Dom}(p)$ to $\{1, \dots, n\}$. Let $\widetilde{L}$ be the set of all $(p, \nu, \sigma) \in (\widetilde{\Sigma}_{\mathcal{V}})^{++}$ such that

$$\nu(i,j) = l \iff (p, \sigma[x \to (i,j)]) \in L_l$$

for all $(i,j) \in \mathrm{Dom}(p)$ and $l \in \{1, \dots, n\}$. Observe that for each $(p, \sigma) \in \Sigma_{\mathcal{V}}{}^{++}$ there is a unique $\nu \in \{1, \dots, n\}$ such that $(p, \nu, \sigma) \in \widetilde{L}$ since the L_l form a partition of N_φ. We prove that $\widetilde{L}$ is definable by a sentence in $\mathrm{FO}(\widetilde{\Sigma}_{\mathcal{V}}^{++})$ by presenting a formula. Let $1 \le l \le n$ and let φ_l be a first-order sentence over $\Sigma_{\mathcal{W}}$ for L_l. We define $\widetilde{\varphi_l} \in \mathrm{FO}((\widetilde{\Sigma}_{\mathcal{W}})^{++})$ as φ_l where all occurrences of $P_{(a,r)}(y)$ (here, $a \in \Sigma, r \in \{0,1\}^{\mathcal{W}}$) are replaced by $\bigvee_{1 \le l \le n} P_{(a,l,r)}(y)$. We obtain sentences $\widetilde{\varphi_l}$ over $\widetilde{\Sigma}_{\mathcal{W}}$. For $(p, \nu, \sigma) \in \widetilde{\Sigma}_{\mathcal{W}}^{++}$, we conclude $(p, \nu, \sigma) \in \mathcal{L}(\widetilde{\varphi_l})$ iff $(p, \sigma) \in \mathcal{L}(\varphi_l)$. Additionally, we define $\widetilde{\varphi_l}'$ as $\widetilde{\varphi_l}$, modified as follows. Occurrences of $P_{(a,l,r)}(y)$ satisfying $r(x) = 1$ become $P_{(a,l,r')}(y) \wedge (x = y)$ and occurrences of $P_{(a,l,r)}(y)$ with $r(x) = 0$ become $P_{(a,l,r')}(y) \wedge \neg(x = y)$, where r' is the restriction of r to $\mathcal{W} \setminus \{x\}$. Then, $\widetilde{\varphi_l}'$ is an FO-formula over the alphabet $\widetilde{\Sigma}_{\mathcal{V}}$ with $\mathrm{Free}(\widetilde{\varphi_l}') = \{x\}$ satisfying for all $(p, \nu, \tau') \in N_{\widetilde{\varphi_l}'}$, that $(p, \nu, \tau') \in \mathcal{L}(\widetilde{\varphi_l}')$ if and only if $(p, \nu, \tau) \in \mathcal{L}(\widetilde{\varphi_l})$. Now, set

$$\widetilde{\varphi} := \forall x. \bigwedge_{1 \le l \le n} \left[(\nu(x) = l) \Leftrightarrow \widetilde{\varphi_l}' \right]$$

where $\nu(x) = l$ stands for $\bigvee_{(a,r') \in \Sigma_{\mathcal{V}}} P_{(a,l,r')}(x)$ and $\Leftrightarrow$ is the standard abbreviation. Now, $\widetilde{\varphi}$ is a first-order sentence over $\widetilde{\Sigma}_{\mathcal{V}}$. We show $\mathcal{L}(\widetilde{\varphi}) = \widetilde{L}$.

Let $(q,\nu,\tau') \in (\widetilde{\Sigma}_{\mathcal{V}})^{++}$, where $q \in \Sigma, \nu \in \{1,\dots,n\}$ and $\tau' \in (\{0,1\}^{\mathcal{V}})^{++}$. Then $(q,\nu,\tau') \models \widetilde{\varphi}$ iff for all $(i,j) \in \mathrm{Dom}(q)$ and all $1 \le l \le n$, $(q,\nu,\tau',[x \to (i,j)]) \in \mathcal{L}\big((\nu(x)=l) \Leftrightarrow \widetilde{\varphi}_l{}'\big)$, where $[x \to (i,j)]$ denotes the assignment defined on $\{x\}$ mapping x to (i,j). Now, $(q,\nu,\tau',[x \to (i,j)]) \in \mathcal{L}(\nu(x)=l)$ iff $\nu(i,j)=l$ and $(q,\nu,\tau',[x \to (i,j)]) \in \mathcal{L}(\widetilde{\varphi}_l{}')$ iff $(q,\nu,\tau) \in \mathcal{L}(\widetilde{\varphi}_l)$ iff $(q,\tau) \in \mathcal{L}(\varphi_l)$ iff $(q,\tau) \in L_l$, where in all cases, τ equals τ' extended by an x-level such that precisely position (i,j) carries 1. Hence the constructed formula $\widetilde{\varphi}$ defines $\widetilde{L}$.

Now, using Proposition 5.3 and Lemma 5.1, there exists an unambiguous 2OTA $\widetilde{A} = \big(Q,I,F,E\big)$ over $\widetilde{\Sigma}_{\mathcal{V}}$ computing $\widetilde{L}$. We obtain a W2OTA $\widetilde{\mathfrak{A}} = \big(Q,I,F,\bar{E}\big)$ over $\widetilde{\Sigma}_{\mathcal{V}}$ disposing weights along $\widetilde{A}$ as:

$$\big(p,q,(a,l,s),r\big) \in E \text{ iff } \big(p,q,(a,l,s),k_l,r\big) \in \bar{E},$$

where $[\![\varphi]\!] = \bigoplus_{l=1,\dots,n} k_l \cdot \mathbb{1}_{L_l}$. Then $\widetilde{\mathfrak{A}}$ is unambiguous. The weight of $(p,\nu,\sigma) \in \widetilde{L}$ in $\widetilde{\mathfrak{A}}$ is $\prod_{1 \le l \le n} k_l^{|\nu^{-1}(l)|}$ and the weight of a picture in $\widetilde{\Sigma}_{\mathcal{V}}^{++} \setminus \widetilde{L}$ is 0. Now, for the projection $\pi : \widetilde{\Sigma}_{\mathcal{V}} \to \Sigma_{\mathcal{V}}$ we compute for $(p,\sigma) \in \Sigma_{\mathcal{V}}^{++}$ and the unique ν such that $(p,\nu,\sigma) \in \widetilde{L}$,

$$\pi(\|\widetilde{\mathfrak{A}}\|)(p,\sigma) = \sum_{\nu} \|\widetilde{\mathfrak{A}}\|(p,\nu,\sigma) = \|\widetilde{\mathfrak{A}}\|(p,\nu,\sigma) = \prod_{1 \le l \le n} k_l^{|\nu^{-1}(l)|}.$$

But,

$$[\![\forall x.\varphi]\!](p,\sigma) = \prod_{(i,j)\in \mathrm{Dom}(p)} [\![\varphi]\!](p,\sigma[(i,j) \to x]) = \prod_{1 \le l \le n} k_l^{|\nu^{-1}(l)|}.$$

Using Proposition 3.4 and Lemma 6.2, we conclude that $[\![\forall x.\varphi]\!]$ is recognizable.

The case $x \notin \mathcal{W}$ is reduced to the above by replacing φ by $\varphi \wedge (x=x)$ and applying Lemma 6.2. □

Clearly, with the lemmas in this section we conclude the follwing theorem.

Theorem 6.4. *We have* $K^{\mathrm{rmso}}\langle\!\langle \Sigma^{++} \rangle\!\rangle \subseteq K^{\mathrm{rec}}\langle\!\langle \Sigma^{++}, W2OTA \rangle\!\rangle$.

As noted before, we now give an example of a formula φ over the Boolean semiring such that the semantics of φ has the form of a step function with recognizable supports but $\forall x.\varphi$ is no recognizable picture series anymore. For this we use the fact that the family of recognizable picture languages is not closed under complement [15].

Example 6.5. We let $\Sigma = \{0,1\}$ and $\Gamma = \Sigma^2$. Consider the following picture language $L \subset \Gamma^{++}$, defined as

$$L = \left\{ p \in \Gamma^{2n \times n} \mid \begin{array}{l} \text{if } \ p_{i,j} \in \{(0,1),(1,1)\} \text{ and } i \le n \\ \text{then } p_{i,j} = p_{i+n,j} \end{array} \right\} \cap N_{\{x\}},$$

where $N_{\{x\}} \subseteq \Sigma_{\{x\}}^{++}$. Intuitively, L comprises pictures over Σ that can be decomposed vertically into two squares such that there is exactly one marking 1 on a position of the upper square carrying the same letter than the respective position of the lower square. Then L is recognizable. Indeed, the language L can be written as

$$L = N_{\{x\}} \cap L_1 \cap (\Gamma^{++} \ominus (L_2 \cap (\Gamma^{++} \obar L_3 \obar \Gamma^{++})) \ominus \Gamma^{++}),$$

where

$$\begin{aligned} L_1 &= \{p \in \Gamma^{++} \mid l_v(p) = 2l_h(p)\} \\ L_2 &= \{p \in \Gamma^{++} \mid l_v(p) = l_h(p) + 1\} \\ L_3 &= \{p \in \Gamma^{++} \mid l_h(p) = 1 \text{ and } p(1,1) = p(l_v(p), 1)\}. \end{aligned}$$

The languages L_1 to L_3 are clearly recognizable (see also the proof of Theorem 7.5 in [15]); with Lemma 6.1 it follows that L is a recognizable picture language. Now, with Theorem 2.2 and applying the constructions of the previous Lemma 6.3, there exists formula $\varphi \in \mathrm{MSO}(\Sigma^{++})$ with $\mathrm{Free}(\varphi) = \{x\}$ satisfying $\mathcal{L}(\varphi) = L$. But

$$[\![\forall x.\varphi]\!] = \{p \in \Sigma^{++} \mid p = s \ominus s \text{ where } s \text{ is a square}\}$$

is a non-recognizable language [15]. Now, consider φ as a weighted formula in $\mathrm{MSO}(\mathbb{B}, \Sigma)$. Then $[\![\varphi]\!] = \mathbb{1}_L$ is recognizable with recognizable support, but $[\![\forall x.\varphi]\!]$ is no recognizable series using the relation between series over $\mathbb{B}$ and their support languages [4].

7 W2OTA-Recognizable Series are WPA-Recognizable

In order to get a circular argument we will now convert a weighted 2-dimensional on-line tessellation automaton into a weighted picture automaton. One could prove this inclusion by defining some intermediate "tiling" device, describing the context of pixels within their computation. Here these tiles are encoded into the states of the new automaton. Let K be a commutative semiring. For the proof of Theorem 7.5 we will first convert a given W2OTA into some "deterministic" device of a certain type via a projection similar to a construction in [26] where we proved a Kleene-Schützenberger Theorem for picture series. However, in the present paper we apply this construction to W2OTA rather than to WPA. The behavior of the constructed deterministic automaton will then be proved to be WPA-recognizable.

Definition 7.1. A weighted 2-dimensional on-line tessellation automaton is called *rule deterministic* if for every input label a of the underlying alphabet there is at most one transition with label a.

Given a rule deterministic W2OTA with transition set E, for $(q_h, q_v, a, k, q) \in E$ as a transition with label a we abreviate (q_h, q_v, a, k, q) by $r(a)$.

Proposition 7.2. *Let $\mathfrak{A}$ be a W2OTA over Σ. There exists a rule deterministic W2OTA $\mathfrak{A}'$ over an alphabet Γ and a projection $\pi : \Gamma \to \Sigma$ satisfying $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$.*

Proof. Let $\mathfrak{A} = (Q, E, I, F)$ be a W2OTA over Σ and K computing S. We set $\Gamma := E$ and define a rule deterministic W2OTA over Γ by letting $\mathfrak{A}' = (Q, E', I, F)$ such that

$$E' := \Big\{ \big(q_h, q_v, (q_h, q_v, a, k, q), k, q\big) \mid (q_h, q_v, a, k, q) \in E \Big\}.$$

Clearly, for every input label $(q_h, q_v, a, k, q) \in \Gamma$ there is at most one transition with label (q_h, q_v, a, k, q) in E'. We define a projection $\pi : \Gamma \to \Sigma$ by letting

$$\pi(q_h, q_v, a, k, q) \mapsto a.$$

We have to prove $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$ $(*)$.

Let $x \in \Sigma^{m\times n}$. If there was no successful run of x in $\mathfrak{A}$, then there is no picture in E^{++} with a successful run in $\mathfrak{A}'$, which is mapped to x by π, so $(*)$ holds. For the other case, let $\{c_1, c_2, \ldots, c_s\} \subseteq E^{++}$ be the set of successful computations for x in $\mathfrak{A}$. These runs belong to successful runs $\{c'_1, c'_2, \ldots, c'_s\} \subseteq E'^{++}$ in $\mathfrak{A}'$ such that

$$\forall 1 \leq i \leq s : \pi(\text{label}(c'_i)) = x, \quad \sum_{1\leq i\leq s} \text{weight}\big(c_i\big) = \sum_{1\leq i\leq s} \text{weight}\big(c'_i\big).$$

Since there cannot be other successful runs in $\mathfrak{A}'$ mapped by the projection π to x, we conclude $(*)$:

$$(\|\mathfrak{A}\|, x) = \sum_{1\leq i\leq s} \text{weight}\big(c_i\big) = \sum_{\pi(x')=x} (\|\mathfrak{A}'\|, x') = (\pi(\|\mathfrak{A}'\|), x).$$

$\square$

Proposition 7.3. *Every picture series that is recognizable by a rule deterministic W2OTA is WPA-recognizable.*

Proof. Let $\mathfrak{A} = (Q, E, I, F)$ be a rule deterministic W2OTA over the alphabet Σ computing a series $\|\mathfrak{A}\| : \Sigma^{++} \to K$. We construct a WPA $\mathfrak{B} = (L, R, F_w, F_n, F_e, F_s)$ over Σ computing $\|\mathfrak{A}\|$ by defining L as the largest subset of $(\Sigma \cup \{\#\})^{2\times 2}$ satisfying for all letters $a, b \in \Sigma$ and $p, q \in \Sigma \mathbin{\dot\cup} \{\#\}$:

If $\Big(\boxed{\begin{smallmatrix} \# & a \\ p & q \end{smallmatrix}} \in L \vee \boxed{\begin{smallmatrix} p & q \\ \# & a \end{smallmatrix}} \in L\Big)$ then $\sigma_h(r(a)) \in I$; If $\Big(\boxed{\begin{smallmatrix} \# & p \\ a & q \end{smallmatrix}} \in L \vee \boxed{\begin{smallmatrix} p & \# \\ q & a \end{smallmatrix}} \in L)$ then $\sigma_n(r(a)) \in I$;

$$\text{If } \boxed{\begin{smallmatrix} a & \# \\ \# & \# \end{smallmatrix}} \in L \text{ then } \sigma(r(a)) \in F.$$

We define

- $F_w = \left\{\boxed{\begin{smallmatrix} \# & a \\ \# & b \end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}, \quad F_n = \left\{\boxed{\begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}$
- $F_e = \left\{\boxed{\begin{smallmatrix} a & \# \\ b & \# \end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}, \quad F_s = \left\{\boxed{\begin{smallmatrix} a & b \\ \# & \# \end{smallmatrix}} \mid a \in \Sigma, b \in \Sigma \cup \{\#\}\right\}$

We set $R = R_{ulc} \cup R_{ue} \cup R_{le} \cup R_m \subseteq \Sigma \times K \times L^4$ (where ulc, ue, le, m stand for "upper left corner", "upper edge", "left edge", "middle", respectively) with $(a, b, c, d, f, g, h, x, y, t, z \in \Sigma \cup \{\#\})$:

- $R_{ulc} = \left\{r = \left(\boxed{\begin{smallmatrix} \# & a \\ \# & c \end{smallmatrix}}, \boxed{\begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix}}, a, \text{ weight}(r(a)), \boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}}\right) \mid a \in \Sigma\right\}$
- $R_{ue} = \left\{r = \left(\boxed{\begin{smallmatrix} a & b \\ h & d \end{smallmatrix}}, \boxed{\begin{smallmatrix} \# & \# \\ b & c \end{smallmatrix}}, b, \text{ weight}(r(b)), \boxed{\begin{smallmatrix} b & c \\ d & f \end{smallmatrix}}\right) \mid a, b \in \Sigma\right\}$
- $R_{le} = \left\{r = \left(\boxed{\begin{smallmatrix} \# & c \\ \# & g \end{smallmatrix}}, \boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}}, c, \text{ weight}(r(c)), \boxed{\begin{smallmatrix} c & d \\ g & h \end{smallmatrix}}\right) \mid a, c \in \Sigma\right\}$
- $R_m = \left\{r = \left(\boxed{\begin{smallmatrix} z & a \\ t & c \end{smallmatrix}}, \boxed{\begin{smallmatrix} x & y \\ a & b \end{smallmatrix}}, a, \text{ weight}(r(a)), \boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}}\right) \mid a, x, z \in \Sigma\right\}$

To prove $\|\mathfrak{B}\| = \|\mathfrak{A}\|$, we observe the following. Given a picture $p \in \Sigma^{++}$ with successful computation $c \in E^{++}$ in $\mathfrak{A}$, for weight(c), the weight of the rule of every pixel of p occurs exactly once in the multiplication. On the other hand, the tiles of an arbitrary picture p are encoded in L. The given construction with its accepting condition defines an unambiguous weighted picture automaton which has a unique successful run for every element in Σ^{++}. Hence for $p \in \Sigma^{++}$ we have

$$\|\mathfrak{B}\|(p) = \prod_{\substack{1\leq i\leq l_v(p)+1\\ 1\leq j\leq l_h(p)+1}} \text{weight}(r(p_{i,j})) = (\|\mathfrak{A}\|, p).$$

□

Similar to Proposition 3.4 we can prove that the family of WPA-recognizable series are closed under projection.

Lemma 7.4 ([4]). *Let* $\pi : \Gamma \to \Sigma$ *and* $S \in K^{\mathrm{rec}}\langle\langle \Gamma^{++}, WPA\rangle\rangle$. *Then* $\pi(S) \in K^{\mathrm{rec}}\langle\langle \Sigma^{++}, WPA\rangle\rangle$.

Theorem 7.5. $K^{\mathrm{rec}}\langle\langle \Sigma^{++}, W2OTA\rangle\rangle \subseteq K^{\mathrm{rec}}\langle\langle \Sigma^{++}, WPA\rangle\rangle$.

Proof. Immediate by Propositions 7.2 and 7.3 and Lemma 7.4. □

8 WPA-Recognizable Picture Series are Definable

We want to show that WPA-recognizable series are REMSO-definable. Similar to [7], for a WPA $\mathfrak{A}$ we construct a weighted EMSO-sentence γ such that $\|\mathfrak{A}\| = [\![\gamma]\!]$. It then remains to prove that γ is restricted. We also need the notion of unambiguous formulas.

Definition 8.1. The class of *unambiguous* formulas in FO(K, Σ) is defined inductively as follows:

1. All atomic formulas of the form $P_a(x), xS_hy, xS_vy, x = y$ or $x \in X$, and their negations are unambiguous.

2. If φ, ψ are unambiguous, then $\varphi \wedge \psi, \forall x.\varphi$ are unambiguous.

3. If φ, ψ are unambiguous and $\mathrm{supp}([\![\varphi]\!]) \cap \mathrm{supp}([\![\psi]\!]) = \emptyset$, then $\varphi \vee \psi$ is unambiguous.

4. Let $\mathcal{V} = \mathrm{Free}(\varphi)$. If φ is unambiguous and for any $(p, \sigma) \in \Sigma_{\mathcal{V}}^{++}$ there is at most one element $(i, j) \in \mathrm{Dom}(p)$ such that $[\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(p, \sigma[x \to (i, j)]) \neq 0$, then $\exists x.\varphi$ is unambiguous.

By qf-MSO$^-(K, \Sigma)$, we denote formulas in MSO(K, Σ) having no quantification and no subformula of the form k.

To make such formulas unambiguous we perform a syntactic transformation $(.)^+$ to them. For the construction, we define transformations $(.)^+$ and $(.)^-$ in a simultaneous induction, as follows: for $\varphi, \psi \in$ qf-MSO$^-(K, \Sigma)$,

1. if φ is atomic or the negation of an atomic formula, then $\varphi^+ = \varphi$ and $\varphi^- = \neg\varphi$ (declaring $\neg\neg\varphi$ as φ),

2. $(\varphi \vee \psi)^+ = \varphi^+ \vee (\varphi^- \wedge \psi^+)$ and $(\varphi \vee \psi)^- = \varphi^- \wedge \psi^-$,

3. $(\varphi \wedge \psi)^- = \varphi^- \vee (\varphi^+ \wedge \psi^-)$ and $(\varphi \wedge \psi)^+ = \varphi^+ \wedge \psi^+$.

Then, obviously, we have $\mathcal{L}(\varphi^+) = \mathcal{L}(\varphi)$ and $\mathcal{L}(\varphi^-) = \Sigma^{++}_{\mathrm{Free}(\varphi)} \setminus \mathcal{L}(\varphi)$. Now, very similar to the proof done in the word case [7, Prop. 5.1], we get:

Lemma 8.2. *Let $\varphi \in \mathrm{FO}(K,\Sigma)$ be unambiguous. Then $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ is a first-order step function. For $\psi \in$ qf-MSO$^-(K,\Sigma)$, the formula ψ^+ is unambiguous.*

Proof. The first assertion is an induction on the structure of φ. Since $\mathcal{L}(\varphi)$ is the language defined by a first-order formula, the constructed semantics are first-order step functions. The transformation $+$ of 1. till 3. above preserves the defined languages and unambiguity of formulas. $\square$

We will use the following abreviations. We set $\min_h(y) := \forall x.\neg(xS_h y)$; $\max_h(y) := \forall x.\neg(yS_h x)$ and $\min_v(y) := \forall x.\neg(xS_v y)$; $\max_v(y) := \forall x.\neg(yS_v x)$. For atomic formulas φ, let $(\varphi \to \psi) := \neg\varphi \vee (\varphi \wedge \psi)$.

Then, the formulas $\min_h(y), \max_h(y), \min_v(y), \max_v(y)$ are restricted, since the assigned semantics are characteristic series with FO-definable supports. Likewise, we have that $[\![((x \in X) \to k)]\!]_{\{x,X\}} = k \cdot \mathbb{1}_{\mathcal{L}((x\in X))} + 1 \cdot \mathbb{1}_{\mathcal{L}(\neg(x \in X))}$ is an FO step function. For set variables $X_1, \ldots, X_l$, we put

$$\mathrm{part}(X_1,\ldots,X_l) := \forall x.\Big(\bigvee_{i=1,\ldots,l} \big[x \in X_i \wedge \bigwedge_{j\neq i} \neg(x \in X_j)\big]\Big),$$

$$\mathrm{pole}_{\mathrm{W}} := \forall x.\big(\big[\min_h(x) \wedge (\bigvee_{q_2^x,q_3^x,q_4^x\in Q,\, q_1^x \in F_w,\, a\in\Sigma} x \in X_{(a,q_1^x,q_2^x,q_3^x,q_4^x)})^+\big] \vee \exists s.(sS_h x)\big),$$

$$\mathrm{pole}_{\mathrm{N}} := \forall x.\big(\big[\min_v(x) \wedge (\bigvee_{q_1^x,q_2^x,q_3^x\in Q,\, q_4^x \in F_n,\, a\in\Sigma} x \in X_{(a,q_1^x,q_2^x,q_3^x,q_4^x)})^+\big] \vee \exists s.(sS_v x)\big),$$

$$\mathrm{pole}_{\mathrm{E}} := \forall x.\big(\big[\max_h(x) \wedge (\bigvee_{q_1^x,q_2^x,q_4^x\in Q,\, q_3^x \in F_e,\, a\in\Sigma} x \in X_{(a,q_1^x,q_2^x,q_3^x,q_4^x)})^+\big] \vee \exists s.(xS_h s)\big),$$

$$\mathrm{pole}_{\mathrm{S}} := \forall x.\big(\big[\max_v(x) \wedge (\bigvee_{q_1^x,q_3^x,q_4^x\in Q,\, q_2^x \in F_s,\, a\in\Sigma} x \in X_{(a,q_1^x,q_2^x,q_3^x,q_4^x)})^+\big] \vee \exists s.(xS_v s)\big).$$

Then, with transformations above and Lemma 8.2, the defined formulas are restricted. For intuition, the formulas $\mathrm{pole}_{\mathrm{W}}, \mathrm{pole}_{\mathrm{S}}, \mathrm{pole}_{\mathrm{E}}$ and $\mathrm{pole}_{\mathrm{N}}$ simulate accepting conditions of the automaton at the respective pole of acceptance.

Theorem 8.3. *We have* $\quad K^{\mathrm{rec}}\langle\!\langle \Sigma^{++}, WPA \rangle\!\rangle \subseteq K^{\mathrm{remso}}\langle\!\langle \Sigma^{++} \rangle\!\rangle$.

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_n, F_e, F_s)$ be a WPA. For $a \in \Sigma, q_1, q_2, q_3, q_4 \in Q$, we set

$$\mu_{(q_1,q_2,q_3,q_4)}(a) = \sum_{(a,k,q_1,q_2,q_3,q_4)\in R} k. \tag{2}$$

Let $\mathcal{V} = \big\{X_{(a,q_1,q_2,q_3,q_4)} \mid (a,q_1,q_2,q_3,q_4) \in \Sigma \times Q^4\big\}$ the set of set variables, $(X_1,\ldots,X_l)$ an enumeration of $\mathcal{V}$. We set

$$\begin{aligned}\alpha\,(X_1,\dots,X_l) := \operatorname{part}(X_1,\dots,X_l) \wedge \bigwedge_{a,q_1,q_2,q_3,q_4} &\forall x.\Big((x \in X_{(a,q_1,q_2,q_3,q_4)}) \to P_a(x)\Big)\\ \wedge \quad \forall x \forall z.\big((x S_v z) \to \bigvee_{q_1^x,\dots,q_4^x,q_2^z,q_3^z,q_4^z \in Q; a,b\in\Sigma} &(x \in X_{(a,q_1^x,q_2^x,q_3^x,q_4^x)}) \wedge (z \in X_{(b,q_3^x,q_2^z,q_3^z,q_4^z)})\Big)^+\\ \wedge \quad \forall y \forall z.\Big((y S_h z) \to \bigvee_{q_1^y,\dots,q_4^y,q_1^z,q_2^z,q_3^z \in Q; c,b\in\Sigma} &(y \in X_{(c,q_1^y,q_2^y,q_3^y,q_4^y)}) \wedge (z \in X_{(b,q_1^z,q_2^z,q_3^z,q_2^y)})\Big)^+.\end{aligned}$$

Intuitively, the formula α qualifies unweighted runs in the automaton $\mathfrak{A}$ as pictures of rules in R. The first line simply models that for every pixel the automaton applies exactly one rule reading that pixel, the two other lines compose the condition concerning the inner rules within a run of a picture automaton, i.e. describe that rules vertically above (resp. horizontally next to) each other have to fit together.

Let $p \in \Sigma^{++}$. There is a bijection between the set of runs (where all weights are replaced by 1) in $\mathfrak{A}$, and the set of $(p,\mathcal{V})$-assignments σ satisfying α, that is $[\![\alpha]\!](p,\sigma) = 1$. Indeed, let $p \in \Sigma^{m\times n}$ and $r \in R^{m\times n}$ be a computation over p in $\mathfrak{A}$. Let $\bar{r}$ be the picture over $\Sigma \times \{1\} \times Q^4$ where all weights are replaced by 1. We set σ_r the $(p,\mathcal{V})$-assignments such that $\sigma_r(X_{(a,q_1,q_2,q_3,q_4)}) := \{(i,j) \mid r(i,j) = (a,1,q_1,q_2,q_3,q_4)\}$, then $[\![\alpha]\!](p,\sigma_r) = 1$. On the other hand, let σ be a $(p,\mathcal{V})$-assignment satisfying α and $(p_{i,j}) = p \in \Sigma^{m\times n}$. For every $(i,j) \in \operatorname{Dom}(p)$ there are unique $q_1,q_2,q_3,q_4 \in Q$ with $(i,j) \in \sigma(X_{(p(i,j),q_1,q_2,q_3,q_4)})$. These rules identify their respective eastern-western and southern-northern states, therefore there is a unique computation c over p (where all weights are replaced by 1), it follows $\sigma_c = \sigma$.

Now, let

$$\begin{aligned}\beta(X_1,\dots,X_l) \quad := \quad & \alpha \wedge \bigwedge_{a,q_1,q_2,q_3,q_4} \forall x.\Big(x \in X_{(a,q_1,q_2,q_3,q_4)}) \to \mu_{(q_1,q_2,q_3,q_4)}(a)\Big)\\ & \wedge \quad \mathrm{pole_W} \ \wedge\ \mathrm{pole_N} \ \wedge\ \mathrm{pole_E} \ \wedge\ \mathrm{pole_S}\,.\end{aligned}$$

The formula β simulates the distribution of weights along the rules. Here, using (2), rules between identical four states reading the same letter are understood as one rule with the respective weight of the sum. Such a transformation leaves the behavior of $\mathfrak{A}$ unchanged. Furthermore β pictures that successful runs have to fulfill initial conditions, i.e. the western state of rules in the first column, the northern state of rules in the first line, the eastern state of rules in the last column, the southern state of rules in the last line are in their respective pole of acceptance.

For $\bar{r}$ with $r \in E^{m\times n}$, let $\sigma_{\bar{r}}$ be the associated $(p,\mathcal{V})$-assignment (described above). We conclude

$$[\![\beta]\!]_{\mathcal{V}}(p,\sigma_{\bar{r}}) = \left(\prod_{a,q_1,q_2,q_3,q_4} \mu(a)_{(q_1,q_2,q_3,q_4)}^{|\sigma_r(X_{(a,q_1,q_2,q_3,q_4)})|}\right) = \sum\{\mathrm{weight}(c) \mid \bar{c} = \bar{r}\}.$$

Finally, we define

$$\gamma := \exists X_1 \cdots \exists X_l.\beta(X_1,\dots,X_l).$$

For $p \in \Sigma^{++}$, we compute

$$\begin{aligned} [\![\gamma]\!](p) &= \sum_{\sigma\ (p,\mathcal{V})\text{-assignment}} [\![\beta]\!]_{\mathcal{V}}(p,\sigma) = \sum_{\bar{r}\colon\ r \text{ run in } \mathfrak{A} \text{ for } p} [\![\beta]\!]_{\mathcal{V}}(p,\sigma_{\bar{r}}) \\ &= \sum_{\bar{r}\colon\ r \text{ run in } \mathfrak{A} \text{ for } p} \ \sum_{c\,:\,\bar{c}=\bar{r}} \operatorname{weight}(c) = (\|\mathfrak{A}\|, p). \end{aligned}$$

Hence, we have $\|\mathfrak{A}\| = [\![\gamma]\!]$. Furthermore, using Lemma 8.2 and remarks above, it follows that the specified formula γ lies in $\mathrm{REMSO}(K,\Sigma)$. □

Now, clearly, we get

Proof of Theorem 4.5. Immediate by Theorems 6.4,7.5 and 8.3. □

With the circular argument given in the last three sections, we proved the coincidence of the family of picture series defined by W2OTA and WPA. We call elements of this class *recognizable* and collect them in $K^{\mathrm{rec}}\langle\!\langle\Sigma^{++}\rangle\!\rangle$.

In the language case of the Boolean semiring, all existential MSO formulas are restricted. If we denote the family of picture series $S \in \mathbb{B}\langle\!\langle\Sigma^{++}\rangle\!\rangle$ which are definable by some EMSO sentence by $\mathbb{B}^{\mathrm{emso}}\langle\!\langle\Sigma^{++}\rangle\!\rangle$, then we have the following consequence of Theorem 4.5;

Corollary 8.4. $\mathbb{B}^{\mathrm{rec}}\langle\!\langle\Sigma^{++}\rangle\!\rangle = \mathbb{B}^{\mathrm{emso}}\langle\!\langle\Sigma^{++}\rangle\!\rangle$.

Proof. It remains to prove direction from right to left. As noted before, we can understand a given weighted formula as a classical formula by replacing 0 and 1 by their equivalents. By induction on the structure of the formulas, for every weighted formula $\varphi \in \mathrm{MSO}(\mathbb{B},\Sigma)$, we have $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)} \in \mathbb{B}\langle\!\langle\Sigma^{++}\rangle\!\rangle$. Therefore, the semantics of a weighted FO-formula clearly is a FO step function, understanding φ as formula for $\operatorname{supp}([\![\varphi]\!])$. □

By this corollary, the classical result in Theorem 2.2 follows now by the fact that a language $L \subseteq \Sigma^{++}$ is recognizable if and only if its characteristic series $\mathbb{1}_L \in \mathbb{B}\langle\!\langle\Sigma^{++}\rangle\!\rangle$ is recognizable [4].

9 Decidability

We now analyse some properties that the family of recognizable picture series does not share with the corresponding family in one dimension. We start by considering the decidability of the restriction of a weighted MSO formula.

In the one-dimensional case of series on words, if the considered semiring is a computable field, it is decidable whether a given weighted MSO formula φ is restricted and in this case one can effectively compute a weighted finite automaton for $[\![\varphi]\!]$ [7]. For the corresponding problem for pictures we have the following. Let Σ be an alphabet.

Proposition 9.1. *Let K be any commutative semiring and let $\varphi \in \mathrm{MSO}(K,\Sigma)$. It is undecidable whether φ is restricted.*

We will prove this proposition by simulating the proof for the fact that the emptiness problem for recognizable picture languages is undecidable [14]. In particular we will use a reduction from the "Post's Correspondence Problem" and simulate the idea in the unweighted case for proving that the language of squares is not FO-definable. We recall the definition of Post's Correspondence Problem (PCP). One instance of PCP of size s is a finite set of pairs of nonempty strings (u_i, v_i) for $i = 1, \dots, s$ $(s \geq 1)$ over Σ. A solution of length n to this instance is a sequence $(i_1, i_2, \dots, i_n)$ $(1 \leq i_j \leq s)$ of indices such that the strings $u_{i_1} u_{i_2} \cdots u_{i_n}$ and $v_{i_1} v_{i_2} \cdots v_{i_n}$ formed by concatenation are identical. It is undecidable whether for a given instance of a PCP, there exists a solution. For picture languages we have;

Lemma 9.2. *It is undecidable whether a given unambiguous tiling system computes an FO-definable language.*

Proof. To every instance I with size s of the PCP we will assign a particular unambiguous tiling recognizable picture language L_I over the alphabet $\Gamma := (\Sigma \times \{0, \dots, s\}) \cup \{0\}$ in such a way that $L_I \in \mathrm{FO}(\Gamma^{++})$ if and only if there is no solution for I. We will do a reduction similar to [14, Theorem 6.1], however we have to construct unambiguous representations. Let $I = \{(u_i, v_i) \mid 1 \leq i \leq s\}$ be an instance of a PCP on Σ. We mark the given words in I as follows. For $1 \leq i \leq s$ we set

$$u'_i := (u_i, 0^{|u_i|-1} \circ i),\ (v_i, 0^{|v_i|-1} \circ i),$$

and denote by U' and V' the sets of all such u'_i and v'_i, respectively. For any sequence $F = (i_1, i_2, \dots, i_n)$, $(1 \leq i_j \leq s)$, of indices, let

$$B_{I,F} = \begin{array}{|c|c|}\hline 0 & v'_{i_1} v'_{i_2} \quad \cdots \quad v'_{i_n} \\ \hline u'_{i_1} & \\ u'_{i_2} & \\ & 0 \\ \vdots & \\ u'_{i_n} & \\ \hline \end{array}$$

the picture over Γ such that the first column is the word $(0 \circ u'_{i_1} u'_{i_2} \cdots u'_{i_n})$, the first line is the word $(0 \circ v'_{i_1} v'_{i_2} \cdots v'_{i_n})$ and all other positions in $B_{I,F}$ have symbol 0. Here, the defined markings specify the end and the position of the corresponding word in the sequence F. Then the set

$$S_I = \big\{ B_{I,F} \mid F \text{ is a sequence of elements in } \{1, \dots, s\} \big\} \subseteq \Gamma^{++},$$

comprising all such $B_{I,F}$ is unambiguous tiling recognizable; indeed, we could define the corresponding local set similar to the one in [14, Proof of Theorem 6.1], but now the specified markings of words in the sequence F ensure unambiguity. We define the language $L_I \subseteq \Gamma^{++}$ as follows:

$$L_I = \{ B_{I,F} \mid u_F = v_F \},$$

where $u_F = u_{i_1} u_{i_2} \cdots u_{i_n}$ and $v_F = v_{i_1} v_{i_2} \cdots v_{i_n}$ if $F = (i_1, i_2, \dots, i_n)$, $(1 \leq i_j \leq s)$. We prove $L_I \in \mathrm{UPLoc}(\Gamma^{++})$. In Example 5 of [14, p. 406] the authors prove that for every recognizable string language W and a fresh symbol 0 not in the underlying alphabet of W, the two-dimensional language A_W whose pictures are squares such that the word in the first row equals the one in the first column and belongs to W while all other positions are filled with letter 0, is recognizable. By following their proof, one even can show that the constructed tiling system is unambiguous. Clearly, a similar construction does also work by letting words of the first row and first column be elements of the recognizable string language $V'^{+} \cup U'^{+}$ and by considering for the equality property only the corresponding first component of the words. Hence, by defining a projection $\pi : \Gamma \to \Sigma \cup \{0\}$, where $(\sigma, i) \in \Sigma \times \{0, \ldots, s\}$ is mapped to σ and $0 \mapsto 0$, the language $L_{U' \cup V'} \subseteq \Gamma^{++}$, defined as

$$L_{U' \cup V'} = \{p \in \Gamma^{n \times n} \mid \pi(p_{1,1} \cdots p_{n,1}) = \pi(p_{1,1} \cdots p_{n,1});\ (p_{1,1} \cdots p_{n,1}) \in U' \cup V'\}$$

is an unambiguous picture language. We conclude that

$$L_I = S_I \cap L_{U' \cup V'}$$

is unambiguous [26].

With the definition of L_I, we have the relation that

$$I \text{ has a solution} \iff L_I \neq \emptyset. \tag{3}$$

Furthermore, we claim that, if $L_I \neq \emptyset$ then $L_I \notin \mathrm{FO}(\Gamma^{++})$. Suppose, the PCP I has a solution and $L_I \in \mathrm{FO}(\Gamma^{++})$. Using Remark 2.1 and Proposition 5.2, we conclude that L_I is locally threshold testable for some $d, t \geq 1$. Now, since with every solution of I there are arbitrary long solutions for I, let

$$Q = \begin{array}{|c|c|}\hline 0 & w_2 \\ \hline w_1 & 0 \\ \hline \end{array}$$

be the assigned picture L_I for a solution of I satisfying $|w_2| > d$. Now, multiplying the solution $(t+1)$ times, we clearly get again a solution. We consider the following two pictures;

$$Q_{t+1} := \begin{array}{|c|c|}\hline 0 & w_2^{t+1} \\ \hline w_1^{t+1} & 0 \\ \hline \end{array} \quad \text{and} \quad Q' := \begin{array}{|c|c|c|}\hline 0 & w_2^{t+1} & w_2^{t+1} \\ \hline w_1^{t+1} & 0 & 0 \\ \hline \end{array}.$$

Then Q_{t+1} is an element of L_I lying in the same (d, t)-block-threshold-equivalence class than Q'. But obviuously, $Q' \notin L_I$ which is a contradiction. With (3), we conclude that:

$$I \text{ has a solution} \iff L_I \notin \mathrm{FO}(\Gamma^{++}). \tag{4}$$

The stated constructions are effective, hence the problem whether a given unambiguous tiling system computes an FO-definable language is undecidable. □

Now, using the constructions of the previous lemma, the proof of Proposition 9.1 is quite simple.

Proof of Proposition 9.1. To every PCP I on Σ we can effectively assign an unambiguous tiling recognizable picture language L_I over an extented alphabet such that

$$I \text{ has a solution} \iff L_I \neq \emptyset \iff L_I \notin \mathrm{FO}(\Gamma^{++}). \tag{5}$$

With Lemma 5.1, there exists an unambiguous 2OTA $\mathfrak{A}_I$ recognizing L_I and therefore, $\mathbb{1}_{L_I} \in K^{\mathrm{rec}}\langle\langle \Gamma^{++}, W2OTA\rangle\rangle$. Immediately, we get that

$$\mathbb{1}_{L_I} \text{ is an FO step function} \iff L_I = \emptyset \iff I \text{ has no solution.} \tag{6}$$

Using Theorems 7.5 and 8.3, for $\mathbb{1}_{L_I}$ we can effectively construct a REMSO(K, Γ) formula φ_I with $[\![\varphi_I]\!] = \mathbb{1}_{L_I}$ and also, from every φ_I we can reconstruct $\mathfrak{A}$. Hence, the asserted claim follows. □

Now, applying relation (6) and the last remark of the proof of Proposition 9.1, concerning the support of picture series, we get the following undecidability results.

Corollary 9.3. *Let K be a commutative semiring and Σ an alphabet. It is undecidable*

1. *whether a given* MSO(K, Σ)*-formula ϕ satisfies supp*$([\![\varphi]\!]) = \emptyset$*;*
2. *whether two given* MSO(K, Σ)*-formulas ϕ and ψ satisfy* $[\![\varphi]\!] = [\![\psi]\!]$*;*
3. *whether a given W2OTA computes a picture series with empty support;*
4. *whether two given W2OTA are equivalent;*
5. *whether a given W2OTA computes a first-order step function.*

10 Conclusion

In [25, 26] we assigned weights to tiling systems, domino systems or weighted (quadrapolic) picture automata and proved for an alphabet Σ and any commutative semiring K the coincidence of the corresponding series with the projections of series defined by rational operations. In [26], we proved that this very class coincides with $K^{\mathrm{rec}}\langle\langle \Sigma^{++}\rangle\rangle$, the family of recognizable picture series. With Theorem 4.5, this implies that the notion of weighted recognizability presented here is robust and extends the main results of [4, 16] to the quantitative setting of picture series (Proposition 8.4).

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