The Complexity of Decomposing Modal and First-Order Theories

Stefan Göller
University of Bremen, Germany
goeller@informatik.uni-bremen.de

Jean Christoph Jung
University of Bremen, Germany
jeanjung@informatik.uni-bremen.de

Markus Lohrey
University of Leipzig, Germany
lohrey@informatik.uni-leipzig.de

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Abstract

We study the satisfiability problem of the logic $\mathbf{K}^2 = \mathbf{K} \times \mathbf{K}$, i.e., the two-dimensional variant of unimodal logic, where models are restricted to asynchronous products of two Kripke frames. Gabbay and Shetman proved in 1998 that this problem is decidable in a tower of exponentials. So far the best known lower bound is NEXP-hardness shown by Marx and Mikulás in 2001.

Our first main result closes this complexity gap: We show that satisfiability in $\mathbf{K}^2$ is nonelementary. More precisely, we prove that it is k-NEXP-complete, where k is the switching depth (the minimal modal rank among the two dimensions) of the input formula, hereby solving a conjecture of Marx and Mikulás. Using our lower-bound technique allows us to derive also nonelementary lower bounds for the two-dimensional modal logics $\mathbf{K4} \times \mathbf{K}$ and $\mathbf{S5}_2 \times \mathbf{K}$ for which only elementary lower bounds were previously known.

Moreover, we apply our technique to prove nonelementary lower bounds for the sizes of Feferman-Vaught decompositions with respect to product for any decomposable logic that is at least as expressive as unimodal $\mathbf{K}$, generalizing a recent result by the first author and Lin. For the three-variable fragment FO^3 of first-order logic, we obtain the following immediate corollaries: (i) the size of Feferman-Vaught decompositions with respect to disjoint sum are inherently nonelementary and (ii) equivalent formulas in Gaifman normal form are inherently nonelementary.

Our second main result consists in providing effective elementary (more precisely, doubly exponential) upper bounds for the two-variable fragment FO^2 of first-order logic both for Feferman-Vaught decompositions and for equivalent formulas in Gaifman normal form.

1 Introduction

1.1 Modal logic and many-dimensional modal logic

Modal logic [1, 2] originated in philosophy and for a long time it was known as "the logic of necessity and possibility". Later, it has been discovered that modal logics are well-suited to talk about relational structures, so called *(Kripke) frames*. Relational structures appear in many branches of computer science; consider for example transition systems in verification, semantic networks in knowledge representation, or attribute value structures in linguistics. This has led to various applications of modal logic in areas such as computer science, mathematics, and artificial intelligence.

Depending on the application, a lot of different modal operators have been introduced in the past, each of them tailored towards expressing different features of the domain. For instance there are modalities that talk about time, space, knowledge, beliefs, etc.

However, it turned out that recent application domains require to express properties that *combine* different modalities, e.g., talk about the evolution of knowledge over time. In order to reflect these requirements in theory, *many-dimensional* modal logics have been studied intensively [3, 4]. A particular way of combining two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ is building their *product* $\mathcal{L}_1 \times \mathcal{L}_2$ [5]. For products, the semantics is given in terms of structures, whose frames are restricted to be asynchronous products of the (one-dimensional) component frames. The interpretation of the atomic propositions is done in an *uninterpreted* way, i.e., it is independent from the component frames.

An important and well-studied problem in this context is *satisfiability checking*, i.e., to decide whether a given formula admits a model. When considering products of modal logics, it has been shown that the computational complexity of satisfiability checking often increases drastically in comparison to the well-behaved component logics. As an example, consider the basic modal logic **K** and its variant **K4** for reasoning over the class of transitive frames. Satisfiability is PSPACE-complete for both **K** and **K4** [6], while for $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K4} \times \mathbf{K}$ only nonelementary upper bounds were known [5]. Even worse, satisfiability becomes undecidable in $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ [7] and $\mathbf{K4} \times \mathbf{K4}$ [8]. To some extent, this can be explained by the grid-like shape of product structures.

1.2 Logical decomposition

Logical decomposition can concisely be summarized as follows: A logic $\mathcal{L}$ admits decomposition w.r.t. some operation op on structures if all $\mathcal{L}$-properties that are interpreted on composed (with respect to the operation op) structures, are already determined by the $\mathcal{L}$-properties of the component structures. Logical decomposition dates back to the work of Mostowski [9] and Feferman and Vaught [10], where it is shown that first-order logic (FO) is decomposable w.r.t. a general product operation, which covers also disjoint union and product. Later, both for more expressive logics and for more sophisticated operations such decomposability results have been proven, see [11] for an excellent survey.

When proving decomposability for a logic $\mathcal{L}$, one often obtains an effective procedure for computing such decompositions: Given a formula φ from $\mathcal{L}$ evaluated on composed structures, one can effectively compute (i) a finite set of formulas $\{\varphi_1, \ldots, \varphi_n\}$, each being evaluated on some specific component, and (ii) a propositional formula β, whose propositions are tests of the form $\mathfrak{S}_i \models \varphi_j$, such that for all composed structures $\mathfrak{S} = \mathsf{op}(\mathfrak{S}_1, \ldots, \mathfrak{S}_k)$: $\mathfrak{S} \models \varphi$ if and only if β evaluates to true. The size of the resulting decomposition is typically nonelementary in the size of the original formula. Dawar et al. proved that this is unavoidable if $\mathcal{L} = \mathsf{FO}$ [12].

Decomposition theorems have powerful implications in computer science logic. Let us mention only four of them.

Firstly, assume some decomposable logic $\mathcal{L}$: Then *decidability of the $\mathcal{L}$-theory* of some composed structure, for instance a product structure, can be derived from the decidability of the $\mathcal{L}$-theories of its component structures.

Secondly, let us mention that *model checking* a fixed $\mathcal{L}$-formula (i.e. the *data complexity*) in a composed structure is not harder than model checking fixed $\mathcal{L}$-formulas on the component structures: If the formula is fixed, also the decomposition is fixed (although possibly large).

Moreover, decompositional methods can be applied for showing decidability of *satisfiability checking*: Instead of asking whether a given formula φ is satisfiable in a composed model, one computes a decomposition for φ, translates the decomposition into disjunctive normal form, and finally checks satisfiability of a conjunction of formulas in their corresponding components. Rabinovich proved that basic modal logic $\mathbf{K}$ is decomposable w.r.t. *interpreted* products [13], i.e. where "interpreted" means that an interpretation of the propositions on the respective component structures is applied. It is worth noting that this, however, does not lead to decidability of $\mathbf{K} \times \mathbf{K}$ w.r.t. the classical (uninterpreted) products mentioned above. To the contrary, satisfiability w.r.t. interpreted products is easily reducible to the uninterpreted version.

Finally, an important application of logical decomposition à la Feferman and Vaught is the (original) proof of *Gaifman's locality theorem* [14] stating that every first-order sentence is equivalent to a boolean combination of basic local sentences, where a basic local sentence admits quantification only relativized to finite neighbourhoods of elements. Gaifman's locality theorem has important applications such as inexpressibility results for first-order logic. For a further and more recent application of Gaifman's locality theorem we mention algorithmic meta-theorems for first-order logic [15], stating that first-order properties can be efficiently evaluated on numerous classes of structures.

1.3 Our contributions and related work

As our first main result we show that (even the interpreted variant of) the satisfiability problem of two-dimensional modal logic $\mathbf{K}^2 = \mathbf{K} \times \mathbf{K}$ has nonelementary complexity, hereby solving a fundamental problem that has been open for more than 10 years. Gabbay and Shetman proved in 1998 that satisfiability in $\mathbf{K}^2$ is decidable in a tower of exponentials [5]. To the best of the authors' knowledge, the best known lower bound has been NEXP-hardness shown by Marx and Mikulás in 2001 [16]. In fact, we prove that satisfiability in $\mathbf{K}^2$ restricted to formulas of switching depth k (the minimal modal rank among the two dimensions) is k-NEXP-complete (where k-NEXP is the set of all problems that can be solved on a nondeterministic Turing machine in k-fold exponential time), hereby confirming a conjecture of Marx and Mikulás [16]. We derive nonelementary lower bounds for the two-dimensional modal logics $\mathbf{K4} \times \mathbf{K}$ and $\mathbf{S5}_2 \times \mathbf{K}$ for which only elementary lower bounds were known [3].

Our lower bound technique allows us to derive a nonelementary lower bound for the size of Feferman-Vaught decompositions w.r.t. product for $\mathbf{K}$. Such a result was already shown in [17]. However, in contrast to [17], our proof technique implies that the nonelementary lower bound carries over to all decomposable logics that are at least as expressive as $\mathbf{K}$. An instance of such a logic is the two-variable fragment FO^2 of first-order logic. Moreover, we prove that the same lower bound holds when relativized to the class of finite trees, answering an open problem formulated in [17].

In the same fashion, we derive the following new results for the three-variable fragment FO^3 of first-order logic: (i) the sizes of Feferman-Vaught decompositions w.r.t. disjoint sum are inherently nonelementary and (ii) equivalent formulas in Gaifman normal form are inherently nonelementary. It is worth mentioning that (i) and (ii) were shown in [12] for full FO. By inspecting the formulas in [12] it turns out that they are in fact FO^4-formulas. However, it seems to be unclear whether the construction from [12] can be adapted so that it yields FO^3-formulas.

Finally, we provide effective doubly exponential (and hence elementary) upper bounds for the two-variable fragment FO^2 of first-order logic both for Feferman-Vaught decompositions and for equivalent formulas in Gaifman normal form. This supports former observations that in many aspects FO^2 is better behaved than FO^3. For instance, in contrast to FO^3 it has a finite model property and satisfiability is decidable in NEXP [18]. We also prove (non-matching) lower bounds of the form $c^{f(n)}$ (for any constant c and any function $f(n) \in o(\sqrt{n})$) for both Feferman-Vaught decompositions and equivalent formulas in Gaifman normal form for FO^2.

An extended abstract of this paper appeared as [19].

2 Preliminaries

For $i, j \in \mathbb{Z}$ let $[i, j]$ be the interval $[i, i+1, \ldots, j]$. By $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, \ldots\}$ we denote the non-negative integers. For a set X we denote by $\mathsf{bool}(X)$ the set of boolean formulas with variables ranging over X. Let $u = u_1 \cdots u_k \in \Sigma^*$ with $u_i \in \Sigma$ for each $i \in [1, k]$. By $|u| \stackrel{\text{def}}{=} k$ we denote the *length* of u. The *tower function* $\mathsf{Tower} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined as $\mathsf{Tower}(0, n) \stackrel{\text{def}}{=} n$ and $\mathsf{Tower}(\ell + 1, n) \stackrel{\text{def}}{=} 2^{\mathsf{Tower}(\ell, n)}$ for each $\ell, n \in \mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{N}$ is *elementary* if it can be formed from the successor function, addition, subtraction, and multiplication using compositions, projections, bounded additions and bounded multiplications (of the form $\sum_{z \le y} g(\overline{x}, z)$ and $\prod_{z \le y} g(\overline{x}, z)$). We will sometimes use the fact that for each elementary function $f : \mathbb{N} \to \mathbb{N}$ there exist some $h_0 \in \mathbb{N}$ such that $f(h) < \mathsf{Tower}(h, 2)$ for each $h \ge h_0$.

2.1 Kripke frames and structures

Let us fix a countable set of *action labels* $\mathbb{A}$ and a countable set of *propositional variables* $\mathbb{P}$. For a finite set $\mathsf{A} \subseteq \mathbb{A}$ of action labels, an A-*frame* is a tuple $\mathfrak{F} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\})$, where W is set of *worlds* and $\xrightarrow{a} \subseteq W \times W$ is a binary (accessibility) relation over W for each $a \in \mathsf{A}$. Most of the time we write $v \xrightarrow{a} w$ instead of $(v, w) \in \xrightarrow{a}$. We say that $\mathfrak{F}$ is a *tree* if

- $W \subseteq U^*$ is a prefix-closed set of words for some set U,
- $\xrightarrow{a} \cap \xrightarrow{b} = \emptyset$ for each $a, b \in \mathsf{A}$ with $a \neq b$, and
- for all $v, w \in W$, we have $v \xrightarrow{a} w$ for some $a \in A$ if and only if there exists $u \in U$ with $w = vu$.

We say that $\mathfrak{F}$ is *finite* if W is finite. In case $\mathfrak{F}$ is a finite tree, the *height* of $\mathfrak{F}$ is defined as $\max\{|w| \mid w \in W\}$.

An (A, P)-*Kripke structure* (or (A, P)-*structure* for short), for a finite set $\mathsf{A} \subseteq \mathbb{A}$ of action labels and a finite set $\mathsf{P} \subseteq \mathbb{P}$ of propositional variables, is a tuple $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$, where $(W, \{\xrightarrow{a} \mid a \in \mathsf{A}\})$ is an A-frame and $W_p \subseteq W$ is an interpretation for each propositional variable $p \in \mathsf{P}$. By $\mathfrak{F}(\mathfrak{S}) \stackrel{\text{def}}{=} (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\})$ we denote the underlying A-frame of $\mathfrak{S}$. We say that $\mathfrak{S}$ is a *tree structure* if $\mathfrak{F}(\mathfrak{S})$ is a tree. By $|\mathfrak{S}| = |W|$ we denote the *size* of $\mathfrak{S}$. For $s \in W$ let $N_{\mathfrak{S}}(s) \stackrel{\text{def}}{=} \{u \in W \mid \exists a \in \mathsf{A} : s \xrightarrow{a} u\}$ be the set of *successors* of s in $\mathfrak{S}$. A *pointed* (A,P)*-structure* is a pair $(\mathfrak{S}, s)$ where $\mathfrak{S}$ is an (A,P)-structure and s is a world of $\mathfrak{S}$. An $(\{a\},\mathsf{P})$-structure is also called *unimodal* and we write $(W, \xrightarrow{a}, \{W_p \mid p \in \mathsf{P}\})$ instead of $(W, \{\xrightarrow{a}\}, \{W_p \mid p \in \mathsf{P}\})$.

2.2 Multimodal logic

Formulas of *multimodal logic* are defined by the following grammar, where a (resp., p) ranges over $\mathbb{A}$ (resp., $\mathbb{P}$):

$$\varphi \quad ::= \quad p \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi \quad | \quad \Diamond_a \varphi$$

We introduce the usual abbreviations $\top = p \vee \neg p$ for some $p \in \mathbb{P}$, $\bot = \neg\top$, $\varphi_1 \vee \varphi_2 = \neg(\neg\varphi_1 \wedge \neg\varphi_2)$, and $\Box_a \varphi = \neg \Diamond_a \neg\varphi$. We say that φ is *over* (A,P) if the set of action labels (resp. the set of propositional variables) that appear in φ is a subset of A (resp. P). For an (A,P)-structure $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$, $w \in W$, and a multimodal logic formula φ over (A,P), we define the satisfaction relation $(\mathfrak{S}, w) \models \varphi$ by structural induction on φ, where $a \in \mathsf{A}$ and $p \in \mathsf{P}$:

$$\begin{aligned}
(\mathfrak{S}, w) \models p &\overset{\text{def}}{\Leftrightarrow} w \in W_p \\
(\mathfrak{S}, w) \models \neg\varphi &\overset{\text{def}}{\Leftrightarrow} (\mathfrak{S}, w) \not\models \varphi \\
(\mathfrak{S}, w) \models \varphi_1 \wedge \varphi_2 &\overset{\text{def}}{\Leftrightarrow} (\mathfrak{S}, w) \models \varphi_1 \text{ and } (\mathfrak{S}, w) \models \varphi_2 \\
(\mathfrak{S}, w) \models \Diamond_a \varphi &\overset{\text{def}}{\Leftrightarrow} \exists w' : w \xrightarrow{a} w' \text{ and } (\mathfrak{S}, w') \models \varphi
\end{aligned}$$

Let φ be a multimodal logic formula over (A,P). An (A,P)-structure $\mathfrak{S}$ is a *model* of φ if $(\mathfrak{S}, w) \models \varphi$ for some world w of $\mathfrak{S}$. We say that φ is *satisfiable* if φ has a model.

If φ is a boolean formula with propositional variables from $\mathsf{P} \subseteq \mathbb{P}$ and $\alpha : \mathsf{P} \to \{0,1\}$ then we write $\alpha \models \varphi$ if φ evaluates to 1 if every propositional variable $p \in \mathsf{P}$ is replaced by the truth value $\alpha(p)$.

2.3 Asynchronous products and many-dimensional modal logic

Fix non-empty, finite, and pairwise disjoint sets $\mathsf{A}_1, \ldots, \mathsf{A}_d \subseteq \mathbb{A}$ of action labels and non-empty, finite, and pairwise disjoint sets $\mathsf{P}_1, \ldots, \mathsf{P}_d \subseteq \mathbb{P}$ of propositional variables. Let $\mathsf{A} = \bigcup_{i \in [1,d]} \mathsf{A}_i$ and $\mathsf{P} = \bigcup_{i \in [1,d]} \mathsf{P}_i$. For A_i-frames $\mathfrak{F}_i = (W_i, \{\xrightarrow{a}_i \mid a \in \mathsf{A}_i\})$ $(i \in [1,d])$ we define the *asynchronous product* $\prod_{i \in [1,d]} \mathfrak{F}_i \overset{\text{def}}{=} (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\})$ to be the A-frame, where

- $W = \prod_{i \in [1,d]} W_i$, and
- for each $\overline{v} = \langle v_1, \ldots, v_d \rangle \in W$ and $\overline{w} = \langle w_1, \ldots, w_d \rangle \in W$ we have $\overline{v} \xrightarrow{a} \overline{w}$ if and only if there is some $i \in [1,d]$ such that $a \in \mathsf{A}_i$, $v_i \xrightarrow{a}_i w_i$ and $v_j = w_j$ for each $j \in [1,d] \setminus \{i\}$.

An (A,P)-structure $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$ is an *uninterpreted product structure* if $\mathfrak{F}(\mathfrak{S}) = \prod_{i \in [1,d]} \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is some A_i-frame. Thus, we do not make any restrictions on how atomic propositions are interpreted.

Next, let us define interpretations of atomic propositions in products, as introduced in [13]. A *(product) interpretation* is a mapping $\sigma : \mathsf{P} \to \mathsf{bool}(\mathsf{P})$. In our lower bound proofs in Section 3, σ will be the *identity interpretation* id with $\mathsf{id}(p) = p$ for all $p \in \mathsf{P}$. Let

$$\mathfrak{S}_i = (W_i, \{\xrightarrow{a}_i \mid a \in \mathsf{A}_i\}, \{W_{p,i} \mid p \in \mathsf{P}_i\})$$

be an $(\mathsf{A}_i, \mathsf{P}_i)$-structure for $i \in [1,d]$. For an interpretation σ, their *σ-product* $\prod^{\sigma}_{i\in[1,d]} \mathfrak{S}_i$ is defined as the (A,P)-structure $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$ such that

- $\mathfrak{F}(\mathfrak{S}) = \prod_{i\in[1,d]} \mathfrak{F}(\mathfrak{S}_i)$ and
- $\langle w_1, \ldots, w_d \rangle \in W_p$ if and only if $\alpha \vDash \sigma(p)$, where $\alpha(q) = 1$ if and only if $w_i \in W_{q,i}$ for each $i \in [1,d]$ and $q \in \mathsf{P}_i$.

If no interpretation is given, we define $\prod_{i\in[1,d]} \mathfrak{S}_i \stackrel{\text{def}}{=} \prod^{\mathsf{id}}_{i\in[1,d]} \mathfrak{S}_i$.

Let us generalize multimodal logic to higher dimensions. A *multimodal formula of dimension $d \geq 1$* (briefly, a *multimodal $\mathbf{K}^d$-formula*) is a formula φ over (A,P) with $\mathsf{A} = \biguplus_{i\in[1,d]} \mathsf{A}_i$ and $\mathsf{P} = \biguplus_{i\in[1,d]} \mathsf{P}_i$ as above. If $|\mathsf{A}_i| = 1$ for all $i \in [1,d]$ then φ is a *unimodal formula of dimension $d \geq 1$* (briefly, a *unimodal $\mathbf{K}^d$-formula*). For a multimodal $\mathbf{K}^d$-formula φ and $i \in [1,d]$ we define $\mathsf{rank}_i(\varphi)$ inductively: $\mathsf{rank}_i(p) \stackrel{\text{def}}{=} 0$ for $p \in \mathsf{P}$, $\mathsf{rank}_i(\neg\varphi) \stackrel{\text{def}}{=} \mathsf{rank}_i(\varphi)$, $\mathsf{rank}_i(\varphi_1 \wedge \varphi_2) \stackrel{\text{def}}{=} \max\{\mathsf{rank}_i(\varphi_1), \mathsf{rank}_i(\varphi_2)\}$, $\mathsf{rank}_i(\Diamond_a\varphi) \stackrel{\text{def}}{=} \mathsf{rank}_i(\varphi)$ for $a \in \mathsf{A} \setminus \mathsf{A}_i$, and $\mathsf{rank}_i(\Diamond_a\varphi) \stackrel{\text{def}}{=} \mathsf{rank}_i(\varphi) + 1$ for $a \in \mathsf{A}_i$. Finally, we define the *switching depth* of φ as $\min\{\mathsf{rank}_i(\varphi) \mid 1 \leq i \leq d\}$ [16]. An *uninterpreted product model* of φ is an uninterpreted product structure $\mathfrak{S}$ (in the above sense) such that for some world $\overline{w}$ of $\mathfrak{S}$ we have $(\mathfrak{S}, \overline{w}) \vDash \varphi$. For an interpretation σ, a *σ-model* is a σ-product structure $\mathfrak{S}$ such that $(\mathfrak{S}, \overline{w}) \vDash \varphi$ for some world $\overline{w}$ of $\mathfrak{S}$. We say φ is *uninterpreted satisfiable* (resp., *σ-satisfiable*) if φ has an uninterpreted (resp., σ-) product model. Let us introduce the following decision problems for multimodal (unimodal) $\mathbf{K}^d$:

MULTIMODAL (RESP. UNIMODAL) $\mathbf{K}^d$-SAT

INPUT: A multimodal (resp. unimodal) $\mathbf{K}^d$-formula φ.
QUESTION: Is φ uninterpreted satisfiable?

We introduce the corresponding variant in the presence of an interpretation σ of the atomic propositions.

MULTIMODAL (RESP. UNIMODAL) $\mathbf{K}^d_\sigma$-SAT

INPUT: A multimodal (resp. unimodal) $\mathbf{K}^d$-formula φ.
QUESTION: Is φ σ-satisfiable?

Since we mainly deal with the unimodal case, we use $\mathbf{K}^d_\sigma$-SAT as an abbreviation for UNIMODAL $\mathbf{K}^d_\sigma$-SAT. The following proposition is not hard to prove, but will be technically useful in Sections 3 and 4. It can be shown for an arbitrary interpretation σ, but we will only need the case $\sigma = \mathsf{id}$.

Proposition 1. *There is a polynomial time many-one reduction from* MULTIMODAL $\mathbf{K}^d_{\mathsf{id}}$-SAT *to* MULTIMODAL $\mathbf{K}^d$-SAT, *which preserves the switching depth.*

Proof. Let $\mathsf{A} = \biguplus_{i\in[1,d]} \mathsf{A}_i$ be the set of action labels and let $\mathsf{P} = \biguplus_{i\in[1,d]} \mathsf{P}_i$ be the set of atomic propositions, and let φ be some formula over (A,P). The idea is to give a formula χ that admits only models which are id-product structures, in particular, φ is id-satisfiable if and only if $\varphi \wedge \chi$ is satisfiable.

We need the following definition: The set of *modal sequences* $\mathsf{ms}(\psi) \subseteq \mathsf{A}^*$ of a formula ψ is inductively defined as follows:

- $\mathsf{ms}(p) \stackrel{\text{def}}{=} \{\varepsilon\}$ for each $p \in \mathsf{P}$,

- $\mathsf{ms}(\neg\psi) \stackrel{\text{def}}{=} \mathsf{ms}(\psi)$,
- $\mathsf{ms}(\psi_1 \wedge \psi_2) \stackrel{\text{def}}{=} \mathsf{ms}(\psi_1) \cup \mathsf{ms}(\psi_2)$,
- $\mathsf{ms}(\Diamond_a \psi) \stackrel{\text{def}}{=} (\{a\}\mathsf{ms}(\psi)) \cup \{\varepsilon\}$.

We note that $\mathsf{ms}(\psi)$ is prefix closed, that $|\mathsf{ms}(\varphi)| \leq |\varphi|$, and that the maximal length of an element of $\mathsf{ms}(\varphi)$ is the modal rank of φ. If $w = a_1 \cdots a_n \in \mathsf{A}^*$ we denote with $\Box_w$ the sequence of boxes $\Box_{a_1} \cdots \Box_{a_n}$. Particularly, $\Box_\varepsilon$ is the empty sequence of boxes. Moreover, define the relation $\xrightarrow{w} \stackrel{\text{def}}{=} \xrightarrow{a_1} \circ \xrightarrow{a_2} \circ \cdots \circ \xrightarrow{a_n}$.

For $i \in [1,d]$ and a word $s \in \mathsf{A}^*$ let $s \setminus i \in (\mathsf{A} \setminus \mathsf{A}_i)^*$ be the word that results from s by removing all occurrences of all symbols from A_i and let $s \restriction i \in \mathsf{A}_i^*$ be the word that results from s by removing all occurrences of all symbols from $\mathsf{A} \setminus \mathsf{A}_i$. We define the following formula χ:

$$\chi \stackrel{\text{def}}{=} \bigwedge_{i \in [1,d]} \bigwedge_{s \in \mathsf{ms}(\varphi)} \bigwedge_{p \in \mathsf{P}_i} \Box_{s \restriction i}((p \to \Box_{s \setminus i}\, p) \wedge (\Diamond_{s \setminus i}\, p \to p))$$

We define $\varphi' \stackrel{\text{def}}{=} \varphi \wedge \chi$.

Note that φ' has the same switching depth as φ and can be constructed in polynomial time from φ. Therefore it suffices to show that φ is id-satisfiable iff φ' is uninterpreted satisfiable.

Since every id-product satisfies χ, it follows that φ' is uninterpreted satisfiable if φ is id-satisfiable. For the other direction let $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$ be an (A, P)-structure such that $\mathfrak{F}(\mathfrak{S}) = \Pi_{i \in [1,d]} \mathfrak{F}_i$ where $\mathfrak{F}_i = (W_i, \{\xrightarrow{a} \mid a \in \mathsf{A}_i\})$ is an A_i-frame for each $i \in [1,d]$ and assume that $(\mathfrak{S}, \overline{w}_0) \models \varphi \wedge \chi$ for some $\overline{w}_0 \in W$. In particular, we have $W = \prod_{i \in [1,d]} W_i$. Let

$$W_R = \{\overline{w} \in W \mid \exists s \in \mathsf{ms}(\varphi) : \overline{w}_0 \xrightarrow{s} \overline{w}\}.$$

We define for each $i \in [1,d]$ an $(\mathsf{A}_i, \mathsf{P}_i)$-structure $\mathfrak{S}_i = (W_i, \{\xrightarrow{a} \mid a \in A_i\}, \{V_p \mid p \in P_i\})$ with $\mathfrak{F}(\mathfrak{S}_i) = \mathfrak{F}_i$ such that $(\prod^{\mathsf{id}}_{i \in [1,d]} \mathfrak{S}_i, \overline{w}_0) \models \varphi$. For giving the interpretations V_p we need the following statement:

Claim 1. For all $\overline{v}, \overline{w} \in W_R$, $i \in [1,d]$, and $p \in \mathsf{P}_i$, if $\overline{v}(i) = \overline{w}(i)$ then $(\overline{v} \in W_p \Leftrightarrow \overline{w} \in W_p)$.

Proof of Claim 1. Since $\overline{v}, \overline{w} \in W_R$ there exist $s, t \in \mathsf{ms}(\varphi)$ such that

$$\overline{w}_0 \xrightarrow{s} \overline{v} \quad \text{and} \quad \overline{w}_0 \xrightarrow{t} \overline{w}.$$

Since we have a product model and $\overline{v}(i) = \overline{w}(i)$, there exists some $\overline{u}$ with

$$\overline{w}_0 \xrightarrow{t \restriction i} \overline{u}, \quad \overline{w}_0 \xrightarrow{s \restriction i} \overline{u}, \quad \overline{u} \xrightarrow{s \setminus i} \overline{v}, \quad \overline{u} \xrightarrow{t \setminus i} \overline{w}.$$

Since $(\mathfrak{S}, \overline{w}_0) \models \chi$, we get $\overline{v} \in W_p \Rightarrow \overline{u} \in W_p \Rightarrow \overline{w} \in W_p$ and analogously $\overline{w} \in W_p \Rightarrow \overline{v} \in W_p$, which proves Claim 1.

Let us define for all $p \in \mathsf{P}_i$

$$V_p = \{\overline{w}(i) \in W_i \mid \overline{w} \in W_R \cap W_p\}.$$

With Claim 1, we get for all $\overline{w} \in W_R$:

$$\overline{w} \in W_p \;\Leftrightarrow\; \overline{w}(i) \in V_p \tag{1}$$

It remains to show that $(\prod_{i\in[1,d]}^{\mathsf{id}} \mathfrak{S}_i, \overline{w}_0) \vDash \varphi$. For a sequence $s \in \mathsf{ms}(\varphi)$ let $\mathsf{sub}_s(\varphi)$ be the set of all subformulas ψ of φ such that in the syntax tree for φ there exists a path to an occurrence of ψ such that s is the sequence of modalities along this path. We prove by induction on the structure of a subformula $\psi \in \mathsf{sub}_s(\varphi)$ that for all $\overline{w} \in W_R$ with $\overline{w}_0 \xrightarrow{s} \overline{w}$:

$$(\mathfrak{S}, \overline{w}) \vDash \psi \quad \Leftrightarrow \quad (\prod_{i\in[1,d]}^{\mathsf{id}} \mathfrak{S}_i, \overline{w}) \vDash \psi.$$

For the induction base consider a propositional variable $p \in \mathsf{P}$ and assume that $p \in \mathsf{P}_i$ and let $\overline{w} \in W_R$. We get:

$$(\mathfrak{S}, \overline{w}) \vDash p \quad \Leftrightarrow \quad \overline{w} \in W_p \quad \overset{(1)}{\Leftrightarrow} \quad \overline{w}(i) \in V_p \quad \Leftrightarrow \quad (\prod_{i\in[1,d]}^{\mathsf{id}} \mathfrak{S}_i, \overline{w}) \vDash p$$

For the induction step, the operators $\wedge$ and $\neg$ are straightforward. Finally, let $\psi = \Diamond_a \theta \in \mathsf{sub}_s(\varphi)$ and assume that $\overline{w}_0 \xrightarrow{s} \overline{w}$. Hence, $\theta \in \mathsf{sub}_{sa}(\varphi)$. We have:

$$\begin{aligned}
(\mathfrak{S}, \overline{w}) \vDash \Diamond_a \theta \quad &\Leftrightarrow \quad \exists \overline{w}' : \overline{w} \xrightarrow{a} \overline{w}' \wedge (\mathfrak{S}, \overline{w}') \vDash \theta \\
&\overset{\text{hyp}}{\Leftrightarrow} \quad \exists \overline{w}' : \overline{w} \xrightarrow{a} \overline{w}' \wedge (\prod_{i\in[1,d]}^{\mathsf{id}} \mathfrak{S}_i, \overline{w}') \vDash \theta \\
&\Leftrightarrow \quad (\prod_{i\in[1,d]}^{\mathsf{id}} \mathfrak{S}_i, \overline{w}) \vDash \Diamond_a \theta
\end{aligned}$$

Since $\varphi \in \mathsf{sub}_\varepsilon(\varphi)$, and $\overline{w}_0 \xrightarrow{\varepsilon} \overline{w}_0$, this shows that $(\prod_{i\in[1,d]}^{\mathsf{id}} \mathfrak{S}_i, \overline{w}_0) \vDash \varphi$.

Overall, we have given a reduction from $\mathbf{K}^d_{\mathsf{id}}$-SAT to $\mathbf{K}^d$-SAT. Note that the constructed formula φ' has the same switching depth as φ. ◻

3 K^2-SAT is hard

The goal of this section is to show a nonelementary lower bound for $\mathbf{K}^2$-SAT and thus to close the complexity gap for this problem. By Proposition 1 it suffices to show this for $\mathbf{K}^2_{\mathsf{id}}$-SAT. As a necessary preliminary step we show how to enforce (nonelementary) big models in $\mathbf{K}^2_{\mathsf{id}}$. Using this, we prove via a standard reduction from appropriate tiling problems that $\mathbf{K}^2_{\mathsf{id}}$-SAT is nonelementary.

In this section, we will only deal with id-products of two structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ (over disjoint sets of propositions and actions). To simplify notation we write $\mathfrak{S}_2 \times \mathfrak{S}_2$ for $\prod_{i\in[1,2]}^{\mathsf{id}} \mathfrak{S}_i$.

Recall the tower function $\mathsf{Tower} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as $\mathsf{Tower}(0, n) = n$ and $\mathsf{Tower}(\ell+1, n) = 2^{\mathsf{Tower}(\ell, n)}$ for each $\ell, n \in \mathbb{N}$. In this section, we construct a family $\{\varphi_{\ell,n} \mid \ell, n \geq 1\}$ of unimodal $\mathbf{K}^2$-formulas such that for each $\ell, n \in \mathbb{N}$ the following is satisfied:

(i) $|\varphi_{\ell,n}| \leq \exp(\ell) \cdot \mathsf{poly}(n)$, and

(ii) if $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \vDash \varphi_{\ell,n}$, then $|\mathfrak{S}|, |\mathfrak{S}'| \geq \mathsf{Tower}(\ell, n)$.

Informally speaking, our intention is that if $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \vDash \varphi_{\ell,n}$ then *both* $(\mathfrak{S}, s)$ *and* $(\mathfrak{S}', s')$ are of a particular structure that we will call (ℓ, n)*-treelike*. Before giving its formal definition,

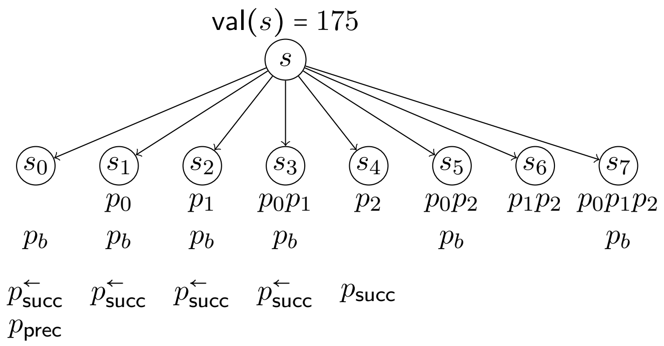

Figure 1: Example of an $(1, 3)$-treelike structure with value 175.

we provide some intuition about when a pointed structure $(\mathfrak{S}, s)$ is treelike (the definition of when $(\mathfrak{S}', s')$ is treelike will be analogous).

Intuitively, think of a pointed structure $(\mathfrak{S}, s)$ to be (ℓ, n)-treelike if it contains a tree of depth ℓ rooted in s (possibly with additional worlds and transitions) such that

- $(\mathfrak{S}, t)$ is $(\ell - 1, n)$-treelike for every successor t of s,
- s has at least $\mathsf{Tower}(\ell, n)$ successors.

For this purpose, we additionally assign a *value* $\mathsf{val}(\mathfrak{S}, s)$ to every (ℓ, n)-treelike structure $(\mathfrak{S}, s)$ and require that for every $i \in [0, m]$ with $m = \mathsf{Tower}(\ell, n) - 1$ there is a successor s_i of s with value i (however, we cannot exclude copies of s_i). For $\ell = 0$, the value is defined by propositional variables $p_0, \dots, p_{n-1}$ which define an n-bit number, where p_0 refers to the least significant bit. For $\ell > 0$, the value is defined using an additional proposition p_b. Intuitively, the worlds $s_0, \dots, s_m$ define a binary number (by convention, the leftmost bit is the least significant bit) $b_0 \cdots b_m$, where $b_i = 1$ precisely when the proposition p_b is satisfied in s_i. Obviously, this number is between 0 and $\mathsf{Tower}(\ell + 1, n) - 1$. Figure 1 gives an example of an $(1, 3)$-treelike structure with value 175. First observe that s has $8 = 2^3 = \mathsf{Tower}(1, 3)$ successors $s_0, \dots, s_7$. Next note that in each s_i the evaluation of the propositions p_j ($j \in [0, 2]$) gives a binary number equal to i. For instance, in s_4 only p_2 is true, hence the corresponding binary number is 001 which is 4. As indicated, the evaluation of p_b gives rise to the binary number $\vec{b} = 11110101$ which equals 175. For enforcing the described treelike structures $(\mathfrak{S}, s)$ we need additional auxiliary propositional variables p_{succ}, $p^{\leftarrow}_{\mathsf{succ}}$, p_{prec}, and $p^{\leftarrow}_{\mathsf{prec}}$. These propositions provide more information about the binary number $\vec{b} = b_0 \cdots b_{\mathsf{Tower}(\ell,n)-1}$ encoded by the successors of s:

- p_{succ} marks the first (from left to right) 0 in $\vec{b}$,
- p_{prec} marks the first (from left to right) 1 in $\vec{b}$, and
- $p^{\leftarrow}_{\mathsf{succ}}$ marks all worlds left of p_{succ},
- $p^{\leftarrow}_{\mathsf{prec}}$ marks all worlds left of p_{prec}.

Inuitively, p_{succ} (resp. p_{prec}) marks the maximal position that changes when $\vec{b}$ is *increased* (resp. *decreased*) by 1. In other words, increasing $\vec{b}$ by 1 can be done by flipping all bits marked with p_{succ} or $p^{\leftarrow}_{\mathsf{succ}}$ and carrying over the remaining ones. We refer again to Figure 1 for a valid evaluation of the auxiliary propositions.

It is worth mentioning that (ℓ, n)-treelike structures are similar to the trees $\tilde{\mathcal{T}}_h(n)$ from [12, Definition 2]. As mentioned above, we add a few more unary predicates (propositional variables) since our structures will be enforced in two-dimensional modal logic instead of first-order logic.

In the following, we formally define (ℓ, n)-treelike structures and their associated values. For this purpose, let us fix the set of action labels $\mathsf{A} \stackrel{\text{def}}{=} \{a, a'\}$ and for each $n \geq 1$ define the set of propositional variables $\mathsf{P}_n \stackrel{\text{def}}{=} \{p_0, \dots, p_{n-1}\} \cup \mathsf{P}_{\mathsf{aux}}$ and $\mathsf{Q}_n \stackrel{\text{def}}{=} \{q_0, \dots, q_{n-1}\} \cup \mathsf{Q}_{\mathsf{aux}}$ with $\mathsf{P}_{\mathsf{aux}} \stackrel{\text{def}}{=} \{p_b, p_{\mathsf{succ}}, p^{\leftarrow}_{\mathsf{succ}}, p_{\mathsf{prec}}, p^{\leftarrow}_{\mathsf{prec}}\}$ and $\mathsf{Q}_{\mathsf{aux}} \stackrel{\text{def}}{=} \{q_b, q_{\mathsf{succ}}, q^{\leftarrow}_{\mathsf{succ}}, q_{\mathsf{prec}}, q^{\leftarrow}_{\mathsf{prec}}\}$. For the sake of simplicity, we call $(\{a\}, \mathsf{P}_n)$-structures (resp. $(\{a'\}, \mathsf{Q}_n)$-structures) *left structures* (resp. *right structures*). We give only the definition of (ℓ, n)-treelike structures for left pointed structures because the definition for right structures is simply obtained by replacing every proposition p_g by q_g, every proposition $p^{\leftarrow}_g$ by $q^{\leftarrow}_g$ and by replacing $\xrightarrow{a}$ by $\xrightarrow{a'}$.

The definition of (ℓ, n)-treelike structures $(\mathfrak{S}, s)$ and their associated *values* $\mathsf{val}_{\ell,n}(\mathfrak{S}, s) \in [0, \mathsf{Tower}(\ell+1, n) - 1]$ is by induction on ℓ. Consider the left pointed structure $(\mathfrak{S}, s)$ where $\mathfrak{S} = (W, \xrightarrow{a}, \{W_p \mid p \in \mathsf{P}_n\})$. Then $(\mathfrak{S}, s)$ is $(0, n)$–treelike. So every left pointed structure over $(\{a\}, \mathsf{P}_n)$ is $(0, n)$-treelike. The *value* of $(\mathfrak{S}, s)$ is

$$\mathsf{val}_{0,n}(\mathfrak{S}, s) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} b_i 2^i \in [0, 2^n - 1],$$

where $b_i = 1$ if $s \in W_{p_i}$ and $b_i = 0$ otherwise.

For $\ell > 0$, $(\mathfrak{S}, s)$ is (ℓ, n)-*treelike* if the following hold, where $m = \mathsf{Tower}(\ell, n) - 1$:

(a) For all $u \in N_{\mathfrak{S}}(s)$, $(\mathfrak{S}, u)$ is $(\ell - 1, n)$-treelike. For $i \in [0, m]$ let

$$N^i_{\mathfrak{S}}(s) \stackrel{\text{def}}{=} \{u \in N_{\mathfrak{S}}(s) \mid \mathsf{val}_{\ell-1,n}(\mathfrak{S}, u) = i\}.$$

(b) $N^i_{\mathfrak{S}}(s) \neq \varnothing$ for every $i \in [0, m]$.

(c) If $u, v \in N^i_{\mathfrak{S}}(s)$ for some $i \in [0, m]$, then $u \in W_p$ if and only if $v \in W_p$ for each $p \in \mathsf{P}_{\mathsf{aux}}$.

(d) If $N_{\mathfrak{S}}(s) \subseteq W_{p_b}$ then $W_{p_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s) = \varnothing$ and $N_{\mathfrak{S}}(s) \subseteq W_{p^{\leftarrow}_{\mathsf{succ}}}$.

(e) If $N_{\mathfrak{S}}(s) \setminus W_{p_b} \neq \varnothing$ and $k \in [0, m]$ is minimal such that $N^k_{\mathfrak{S}}(s) \setminus W_{p_b} \neq \varnothing$, then for all $v \in N_{\mathfrak{S}}(s)$: $v \in W_{p_{\mathsf{succ}}}$ iff $\mathsf{val}_{\ell-1,n}(\mathfrak{S}, v) = k$ and $v \in W_{p^{\leftarrow}_{\mathsf{succ}}}$ iff $\mathsf{val}_{\ell-1,n}(\mathfrak{S}, v) < k$.

(f) If $N_{\mathfrak{S}}(s) \cap W_{p_b} = \varnothing$ then $W_{p_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s) = \varnothing$ and $N_{\mathfrak{S}}(s) \subseteq W_{p^{\leftarrow}_{\mathsf{prec}}}$.

(g) If $N_{\mathfrak{S}}(s) \cap W_{p_b} \neq \varnothing$ and $k \in [0, m]$ is minimal such that $N^k_{\mathfrak{S}}(s) \cap W_{p_b} \neq \varnothing$, then for all $v \in N_{\mathfrak{S}}(s)$: $v \in W_{p_{\mathsf{prec}}}$ iff $\mathsf{val}_{\ell-1,n}(\mathfrak{S}, v) = k$ and $v \in W_{p^{\leftarrow}_{\mathsf{prec}}}$ iff $\mathsf{val}_{\ell-1,n}(\mathfrak{S}, v) < k$.

Note that we make no restriction on the valuation of propositions in the world s. Moreover, also the set $W_{p_b} \cap N_{\mathfrak{S}}(s)$ is arbitrary, but this set uniquely determines the sets $W_{p_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s)$, $W_{p^{\leftarrow}_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s)$, $W_{p_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s)$, and $W_{p^{\leftarrow}_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s)$.

Finally, we define the *value* of $(\mathfrak{S}, s)$ as follows: For $i \in [0, m]$ let $b_i = 0$ if $W_{p_b} \cap N^i_{\mathfrak{S}}(s) = \varnothing$ and $b_i = 1$ otherwise. Then,

$$\mathsf{val}_{\ell,n}(\mathfrak{S}, s) \stackrel{\text{def}}{=} \sum_{i=0}^{m} b_i 2^i \in [0, 2^{m+1} - 1] = [0, \mathsf{Tower}(\ell+1, n) - 1].$$

Observe that this definition does not require a *unique* successor world s_i for each value i. In fact, one cannot enforce this in modal logic. Also note that there is always a tree of height ℓ that is (ℓ, n)-treelike.

The indices ℓ and n in $\mathsf{val}_{\ell,n}(\mathfrak{S}, s)$ are necessary, since, for instance, an (ℓ, n)-treelike structure $(\mathfrak{S}, s)$ is also $(0, n)$-treelike but in general $\mathsf{val}_{\ell,n}(\mathfrak{S}, s) \neq \mathsf{val}_{0,n}(\mathfrak{S}, s)$. Nevertheless we will omit these indices. This is possible because of the following conventions: When we talk about an (ℓ, n)-treelike structure $(\mathfrak{S}, s)$, then $\mathsf{val}(\mathfrak{S}, s)$ will always refer to $\mathsf{val}_{\ell,n}(\mathfrak{S}, s)$. Moreover, when we talk about a successor node s' of s, then $\mathsf{val}(\mathfrak{S}, s')$ will always refer to $\mathsf{val}_{\ell-1,n}(\mathfrak{S}, s)$.

We will construct a family of unimodal $\mathbf{K}^2$-formulas $(\varphi_{\ell,n})_{\ell,n\geq 0}$ that admit only asynchronous products of (ℓ, n)-treelike structures as models. In order to emphasize the two dimensions that we have in formulas over $(\{a\} \uplus \{a'\}, \mathsf{P}_n \uplus \mathsf{Q}_n)$, we write $\diamondminus$ and $\diamondbar$ instead of $\Diamond_a$ and $\Diamond_{a'}$, respectively, to refer to the modality of the first and second dimension of the product, and similarly for box formulas.

Before we define the formulas $\varphi_{\ell,n}$, we introduce auxiliary formulas $\mathsf{eq}_{\ell,n}$, $\mathsf{first}_{\ell,n}$, $\mathsf{last}_{\ell,n}$, and $\mathsf{succ}_{\ell,n}$ whose names indicate their meaning on the asynchronous product of two (ℓ, n)-treelike structures (see also Lemma 2 below). For $\ell = 0$ they are as follows:

$$\begin{aligned}
\mathsf{eq}_{0,n} &\stackrel{\text{def}}{=} \bigwedge_{i\in[0,n-1]} p_i \leftrightarrow q_i \\
\mathsf{first}_{0,n} &\stackrel{\text{def}}{=} \bigwedge_{i\in[0,n-1]} \neg p_i \wedge \neg q_i \\
\mathsf{last}_{0,n} &\stackrel{\text{def}}{=} \bigwedge_{i\in[0,n-1]} p_i \wedge q_i \\
\mathsf{succ}_{0,n} &\stackrel{\text{def}}{=} \bigvee_{i\in[0,n-1]} (\neg p_i \wedge q_i \wedge \bigwedge_{j\in[0,i-1]} (p_j \wedge \neg q_j) \wedge \bigwedge_{j\in[i+1,n-1]} p_j \leftrightarrow q_j)
\end{aligned}$$

For $\ell > 0$ we define them as follows:

$$\begin{aligned}
\mathsf{eq}_{\ell,n} &\stackrel{\text{def}}{=} \boxminus \boxbar \left(\mathsf{eq}_{\ell-1,n} \to (p_b \leftrightarrow q_b)\right) \\
\mathsf{first}_{\ell,n} &\stackrel{\text{def}}{=} \boxminus \neg p_b \wedge \boxbar \neg q_b \\
\mathsf{last}_{\ell,n} &\stackrel{\text{def}}{=} \boxminus p_b \wedge \boxbar q_b \\
\mathsf{succ}_{\ell,n} &\stackrel{\text{def}}{=} \diamondminus \neg p_b \wedge \boxminus \boxbar (\mathsf{eq}_{\ell-1,n} \to ((p_{\mathsf{succ}} \leftrightarrow q_{\mathsf{prec}}) \wedge ((\neg \overleftarrow{p}_{\mathsf{succ}} \wedge \neg p_{\mathsf{succ}}) \to (p_b \leftrightarrow q_b))))
\end{aligned}$$

In order to show the intuition of the introduced formulas we prove the following lemma.

Lemma 2. *Let $\ell \geq 0$ and let $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ be left and right (ℓ, n)-treelike structures. Then the following holds:*

(a) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{eq}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$.

(b) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{first}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s') = 0$.

(c) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{last}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s') = \mathsf{Tower}(\ell + 1, n) - 1$.

(d) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{succ}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}', s') = \mathsf{val}(\mathfrak{S}, s) + 1$

Proof. We show the statement by induction on ℓ. For the induction base let $\ell = 0$. For (a) we have $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{eq}_{0,n}$ if and only if $s \in W_{p_i} \Leftrightarrow s' \in W_{q_i}$ for all $i \in [0, n-1]$ if and only if $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$. Both (b) and (c) can be proven in analogy to (a). For (d) we have $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{succ}_{0,n}$, if and only if there is some $i \in [0, n-1]$ such that $s \notin W_{p,i}$, $s' \in W_{q_i}$, $s \in W_{p_j}$ and $s' \notin W_{q_j}$ for each $j \in [0, i-1]$ and $s \in W_{p_j} \Leftrightarrow s' \in W_{q_j}$ for each $j \in [i+1, n-1]$. This is equivalent to $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{succ}_{0,n}$.

For the induction step let $\ell > 0$. By definition of val in treelike structures, we have that $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$ iff

$$\forall u \in N_{\mathfrak{S}}(s), u' \in N_{\mathfrak{S}'}(s') : \ \mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}', u') \Rightarrow (u \in W_{p_b} \Leftrightarrow u' \in W_{q_b}).$$

By induction, this is equivalent to

$$\forall u \in N_{\mathfrak{S}}(s), u' \in N_{\mathfrak{S}'}(s') : \ (\mathfrak{S} \times \mathfrak{S}', \langle u, u'\rangle) \models \mathsf{eq}_{\ell-1,n} \Rightarrow (u \in W_{p_b} \Leftrightarrow u' \in W_{q_b}),$$

This is equivalent to

$$(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \boxminus \boxdot \left(\mathsf{eq}_{\ell-1,n} \to (p_b \leftrightarrow q_b)\right),$$

i.e., to $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{eq}_{\ell,n}$.

For (b) we have $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{first}_{\ell,n}$, i.e., $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \boxminus\neg p_b \wedge \boxdot\neg q_b$ if and only if $u \notin W_{p_b}$ and $u' \notin W_{q_b}$ for all $u \in N_{\mathfrak{S}}(s), u' \in N_{\mathfrak{S}'}(s')$. By definition of val, this is equivalent to $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s') = 0$.

Point (c) is proven analogously to (b).

Let us finally prove (d). Assume first that $\mathsf{val}(\mathfrak{S}', s') = \mathsf{val}(\mathfrak{S}, s) + 1$ and hence $\mathsf{val}(\mathfrak{S}, s) < \mathsf{Tower}(\ell+1, n) - 1$. Let $k \le \mathsf{Tower}(\ell, n) - 1$ be the position of the first 0 in the binary representation of $\mathsf{val}(\mathfrak{S}, s)$. Let $u \in N_{\mathfrak{S}}(s)$ such that $\mathsf{val}(\mathfrak{S}, u) = k$, which exists by the definition of (ℓ, n)-treelike structures. Since the k-th bit of $\mathsf{val}(\mathfrak{S}, s)$ is zero, we have $u \notin W_{p_b}$. Hence, we get $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \diamondminus\neg p_b$. Moreover, k is minimal such that there is $u \in N_{\mathfrak{S}}(s) \setminus W_{p_b}$ with $\mathsf{val}(\mathfrak{S}, u) = k$. By point (e) from the definition of (ℓ, n)-treelike structures we get

$$\begin{aligned} N_{\mathfrak{S}}(s) \cap W_{p_{\mathsf{succ}}} &= \{u \in N_{\mathfrak{S}}(s) \mid \mathsf{val}(\mathfrak{S}, u) = k\} \text{ and} \\ N_{\mathfrak{S}}(s) \setminus (W_{p^{\leftarrow}_{\mathsf{succ}}} \cup W_{p_{\mathsf{succ}}}) &= \{v \in N_{\mathfrak{S}}(s) \mid \mathsf{val}(\mathfrak{S}, v) > k\}. \end{aligned}$$

Since $\mathsf{val}(\mathfrak{S}', s') = \mathsf{val}(\mathfrak{S}, s) + 1$, basic arithmetic shows that k is the position of the first 1 in $\mathsf{val}(\mathfrak{S}', s')$ and that for all $k' > k$, the k'-th bits in the binary representations of $\mathsf{val}(\mathfrak{S}, s)$ and $\mathsf{val}(\mathfrak{S}', s')$ are equal. Point (g) from the definition of (ℓ, n)-treelike structures implies

$$W_{q_{\mathsf{prec}}} \cap N_{\mathfrak{S}'}(s') = \{u' \in N_{\mathfrak{S}'}(s') \mid \mathsf{val}(\mathfrak{S}', u') = k\}.$$

Let now $v \in N_{\mathfrak{S}}(s), v' \in N_{\mathfrak{S}'}(s')$ be such that $(\mathfrak{S} \times \mathfrak{S}', \langle v, v'\rangle) \models \mathsf{eq}_{\ell-1,n}$. By part (a) of the Lemma, $\mathsf{val}(\mathfrak{S}, v) = \mathsf{val}(\mathfrak{S}', v')$. Hence, by the above consideration, $v \in W_{p_{\mathsf{succ}}}$ iff $v' \in W_{q_{\mathsf{prec}}}$. Moreover, if $v \notin W_{p^{\leftarrow}_{\mathsf{succ}}} \cup W_{p_{\mathsf{succ}}}$ then $k' := \mathsf{val}(\mathfrak{S}, v) = \mathsf{val}(\mathfrak{S}', v') > k$. Since the k'-th bits in the binary representations of $\mathsf{val}(\mathfrak{S}, s)$ and $\mathsf{val}(\mathfrak{S}', s')$ coincide, we have $v \in W_{p_b}$ iff $v' \in W_{q_b}$. This shows that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{succ}_{\ell,n}$ holds.

For the other direction assume that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \mathsf{succ}_{\ell,n}$, i.e.,

(i) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \diamondminus\neg p_b$ and

(ii) for all $u \in N_{\mathfrak{S}}(s), u' \in N_{\mathfrak{S}'}(s')$, if $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \mathsf{eq}_{\ell-1,n}$ then

$$(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models (p_{\mathsf{succ}} \leftrightarrow q_{\mathsf{prec}}) \wedge ((\neg p^{\leftarrow}_{\mathsf{succ}} \wedge \neg p_{\mathsf{succ}}) \rightarrow (p_b \leftrightarrow q_b)).$$

By (i), we have $\mathsf{val}(\mathfrak{S}, s) < \mathsf{Tower}(\ell+1, n) - 1$. Thus, by definition of (ℓ, n)-treelike structures, $W_{p_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s) \neq \varnothing$. By part (a) of the Lemma, $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \mathsf{eq}_{\ell-1,n}$ if and only if $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}', u')$. From (ii), more precisely from $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models p_{\mathsf{succ}} \leftrightarrow q_{\mathsf{prec}}$, it follows that there is a k such that $W_{p_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s) = \{v \in N_{\mathfrak{S}}(s) \mid \mathsf{val}(\mathfrak{S}, v) = k\}$ and $W_{q_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s) = \{v' \in N_{\mathfrak{S}'}(s') \mid \mathsf{val}(\mathfrak{S}', v') = k\}$. Hence, by definition of (ℓ, n)-treelike structures, $N_{\mathfrak{S}}(s) \setminus (W_{p^{\leftarrow}_{\mathsf{succ}}} \cup W_{p_{\mathsf{succ}}}) = \{v \in N_{\mathfrak{S}}(s) \mid \mathsf{val}(\mathfrak{S}, v) > k\}$.

By (ii), for all $v \in N_{\mathfrak{S}}(s) \setminus (W_{p^{\leftarrow}_{\mathsf{succ}}} \cup W_{p_{\mathsf{succ}}})$, and $v' \in N_{\mathfrak{S}'}(s')$ with $\mathsf{val}(\mathfrak{S}, v) = \mathsf{val}(\mathfrak{S}', v')$ we have $v \in W_{p_b}$ iff $v' \in W_{q_b}$. Thus, for every $k' > k$, the k'-th bits in the binary representations of $\mathsf{val}(\mathfrak{S}, s)$ and $\mathsf{val}(\mathfrak{S}', s')$ coincide. The definition of (ℓ, n)-treelike structures yields that the k-th bit of $\mathsf{val}(\mathfrak{S}, s)$ ($\mathsf{val}(\mathfrak{S}', s')$, respectively) is 0 (1, respectively) and all lower bits are 1 (0, respectively). Thus, overall we obtain by definition of val that $\mathsf{val}(\mathfrak{S}', s') = \mathsf{val}(\mathfrak{S}, s) + 1$. □

Now we give a family of formulas $\varphi_{\ell,n}$ with the idea that every model of $\varphi_{\ell,n}$ is the product of a left (ℓ, n)-treelike structure and a right (ℓ, n)-treelike structure with the same value.

Definition 3. *Set $\varphi_{0,n} = \mathsf{eq}_{0,n}$ and define $\varphi_{\ell,n}$, by induction on ℓ, as the conjunction of the following formulas:*

(1) $\diamondminus \diamondvert (\varphi_{\ell-1,n} \wedge \mathsf{first}_{\ell-1,n})$

(2) $\boxminus \diamondvert \varphi_{\ell-1,n}$

(3) $\boxvert \diamondminus \varphi_{\ell-1,n}$

(4) $\boxminus \big(\boxvert \neg \mathsf{last}_{\ell-1,n} \rightarrow \diamondvert \mathsf{succ}_{\ell-1,n}\big)$

(5) $\boxminus \boxvert \,(\mathsf{eq}_{\ell-1,n} \rightarrow \bigwedge_{p_g \in \mathsf{P_{aux}}} (p_g \leftrightarrow q_g))$

(6) $\boxminus \boxvert \,(((p_{\mathsf{succ}} \vee p^{\leftarrow}_{\mathsf{prec}}) \rightarrow \neg p_b) \wedge ((p^{\leftarrow}_{\mathsf{succ}} \vee p_{\mathsf{prec}}) \rightarrow p_b)))$

(7) $\boxminus \boxvert \,(\mathsf{succ}_{\ell-1,n} \rightarrow \bigwedge_{x \in \{\mathsf{succ}, \mathsf{prec}\}} ((q_x \vee q^{\leftarrow}_x) \rightarrow p^{\leftarrow}_x) \wedge (p^{\leftarrow}_x \rightarrow (q^{\leftarrow}_x \vee q_x))$

(8) $\diamondminus \diamondvert (p_{\mathsf{succ}} \vee p^{\leftarrow}_{\mathsf{succ}}) \;\wedge\; \diamondminus \diamondvert (p_{\mathsf{prec}} \vee p^{\leftarrow}_{\mathsf{prec}})$

Some remarks regarding the intuition of the formulas are appropriate. In the following explanation we will, in analogy to left and right structures, distinguish left and right worlds.

Formulas (2) and (3) together imply condition (a) from the definiton of (ℓ, n)-treelike structures (every successor is $(\ell - 1, n)$-treelike). Condition (b), the existence of successor worlds for each value $k \in [0, \mathsf{Tower}(\ell, n) - 1]$, is enforced by induction on k: Formula (1) enforces a left $(\ell - 1, n)$-treelike structure with value 0, thus establishing the induction base. Formula (4) enforces for every left world with value $k < \mathsf{Tower}(\ell, n) - 1$ a right world with value $k + 1$. Formula (3) enforces a left world having the same value $k + 1$; this yields the induction step. Formula (5) enforces condition (c). The remaining conditions (d)-(g) from the definiton of (ℓ, n)-treelike structures can be reformulated as follows:

(i) If a left successor world satisfies p_{succ} or $p^{\leftarrow}_{\mathsf{prec}}$ (resp., $p^{\leftarrow}_{\mathsf{succ}}$ or p_{prec}), then it does not satisfy p_b (resp., it satisfies p_b).

(ii) If p_{succ} or $p^{\leftarrow}_{\mathsf{succ}}$ (resp., p_{prec} or $p^{\leftarrow}_{\mathsf{prec}}$) is satisfied in a left successor world of value $k > 0$, then $p^{\leftarrow}_{\mathsf{succ}}$ (resp., $p^{\leftarrow}_{\mathsf{prec}}$) is satisfied in all left successor worlds with value $k - 1$.

(iii) If $p^{\leftarrow}_{\mathsf{succ}}$ (resp., $p^{\leftarrow}_{\mathsf{prec}}$) is satisfied in a left successor world of value $k < \mathsf{Tower}(\ell, n) - 1$, then $p^{\leftarrow}_{\mathsf{succ}}$ or p_{succ} (resp., $p^{\leftarrow}_{\mathsf{prec}}$ or p_{prec}) is satisfied in every left successor world of value $k + 1$.

(iv) There is a left successor world satisfying either p_{succ} or $p^{\leftarrow}_{\mathsf{succ}}$ (resp. p_{prec} or $p^{\rightarrow}_{\mathsf{prec}}$).

Clearly, (i) (resp. (iv)) is expressed by formula (6) (resp. (8)). Finally, formula (5) and (7) yield (ii) and (iii). For instance, if a left world with value $k > 0$ satisfies $p^{\leftarrow}_{\mathsf{succ}}$ or p_{succ}, then by formula (5) the corresponding right world satisfies $q^{\leftarrow}_{\mathsf{succ}}$ or q_{succ}. Formula (7) implies that $p^{\leftarrow}_{\mathsf{succ}}$ is satisfied in every left world with value $k - 1$. We are now ready to present our main theorem.

Theorem 4. *For every $\ell \geq 0$ the following holds:*

(a) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \varphi_{\ell,n}$ iff $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are (ℓ, n)-treelike structures with $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$.

(b) $|\varphi_{\ell,n}| \leq 3^{\ell} \cdot \mathsf{poly}(\ell, n)$ and the formula $\varphi_{\ell,n}$ is computable in time $3^{\ell} \cdot \mathsf{poly}(\ell, n)$.

(c) The switching depth of $\varphi_{\ell,n}$ is ℓ.

Proof. In the following, (1), (2), ... , (8) refer to the formulas from Definition 3.

(a) We show the statement by induction on ℓ. For the *induction base* let $\ell = 0$. Recall that every pointed structure is $(0, n)$-treelike. Moreover, $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \varphi_{0,n}$ if and only if $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{eq}_{0,n}$ if and only if $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$ (by Lemma 2(a)).

For the *induction step* assume that $\ell \geq 1$ and that (a) holds for $\ell - 1$. We deal with the two directions in (a) separately. First, we show that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \varphi_{\ell,n}$ holds if $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are (ℓ, n)-treelike structures with $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$. Observe that the definition of treelike structures implies that all $(\mathfrak{S}, u)$ and $(\mathfrak{S}', u')$ are $(\ell - 1, n)$-treelike structures for all $u \in N_{\mathfrak{S}}(s)$ and $u' \in N_{\mathfrak{S}'}(s')$.

It is routine to show that formulas (1) to (8) hold in $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle)$. For (1) note that by definition of (ℓ, n)-treelike structures, there are $u \in N_{\mathfrak{S}}(s)$, $u' \in N_{\mathfrak{S}'}(s')$ with $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}', u') = 0$. By induction hypothesis, $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \varphi_{\ell-1,n}$ and by Lemma 2(b), $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \mathsf{first}_{\ell-1,n}$. Thus formula (1) holds in $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle)$. Formula (2) is satisfied since by definition of (ℓ, n)-treelike structures, for each $u \in N_{\mathfrak{S}}(s)$ there is $u' \in N_{\mathfrak{S}'}(s')$ with $\mathsf{val}(\mathfrak{S}', u) = \mathsf{val}(\mathfrak{S}', u')$. Now $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \varphi_{\ell-1,n}$ by induction hypothesis, which implies $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \boxminus \mathbin{\Diamond\mkern-10mu\mid}\ \varphi_{\ell-1,n}$. Formula (3) is shown analogously. Consider now formula (4). Let $u \in N_{\mathfrak{S}}(s)$ be such that $(\mathfrak{S} \times \mathfrak{S}', \langle u, s' \rangle) \models \boxbar \neg \mathsf{last}_{\ell-1,n}$, i.e., $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \neg \mathsf{last}_{\ell-1,n}$ for all $u' \in N_{\mathfrak{S}'}(s')$. Lemma 2(c) yields that $k := \mathsf{val}(\mathfrak{S}, u) < \mathsf{Tower}(\ell, n) - 1$. By definition of (ℓ, n)-treelike structures there is some $u' \in N_{\mathfrak{S}'}(s')$ such that $\mathsf{val}(\mathfrak{S}', u') = k + 1$. By Lemma 2(d), we obtain $(\mathfrak{S} \times \mathfrak{S}', \langle u, u' \rangle) \models \mathsf{succ}_{\ell-1,n}$, hence (4) is satisfied. Formula (5) is satisfied due to $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$ and the definition of (ℓ, n)-treelike structures. Formula (6) is satisfied since, by definition of (ℓ, n)-treelike structures, for every $u \in N_{\mathfrak{S}}(s)$, we have $u \in W_{p_{\mathsf{succ}}} \cup W_{p^{\leftarrow}_{\mathsf{prec}}} \Rightarrow u \notin W_{p_b}$ and $u \in W_{p_{\mathsf{prec}}} \cup W_{p^{\leftarrow}_{\mathsf{succ}}} \Rightarrow u \in W_{p_b}$. For (7) we show exemplarily that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \boxminus \boxbar ((\mathsf{succ}_{\ell-1,n} \wedge (q_{\mathsf{succ}} \vee q^{\leftarrow}_{\mathsf{succ}})) \to p^{\leftarrow}_{\mathsf{succ}})$ holds. Hence, let $u \in N_{\mathfrak{S}}(s)$, $u' \in N_{\mathfrak{S}'}(s')$ such that $(\mathfrak{S} \times \mathfrak{S}', \langle u, u'\rangle) \models \mathsf{succ}_{\ell-1,n} \wedge (q_{\mathsf{succ}} \vee q^{\leftarrow}_{\mathsf{succ}})$. Thus, by Lemma 2(d), $\mathsf{val}(\mathfrak{S}, u) = k$ and $\mathsf{val}(\mathfrak{S}', u') = k+1$ for some $k \in [0, \mathsf{Tower}(\ell, n) - 2]$. Moreover, $u' \in W_{q_{\mathsf{succ}}} \cup W_{q^{\leftarrow}_{\mathsf{succ}}}$. By definition of (ℓ, n)-treelike structures, there is $v' \in N_{\mathfrak{S}'}(s')$ with $\mathsf{val}(\mathfrak{S}', v') = k = \mathsf{val}(\mathfrak{S}, u)$. Since $u' \in W_{q_{\mathsf{succ}}} \cup W_{q^{\leftarrow}_{\mathsf{succ}}}$ we must have $v' \in W_{q^{\leftarrow}_{\mathsf{succ}}}$ (again by the definition of (ℓ, n)-treelike structures). By Lemma 2(a), we have $(\mathfrak{S} \times \mathfrak{S}', \langle u, v'\rangle) \models \mathsf{eq}_{\ell-1,n}$. Since $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$, we have $u \in W_{p^{\leftarrow}_{\mathsf{succ}}}$ iff $v' \in W_{q^{\leftarrow}_{\mathsf{succ}}}$ (by formula (5), which was already shown to hold). Hence $(\mathfrak{S} \times \mathfrak{S}', \langle u, u'\rangle) \models p^{\leftarrow}_{\mathsf{succ}}$ as required by formula (7). Finally, for (8) note that the definition of (ℓ, n)-treelike structures implies that there exist $u, v \in N_{\mathfrak{S}}(s)$ with $u \in W_{p^{\leftarrow}_{\mathsf{succ}}} \cup W_{p_{\mathsf{succ}}}$ and $v \in W_{p^{\leftarrow}_{\mathsf{prec}}} \cup W_{p_{\mathsf{prec}}}$.

Let us now deal with the other direction in (a), i.e., that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \varphi_{\ell,n}$ implies that $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are (ℓ, n)-treelike structures with $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$. Thus, assume that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \varphi_{\ell,n}$. We have to show that $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are (ℓ, n)-treelike structures with $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$.

Let $u \in N_{\mathfrak{S}}(s)$. By formula (2) there is some $u' \in N_{\mathfrak{S}'}(s')$ with $(\mathfrak{S} \times \mathfrak{S}', \langle u, u'\rangle) \models \varphi_{\ell-1,n}$. By the induction hypothesis, $(\mathfrak{S}, u)$ and $(\mathfrak{S}', u')$ are $(\ell-1, n)$-treelike structures. Analogously (using formula (3)), one can show that for every $u' \in N_{\mathfrak{S}'}(s')$, $(\mathfrak{S}', u')$ is $(\ell-1, n)$-treelike. We have shown point (a) from the definition of (ℓ, n)-treelike structures.

Next, let $T = \mathsf{Tower}(\ell, n) - 1$. By induction on i, we prove the following claim, which states point (b) from the definition of (ℓ, n)-treelike structures.

Claim 1. For every $i \in [0, T]$ there are $u \in N_{\mathfrak{S}}(s), u' \in N_{\mathfrak{S}'}(s')$ such that $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}', u') = i$.

Proof of Claim 1. We prove the claim by induction on i. For $i = 0$ the statement holds by Lemma 2(b) since formula (1) is satisfied. For the inductive step, assume that there are states $v \in N_{\mathfrak{S}}(s)$, $v' \in N_{\mathfrak{S}'}(s')$ such that $\mathsf{val}(\mathfrak{S}, v) = \mathsf{val}(\mathfrak{S}', v') = i < T$. Formula (4) implies that $(\mathfrak{S} \times \mathfrak{S}', \langle v, s'\rangle) \models \boxbar\neg\mathsf{last}_{\ell-1,n} \to \diamondbar\mathsf{succ}_{\ell-1,n}$. By Lemma 2(c), $(\mathfrak{S} \times \mathfrak{S}', \langle v, x'\rangle) \models \neg\mathsf{last}_{\ell-1,n}$ holds for all $x' \in N_{\mathfrak{S}'}(s')$. Hence, there exists $u' \in N_{\mathfrak{S}'}(s')$ such that $(\mathfrak{S} \times \mathfrak{S}', \langle v, u'\rangle) \models \mathsf{succ}_{\ell-1,n}$. Thus, $\mathsf{val}(\mathfrak{S}', u') = \mathsf{val}(\mathfrak{S}, v) + 1 = i + 1$ by Lemma 2(d). Finally, by formula (3), there exists $u \in N_{\mathfrak{S}}(s)$ such that $(\mathfrak{S} \times \mathfrak{S}', \langle u, u'\rangle) \models \varphi_{\ell-1,n}$. By induction, this implies $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}', u') = i + 1$. This finishes the proof of Claim 1.

Claim 2. If $u \in N_{\mathfrak{S}}(s), u' \in N_{\mathfrak{S}'}(s')$ and $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}', u')$, then for all $p_g \in \mathsf{P}_{\mathsf{aux}}$, $u \in W_{p_g}$ iff $u' \in W_{q_g}$.

Proof of Claim 2. Let u and u' as in Claim 2. By Lemma 2(a), $(\mathfrak{S} \times \mathfrak{S}', \langle u, u'\rangle) \models \mathsf{eq}_{\ell-1,n}$. Formula (5) implies $u \in W_{p_g}$ iff $u' \in W_{q_g}$ for all $p_g \in \mathsf{P}_{\mathsf{aux}}$.

By Claim 2 it suffices to show the remaining points (c)–(g) from the definition of (ℓ, n)-treelike structures only for $(\mathfrak{S}, s)$ (then they also hold for $(\mathfrak{S}', s')$).

By the following claim, successors of s with the same value satisfy the same propositions; it implies point (c) from the definition of (ℓ, n)-treelike structures.

Claim 3. If $u_1, u_2 \in N_{\mathfrak{S}}(s)$ and $\mathsf{val}(\mathfrak{S}, u_1) = \mathsf{val}(\mathfrak{S}, u_2)$, then for all $p \in \mathsf{P}_{\mathsf{aux}}$, $u_1 \in W_p$ iff $u_2 \in W_p$.

Proof of Claim 3. Let u_1, u_2 be as required. By formula (2) there is $v' \in N_{\mathfrak{S}'}(s')$ with $(\mathfrak{S} \times \mathfrak{S}', \langle u_1, v'\rangle) \models \varphi_{\ell-1,n}$. Hence $\mathsf{val}(\mathfrak{S}', v') = \mathsf{val}(\mathfrak{S}, u_1) = \mathsf{val}(\mathfrak{S}, u_2)$. By Claim 2, for every $p_g \in \mathsf{P}_{\mathsf{aux}}$, we have $u_1 \in W_{p_g}$ iff $v' \in W_{q_g}$ iff $u_2 \in W_{p_g}$.

From formula (6), we get the following implications for all $u \in N_{\mathfrak{S}}(s)$:

$$u \in W_{p_{\mathsf{succ}}} \cup W_{p^{\leftarrow}_{\mathsf{prec}}} \;\Rightarrow\; u \notin W_{p_b}, \tag{i}$$

$$u \in W_{p_{\mathsf{prec}}} \cup W_{p^{\leftarrow}_{\mathsf{succ}}} \;\Rightarrow\; u \in W_{p_b}. \tag{ii}$$

Next, let $x \in \{\mathsf{succ}, \mathsf{prec}\}$, and $u, v \in N_{\mathfrak{S}}(s)$ such that $v \in W_{p_x} \cup W_{p_x^{\leftarrow}}$, $\mathsf{val}(\mathfrak{S}, u) < T$, and $\mathsf{val}(\mathfrak{S}, v) = \mathsf{val}(\mathfrak{S}, u) + 1$. By Claim 1, there exists $v' \in N_{\mathfrak{S}'}(s')$ with $\mathsf{val}(\mathfrak{S}', v') = \mathsf{val}(\mathfrak{S}, u) + 1 = \mathsf{val}(\mathfrak{S}, v)$. With Claim 2, we get $v' \in W_{q_x} \cup W_{q_x^{\leftarrow}}$. Lemma 2(d) implies $(\mathfrak{S} \times \mathfrak{S}', \langle u, v'\rangle) \models \mathsf{succ}_{\ell-1,n} \wedge (q_x \vee q_x^{\leftarrow})$. By formula (7), we must have $(\mathfrak{S} \times \mathfrak{S}', \langle u, v'\rangle) \models p_x^{\leftarrow}$, i.e., $u \in W_{p_x^{\leftarrow}}$. Using induction, we get the following implication:

$$(u, v \in N_{\mathfrak{S}}(s) \;\wedge\; \mathsf{val}(\mathfrak{S}, v) > \mathsf{val}(\mathfrak{S}, u) \wedge\; v \in W_{p_x} \cup W_{p_x^{\leftarrow}}) \;\Rightarrow\; u \in W_{p_x^{\leftarrow}} \tag{iii}$$

In the same way, formula (7) implies the following implication:

$$(u, v \in N_{\mathfrak{S}}(s) \;\wedge\; \mathsf{val}(\mathfrak{S}, v) > \mathsf{val}(\mathfrak{S}, u) \;\wedge\; u \in W_{p_x^{\leftarrow}}) \;\Rightarrow\; v \in W_{p_x} \cup W_{p_x^{\leftarrow}} \tag{iv}$$

Assume that there exist $u, v \in N_{\mathfrak{S}}(s)$ with $\mathsf{val}(\mathfrak{S}, u) < \mathsf{val}(\mathfrak{S}, v)$ and $u, v \in W_{p_x}$. With (iii) we obtain $u \in W_{p_x} \cap W_{p_x^{\leftarrow}}$. With (i) and (ii) we obtain $u \in W_{p_b}$ and $u \notin W_{p_b}$, a contradiction. Hence, $u, v \in W_{p_x}$ implies $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}, v)$.

If there is no $u \in N_{\mathfrak{S}}(s) \cap W_{p_x}$, then, by formula (8), there must exist $u \in N_{\mathfrak{S}}(s) \cap W_{p_x^{\leftarrow}}$. Then, (iii) and (iv) imply $N_{\mathfrak{S}}(s) \subseteq W_{p_x^{\leftarrow}}$. Moreover, if $x = \mathsf{succ}$ (resp. $x = \mathsf{prec}$), then by (ii) $N_{\mathfrak{S}}(s) \subseteq W_{p_b}$ (resp. by (i) $N_{\mathfrak{S}}(s) \cap W_{p_b} = \varnothing$). Hence, in this case we have verified points (d)–(g) from the definition of (ℓ, n)-treelike structures.

Now assume that there is a unique $k \in [0, T]$ such that $u \in W_{p_x}$ for all $u \in N_{\mathfrak{S}}(s)$ with $\mathsf{val}(\mathfrak{S}, u) = k$. By (i) and (ii), for all $u \in N_{\mathfrak{S}}(s)$ with $\mathsf{val}(\mathfrak{S}, u) = k$ we have $u \notin W_{p_x^{\leftarrow}}$ and $u \notin W_{p_b}$ if $x = \mathsf{succ}$ (resp. $u \in W_{p_b}$ if $x = \mathsf{prec}$). Moreover, (i), (ii), and (iii) imply for all $v \in N_{\mathfrak{S}}(s)$ with $\mathsf{val}(\mathfrak{S}, v) < k$: $v \in W_{p_x^{\leftarrow}}$ and $v \in W_{p_b}$ if $x = \mathsf{succ}$ (resp. $v \notin W_{p_b}$ if $x = \mathsf{prec}$). If there would exists $v \in W_{p_x^{\leftarrow}}$ with $v \in N_{\mathfrak{S}}(s)$ and $\mathsf{val}(\mathfrak{S}, v) > k = \mathsf{val}(\mathfrak{S}, u)$, then $u \in W_{p_x^{\leftarrow}} \cap W_{p_x}$ by (iii), which leads to a contradiction with (i) and (ii). Hence, we have $v \notin W_{p_x^{\leftarrow}}$ for all $v \in N_{\mathfrak{S}}(s)$ with $\mathsf{val}(\mathfrak{S}, v) > k$. Again, we have verified points (d)–(g) from the definition of (ℓ, n)-treelike structures.

We have shown that $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \varphi_{\ell,n}$ implies that $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are (ℓ, n)-treelike structures. Moreover, by Claim 2, $(\mathfrak{S} \times \mathfrak{S}', \langle s, s'\rangle) \models \varphi_{\ell,n}$ implies also $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$. This finishes the proof of part (a) of the theorem.

(b) Again, we show the statement by induction on ℓ starting with $\ell = 0$. For $\varphi_{0,n} = \bigwedge_{i \in [0,n-1]} p_i \leftrightarrow q_i \wedge \boxminus\bot \wedge \boxbar\bot$ the statement is trivial. Let now be $\ell > 0$. The formula $\varphi_{\ell-1,n}$ occurs 3 times in $\varphi_{\ell,n}$. The auxiliary formulas $\mathsf{succ}_{\ell-1,n}$, $\mathsf{eq}_{\ell-1,n}$, $\mathsf{last}_{\ell-1,n}$, and $\mathsf{first}_{\ell-1,n}$ are all polynomially sized in ℓ and n. Thus, overall we get $|\varphi_{\ell,n}| = 3 \cdot |\varphi_{\ell-1,n}| + \mathsf{poly}(\ell, n)$. Thus, we obtain by induction hypothesis $|\varphi_{\ell,n}| = 3^{\ell} \cdot \mathsf{poly}(\ell, n)$.

(c) This is an immediate consequence of Definition 3. □

We are finally ready to proceed to the main result of this section. By making use of the models that are enforced by the formulas $\varphi_{\ell,n}$, we can encode big numbers. In the proof of the following proposition we use these numbers to encode big tiling problems. Let ℓ-NEXP denote the class of all problems decidable by a nondeterministic $\mathsf{Tower}(\ell, \mathsf{poly}(n))$-time bounded Turing machine.

Proposition 5. *The folowing holds:*

- *For each $\ell \geq 1$, $\mathbf{K}^2_{\mathsf{id}}$-SAT restricted to formulas of switching depth ℓ is ℓ-NEXP-hard under polynomial time many-one reductions.*
- *In particular, $\mathbf{K}^2_{\mathsf{id}}$-SAT is nonelementary.*

For the proof of Proposition 5, we need to introduce tilings and the tiling problem. A *tiling system* is a tuple $S = (\Theta, \mathbb{H}, \mathbb{V})$, where Θ is a finite set of *tile types*, $\mathbb{H} \subseteq \Theta \times \Theta$ is a *horizontal matching relation*, and $\mathbb{V} \subseteq \Theta \times \Theta$ is a *vertical matching relation*. A mapping $\tau : [0, k-1] \times [0, k-1] \to \Theta$ (where $k \geq 1$) is a *k-solution for S* if for all $(x, y) \in [0, k-1] \times [0, k-1]$ the following holds:

- if $x < k - 1$, $\tau(x, y) = \theta$, and $\tau(x + 1, y) = \theta'$, then $(\theta, \theta') \in \mathbb{H}$, and
- if $y < k - 1$, $\tau(x, y) = \theta$, and $\tau(x, y + 1) = \theta'$, then $(\theta, \theta') \in \mathbb{V}$.

Let $\mathsf{Sol}_k(S)$ denote the set of all k-solutions for S. Let $w = \theta_0 \cdots \theta_{n-1} \in \Theta^n$ be a word and let $k \geq n$. With $\mathsf{Sol}_k(S, w)$ we denote the set of all $\tau \in \mathsf{Sol}_k(S)$ such that $\tau(x, 0) = \theta_x$ for all $x \in [0, n-1]$. For each tiling system $S = (\Theta, \mathbb{H}, \mathbb{V})$ we define its ℓ-EXP*-tiling problem* as follows:

<u>ℓ-EXP-TILING PROBLEM FOR $S = (\Theta, \mathbb{H}, \mathbb{V})$</u>
INPUT: A word $w \in \Theta^n$.
QUESTION: Does $\mathsf{Sol}_{\mathsf{Tower}(\ell,n)}(S, w) \neq \emptyset$ hold?

The following result is folklore. In general, from a nondeterministic $t(n)$-time bounded Turing machine M one can construct a tiling system S_M which simulates M in the following sense (see e.g. [20]): From an input x of length n for M one can construct a word w_x of length n over the tile types of S_M such that M accepts x iff $\mathsf{Sol}_{t(n)}(S_M, w_x) \neq \emptyset$.

Theorem 6 (folklore)**.** *For each $\ell \geq 1$, there exists a fixed tiling system S_ℓ such that the ℓ-EXP-TILING PROBLEM FOR S_ℓ is hard for ℓ-NEXP under polynomial time many-one reductions.*

We can finally prove Proposition 5.

Proof of Proposition 5. Due to technical reasons, we only sketch the proof for $\ell \geq 3$. Let $S_\ell = (\Theta, \mathbb{H}, \mathbb{V})$ be some tiling system such that the ℓ-EXP-TILING PROBLEM FOR S_ℓ is hard for ℓ-NEXP (Theorem 6). We give a polynomial time many-one reduction from the ℓ-EXP-TILING PROBLEM FOR S_ℓ to $\mathbf{K}^2_{\mathsf{id}}$-SAT restricted to formulas of switching depth ℓ. We add to the set of propositions P_n from the previous section, all tile types from Θ and two additional propositions x and y. To Q_n we add copies θ' ($\theta \in \Theta$), x', and y' of these propositions.

Let $w = \theta_0 \cdots \theta_{n-1}$ be an input word for the ℓ-EXP-tiling problem for S_ℓ and let $m = \mathsf{Tower}(\ell - 1, n) - 1$. A *grid element structure* is a pointed structure $(\mathfrak{S}, s)$ where W is the set of worlds of $\mathfrak{S}$ such that the following conditions hold:

- There exists a unique $\theta \in \Theta$ with $s \in W_\theta$.
- For every $u \in N_{\mathfrak{S}}(s)$, $(\mathfrak{S}, u)$ is $(\ell-2, n)$-treelike and in addition, either $u \in W_x$ or $u \in W_y$.

- For every $i \in [0, m]$ there exist $u, v \in N_{\mathfrak{S}}(s)$ such that $\mathsf{val}(\mathfrak{S}, u) = i = \mathsf{val}(\mathfrak{S}, v)$, $u \in W_x$, and $v \in W_y$.
- If $u, v \in N_{\mathfrak{S}}(s)$, $\mathsf{val}(\mathfrak{S}, u) = \mathsf{val}(\mathfrak{S}, v)$, and $u \in W_x$ iff $v \in W_x$, then $u \in W_{p_b}$ iff $v \in W_{p_b}$.

For $i \in [0, m]$ we define $X_i, Y_i \in \{0,1\}$ as follows: $X_i = 1$ iff there exists $u \in N_{\mathfrak{S}}(s) \cap W_x \cap W_{p_b}$ with $\mathsf{val}(\mathfrak{S}, u) = i$ and $Y_i = 1$ iff there exists $u \in N_{\mathfrak{S}}(s) \cap W_y \cap W_{p_b}$ with $\mathsf{val}(\mathfrak{S}, u) = i$. We think of $(\mathfrak{S}, s)$ to correspond to the grid element $\mathsf{El}(\mathfrak{S}, s) \overset{\text{def}}{=} (X, Y, \theta) \in [0, \mathsf{Tower}(\ell, n) - 1]^2 \times \Theta$ with $s \in W_\theta$, $X = \sum_{i=0}^m X_i \cdot 2^i$ and $Y = \sum_{i=0}^m Y_i \cdot 2^i$. Grid element structures for the second dimension are defined analogously using the propositions in $\mathsf{Q}_n \cup \Theta' \cup \{x', y'\}$.

Using the formulas from Section 3 for level $\ell - 2$, it is easy to write down a formula gridel of switching depth $\ell - 1$ such that for two pointed structures $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ we have $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{gridel}$ if and only if $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are grid element structures with $\mathsf{El}(\mathfrak{S}, s) = \mathsf{El}(\mathfrak{S}', s')$. For this, it is useful to have for $z \in \{x, y\}$ the abbreviations $\diamondminus_z \psi = \diamondminus(z \wedge \psi)$, $\boxminus_z \psi = \boxminus(z \to \psi)$, $\diamondbar_{z'} \psi = \diamondbar(z' \wedge \psi)$, and $\boxbar_{z'} \psi = \boxbar(z' \to \psi)$. Then, for $z \in \{x, y\}$ we can define relativized formulas $\varphi^z_{\ell-1,n}$, $\mathsf{eq}^z_{\ell-1,n}$, $\mathsf{first}^z_{\ell-1,n}$, $\mathsf{last}^z_{\ell-1,n}$, and $\mathsf{succ}^z_{\ell-1,n}$ by replacing in the definitions of the subformulas $\varphi_{\ell-1,n}$, $\mathsf{eq}_{\ell-1,n}$, $\mathsf{first}_{\ell-1,n}$, $\mathsf{last}_{\ell-1,n}$, and $\mathsf{succ}_{\ell-1,n}$ every modality $\diamondminus$ (resp., $\boxminus$, $\diamondbar$, $\boxbar$) by $\diamondminus_z$ (resp., $\boxminus_z$, $\diamondbar_{z'}$, $\boxbar_{z'}$). All occurrences of $\varphi_{\ell-2,n}$, $\mathsf{eq}_{\ell-2,n}$, $\mathsf{first}_{\ell-2,n}$, $\mathsf{last}_{\ell-2,n}$, and $\mathsf{succ}_{\ell-2,n}$ are not changed, i.e., we do not replace modalities within these formulas. Then gridel is the formula

$$\bigvee_{\theta \in \Theta} \Big(\theta \wedge \theta' \wedge \bigwedge_{\kappa \in \Theta \setminus \{\theta\}} (\neg\kappa \wedge \neg\kappa')\Big) \;\wedge\; \boxminus \boxbar \big((x \oplus y) \;\wedge\; (x' \oplus y')\big) \;\wedge\; \varphi^x_{\ell-1,n} \wedge \varphi^y_{\ell-1,n},$$

of switching depth $\ell - 1$, where $\oplus$ denotes exclusive or.

A *tiling structure* is a pointed structure $(\mathfrak{S}, s)$ such that the following holds:

- For every $u \in N_{\mathfrak{S}}(s)$, $(\mathfrak{S}, u)$ is a grid element structure.
- For every $X, Y \in [0, \mathsf{Tower}(\ell, n) - 1]^2$ there exists $u \in N_{\mathfrak{S}}(s)$ such that $\mathsf{El}(\mathfrak{S}, u) = (X, Y, \theta)$ for some $\theta \in \Theta$.
- If $u, v \in N_{\mathfrak{S}}(s)$ with $\mathsf{El}(\mathfrak{S}, u) = (X, Y, \theta_1)$ and $\mathsf{El}(\mathfrak{S}, v) = (X, Y, \theta_2)$, then $\theta_1 = \theta_2$.
- The unique mapping $\tau_{(\mathfrak{S},s)} : [0, \mathsf{Tower}(\ell, n) - 1]^2 \to \Theta$ with $\tau_{(\mathfrak{S},s)}(X, Y) = \theta$ if there exists $u \in N_{\mathfrak{S}}(s)$ with $\mathsf{El}(\mathfrak{S}, u) = (X, Y, \theta)$ (which is well defined by the previous points) is a $\mathsf{Tower}(\ell, n)$-solution of S_ℓ.

Again, the definition for the second dimension is analogous.

Using the formula gridel and a construction similar to those for $\varphi_{\ell,n}$, it is easy to write down a formula tiling of switching depth ℓ such that for all pointed structures $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ we have $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{tiling}$ if and only if $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ are tiling structures with $\tau_{(\mathfrak{S},s)} = \tau_{(\mathfrak{S}',s')}$. We take for tiling the conjunction of the following formulas:

(1) $\diamondminus \diamondbar (\mathsf{gridel} \wedge \mathsf{first}^x_{\ell-1,n} \wedge \mathsf{first}^y_{\ell-1,n})$

(2) $\boxminus \diamondbar \mathsf{gridel}$

(3) $\boxbar \diamondminus \mathsf{gridel}$

(4) $\boxbar \boxminus ((\mathsf{eq}^x_{\ell-1,n} \wedge \mathsf{eq}^y_{\ell-1,n}) \to \mathsf{gridel})$

(5) $\boxminus(\boxdot\neg\mathsf{last}^x_{\ell-1,n} \to \diamondplus(\mathsf{succ}^x_{\ell-1,n} \wedge \mathsf{eq}^y_{\ell-1,n} \wedge \bigvee_{(\theta_1,\theta_2)\in\mathbb{H}} (\theta_1 \wedge \theta_2')))$

(6) $\boxminus(\boxdot\neg\mathsf{last}^y_{\ell-1,n} \to \diamondplus(\mathsf{succ}^y_{\ell-1,n} \wedge \mathsf{eq}^x_{\ell-1,n} \wedge \bigvee_{(\theta_1,\theta_2)\in\mathbb{V}} (\theta_1 \wedge \theta_2')))$

Finally, it is straightforward to write down a formula φ_w expressing that $\tau_{(\mathfrak{S},s)}(x,0) = \theta_x$ for all $x \in [0, n-1]$. Hence, $\mathsf{Sol}_{\mathsf{Tower}(\ell,n)}(S_\ell, w) \neq \emptyset$ if and only if $\mathsf{tiling} \wedge \varphi_w$ is id-satisfiable. This formula has switching depth ℓ, which concludes the proof. □

The following theorem is an immediate consequence of Proposition 1 and Proposition 5.

Corollary 7. *The following holds:*

- *For each $\ell \geq 1$,* $\mathbf{K}^2$-SAT *restricted to formulas of switching depth ℓ is ℓ-*NEXP*-hard under polynomial time many-one reductions.*
- *In particular,* $\mathbf{K}^2$-SAT *is nonelementary.*

4 Hardness results for $\mathbf{K4} \times \mathbf{K}$ and $\mathbf{S5}_2 \times \mathbf{K}$

In this section, we prove further nonelementary lower bound results for the satisfiability problem of two-dimensional modal logics on restricted classes of frames. We hereby close nonelementary complexity gaps that were stated as open problems in [8]. Although in [8] uninterpreted product models for these logics are of interest, we prove our lower bounds for the id-interpretation only: For each of the logics studied here, the id-interpretation case can be reduced in polynomial time to the uninterpreted case in analogy to Proposition 1, because the proof of Proposition 1 makes no assumptions about the underlying frames.

We define the following logics:

- $\mathbf{K4} \times \mathbf{K}$: Two-dimensional unimodal logic restricted to product models $\mathfrak{S} \times \mathfrak{S}'$, where $\mathfrak{F}(\mathfrak{S})$ is *transitive*.
- $\mathbf{S5} \times \mathbf{K}$: Two-dimensional unimodal logic restricted to product models $\mathfrak{S} \times \mathfrak{S}'$ such that if $\mathfrak{F}(\mathfrak{S}) = (W, \equiv)$, then $\equiv$ is an *equivalence relation*.
- $\mathbf{S5}_2 \times \mathbf{K}$: Two-dimensional modal logic that is bimodal in the first dimension and unimodal in the second dimension restricted to models $\mathfrak{S} \times \mathfrak{S}'$, where $\mathfrak{F}(\mathfrak{S}) = (W, \equiv_1, \equiv_2)$ implies that both $\equiv_1$ and $\equiv_2$ are equivalence relations.

Let us start with $\mathbf{K4} \times \mathbf{K}$. We adapt the straightforward reduction from $\mathbf{K}$ to $\mathbf{K4}$ to the two-dimensional case. When following a transition in a $\mathbf{K4}$-frame one has no control over how far one is actually going due to transitivity of the frame. The idea for the reduction is to introduce additional propositions $h_0, \ldots, h_n$ and enforce *levels* in the models. Intuitively, h_i is true in w' precisely when w' is in level i seen from the world w where the formula is evaluated. Following a transition is then restricted to increase the level only by 1.

Let φ be a unimodal $\mathbf{K}^2$-formula with $\mathsf{rank}_1(\varphi) = r$ and define for every $0 \le k \le r$ the translation function t_k by taking

$$\begin{aligned}
t_k(p) &\overset{\text{def}}{=} H_k \wedge p \\
t_k(\neg\psi) &\overset{\text{def}}{=} H_k \wedge \neg t_k(\psi) \\
t_k(\psi_1 \wedge \psi_2) &\overset{\text{def}}{=} t_k(\psi_1) \wedge t_k(\psi_2) \\
t_k(\Diamond\psi) &\overset{\text{def}}{=} \Diamond t_k(\psi) \\
t_k(\diamondminus\psi) &\overset{\text{def}}{=} H_k \wedge \diamondminus(H_{k+1} \wedge t_{k+1}(\psi))
\end{aligned}$$

where $H_k \overset{\text{def}}{=} h_k \wedge \bigwedge_{i \ne k} \neg h_i$ and $k < r$ in the definition of $t_k(\diamondminus\psi)$. We show that the translation is satisfiability preserving. More precisely, we prove the following lemma.

Lemma 8. *For every unimodal $\mathbf{K}^2$-formula φ we have: φ is* id*-satisfiable in $\mathbf{K}^2$ iff $t_0(\varphi)$ is* id*-satisfiable in $\mathbf{K4} \times \mathbf{K}$.*

Proof. For the remainder of the proof assume that φ is defined over $\mathsf{P} = \mathsf{P}_1 \cup \mathsf{P}_2$ for disjoint P_1 and P_2. Moreover set $r \overset{\text{def}}{=} \mathsf{rank}_1(\varphi)$. As in Section 3 we will write $\mathfrak{S}_2 \times \mathfrak{S}_2$ for $\prod^{\mathsf{id}}_{i\in[1,2]} \mathfrak{S}_i$.

Assume first that φ is id-satisfiable. Thus, there are structures

$$\mathfrak{S}_i = (W_i, \longrightarrow_i, \{W_{i,p} \mid p \in \mathsf{P}_i\})$$

$(i \in \{1,2\})$ and $\overline{w} = \langle w_1, w_2 \rangle \in W_1 \times W_2$ such that $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{w}) \models \varphi$. Without loss of generality we assume that $\mathfrak{S}_1$ is a tree with root w_1. Define

$$\mathfrak{S}_1' = (W_1, \longrightarrow_1^+, \{W_{1,p}' \mid p \in \mathsf{P} \cup \{h_0, \ldots, h_r\}\}),$$

where

- $\longrightarrow_1^+$ is the transitive closure of $\longrightarrow_1$,
- $W_{1,p}' \overset{\text{def}}{=} W_{1,p}$ for all $p \in \mathsf{P}_1$, and
- $W_{1,h_i}' \overset{\text{def}}{=} V_i$, where V_i is defined to be the set of worlds w' such that the (unique) path in $\mathfrak{S}_1$ from w_1 to w' has length i, i.e., consists of i transitions.

We prove by structural induction that for each subformula ψ of φ with $\mathsf{rank}_1(\psi) = i$ we have

$$(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \psi \quad \Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi)$$

for each $\overline{x} \in V_{r-i} \times W_2$.

For the induction base, assume $\psi = p$ for some atomic proposition $p \in \mathsf{P}_1 \cup \mathsf{P}_2$. Thus, $i \overset{\text{def}}{=} \mathsf{rank}_1(\psi) = 0$. Let us fix some arbitrary $\overline{x} = \langle x_1, x_2 \rangle \in V_r \times W_2$. By definition of $\mathfrak{S}_1'$, we have that $\overline{x} \in W_{1,h_r}' \times W_2$ and hence $(\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models H_r$. Finally, the following equivalences hold, where we assume that $p \in \mathsf{P}_j$ $(j \in \{1,2\})$: $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \psi$ if and only if $x_j \in W_{j,p}$ if and only $(\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models H_r \wedge p = t_r(\psi)$.

For the induction step, assume ψ is not atomic. Let $i \overset{\text{def}}{=} \mathsf{rank}_1(\psi)$ and let us fix some $\overline{x} \in V_{r-i} \times W_2$. Note that $(\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models H_{r-i}$ by definition of $\mathfrak{S}_1'$. We make a case distinction on the structure of ψ.

Case $\psi = \neg\chi$. Then we have

$$\begin{aligned}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \psi \quad &\Leftrightarrow \quad (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \not\models \chi \\
&\overset{\mathsf{rank}_1(\chi)=i,\text{hyp}}{\Leftrightarrow} \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \not\models t_{r-i}(\chi) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \neg t_{r-i}(\chi) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models H_{r-i} \wedge \neg t_{r-i}(\chi) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi).
\end{aligned}$$

Case $\psi = \chi_1 \wedge \chi_2$. Then we have

$$\begin{aligned}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \psi \quad &\Leftrightarrow \quad (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \chi_1 \text{ and } (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \chi_2 \\
&\overset{\mathsf{rank}_1(\chi_1)=\mathsf{rank}_1(\chi_2)=i,\text{hyp}}{\Leftrightarrow} \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\chi_1) \text{ and} \\
&\qquad\qquad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\chi_2) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\chi_1) \wedge t_{r-i}(\chi_2) \\
&\Leftrightarrow \quad (S_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi).
\end{aligned}$$

Case $\psi = \diamondsuit\chi$. Then we have

$$\begin{aligned}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \psi \quad &\Leftrightarrow \quad \exists \overline{y} \in V_{r-i} \times W_2 : \overline{x} \longrightarrow_2 \overline{y} \text{ and } (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{y}) \models \chi \\
&\overset{\mathsf{rank}_1(\chi)=i,\text{hyp}}{\Leftrightarrow} \quad \exists \overline{y} \in V_{r-i} \times W_2 : \overline{x} \longrightarrow_2 \overline{y} \text{ and } (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{y}) \models t_{r-i}(\chi) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \diamondsuit t_{r-i}(\chi) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi).
\end{aligned}$$

Case $\psi = \Diamond\chi$. Then $\mathsf{rank}_1(\chi) = \mathsf{rank}_1(\psi) - 1 = i - 1$ and we have

$$\begin{aligned}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \psi \quad &\Leftrightarrow \quad \exists \overline{y} \in V_{r-(i-1)} \times W_2 : \overline{x} \longrightarrow_1 \overline{y} \text{ and } (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{y}) \models \chi \\
&\overset{\mathsf{rank}_1(\chi)=i-1,\text{hyp}}{\Leftrightarrow} \quad \exists \overline{y} \in V_{r-i+1} \times W_2 : \overline{x} \longrightarrow_1 \overline{y} \text{ and} \\
&\qquad\qquad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{y}) \models t_{r-i+1}(\chi) \\
&\overset{\text{Def.}V_{r-i+1}}{\Leftrightarrow} \quad \exists \overline{y} \in V_{r-i+1} \times W_2 : \overline{x} \longrightarrow_1^+ \overline{y} \text{ and} \\
&\qquad\qquad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{y}) \models H_{r-i+1} \wedge t_{r-i+1}(\chi) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \Diamond(H_{r-i+1} \wedge t_{r-i+1}(\chi)) \\
&\overset{\overline{x}\in V_{r-i}}{\Leftrightarrow} \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models H_{r-i} \wedge \Diamond(H_{r-i+1} \wedge t_{r-i+1}(\chi)) \\
&\Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi).
\end{aligned}$$

Since $\mathsf{rank}_1(\varphi) = r$, $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{w}) \models \varphi$, and $\overline{w} \in V_0 \times W_2$ we get $(\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{w}) \models t_0(\varphi)$. Hence $t_0(\varphi)$ is id-satisfiable in $\mathbf{K4} \times \mathbf{K}$.

For the other direction assume that $t_0(\varphi)$ is id-satisfiable in $\mathbf{K4} \times \mathbf{K}$. Thus, there are a transitive structure $\mathfrak{S}_1 = (W_1, \longrightarrow_1, \{W_{1,p} \mid p \in \mathsf{P}_1 \cup \{h_0, \ldots, h_r\}\})$ and a structure $\mathfrak{S}_2 = (W_2, \longrightarrow_2, \{W_{2,p} \mid p \in \mathsf{P}_2\})$ and $\overline{w} = \langle w_1, w_2 \rangle \in W_1 \times W_2$ such that $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{w}) \models t_0(\varphi)$. For each $0 \leq i \leq r$ we set

$$T_i \overset{\text{def}}{=} W_{1,h_i} \smallsetminus (\bigcup_{j \neq i} W_{1,h_j}),$$

which is exactly the semantics of the formulas H_i in $\mathfrak{S}_1$. Now, define the structure $\mathfrak{S}_1' = (W_1', \longrightarrow_1', \{W_{1,p}' \mid p \in \mathsf{P}_1)$ by taking

- $W_1' = \bigcup_{0 \le i \le r} T_i$,
- $\longrightarrow_1' = \longrightarrow_1 \cap (\bigcup_{0 \le i < r} T_i \times T_{i+1})$, and
- $W_{1,p}' = W_{1,p} \cap W_1'$ for all $p \in \mathsf{P}_1$.

We prove by structural induction that for each subformula ψ of φ with $\mathsf{rank}_1(\psi) = i$ and each $\overline{x} \in T_{r-i} \times W_2$ we have

$$(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi) \quad \Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \psi$$

For the induction base assume $\psi = p$ for some atomic proposition $p \in \mathsf{P}_1 \cup \mathsf{P}_2$. Hence, $i \overset{\text{def}}{=} \mathsf{rank}_1(\psi) = 0$ and $t_r(\psi) = H_r \wedge p$. Let us assume some $\overline{x} \in T_r \times W_2$. By definition of T_r, we have $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models h_r$ and $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \not\models h_j$ for each $j \in \{1, \ldots, r-1\}$. Thus, $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models H_r$. Moreover we have $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models p$ if and only if $(\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models p$ by definition of $W_{1,p}'$. Thus,

$$(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi) \quad \Leftrightarrow \quad (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models H_{r-i} \wedge \psi \quad \Leftrightarrow \quad (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \psi.$$

For the induction step, assume ψ is not atomic. Let $i \overset{\text{def}}{=} \mathsf{rank}_1(\psi)$ and fix some $\overline{x} = \langle x_1, x_2 \rangle \in T_{r-i} \times W_2$. Note that we have $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models H_{r-i}$, by definition of T_{r-i}.

Case $\psi = \neg\chi$. Then we have

$$\begin{array}{rcl}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi) & \Leftrightarrow & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models H_{r-i} \wedge \neg t_{r-i}(\chi) \\
& \overset{x_1 \in T_{r-i}}{\Leftrightarrow} & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \neg t_{r-i}(\chi) \\
& \Leftrightarrow & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \not\models t_{r-i}(\chi) \\
& \overset{\mathsf{rank}_1(\chi)=i,\text{hyp}}{\Leftrightarrow} & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \not\models \chi \\
& \Leftrightarrow & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \neg\chi \\
& \Leftrightarrow & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \psi.
\end{array}$$

Case $\psi = \chi_1 \wedge \chi_2$. Note that $\mathsf{rank}_1(\psi) = \mathsf{rank}_1(\chi_1) = \mathsf{rank}_1(\chi_2) = i$. Then we have

$$\begin{array}{rcl}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi) & \Leftrightarrow & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\chi_1) \wedge t_{r-i}(\chi_2) \\
& \Leftrightarrow & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\chi_1) \text{ and} \\
& & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\chi_2) \\
& \overset{\mathsf{rank}_1(\chi_1)=\mathsf{rank}_1(\chi_2)=i,\text{hyp}}{\Leftrightarrow} & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \chi_1 \text{ and} \\
& & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \chi_2 \\
& \Leftrightarrow & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \chi_1 \wedge \chi_2 \\
& \Leftrightarrow & (\mathfrak{S}_1' \times \mathfrak{S}_2, \overline{x}) \models \psi.
\end{array}$$

Case $\psi = \diamondvert\chi$. Then $\mathsf{rank}_1(\psi) = \mathsf{rank}_1(\chi) = i$ and we have

$$
\begin{array}{lcl}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi) & \Leftrightarrow & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \diamondvert t_{r-i}(\chi) \\
& \Leftrightarrow & \exists \overline{y} \in T_{r-i} \times W_2 : \overline{x} \longrightarrow_2 \overline{y} \text{ and } (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{y}) \models t_{r-i}(\chi) \\
& \overset{\mathsf{rank}_1(\chi)=i,\text{hyp}}{\Leftrightarrow} & \exists \overline{y} \in T_{r-i} \times W_2 : \overline{x} \longrightarrow_2 \overline{y} \text{ and } (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{y}) \models \chi \\
& \Leftrightarrow & (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{x}) \models \diamondvert\chi \\
& \Leftrightarrow & (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{x}) \models \psi.
\end{array}
$$

Case $\psi = \diamondminus\chi$. Then $\mathsf{rank}_1(\chi) = \mathsf{rank}_1(\psi) - 1 = i - 1$ and we have

$$
\begin{array}{lcl}
(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models t_{r-i}(\psi) & \Leftrightarrow & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models H_{r-i} \wedge \diamondminus(H_{r-i+1} \wedge t_{r-i+1}(\chi)) \\
& \overset{x_1 \in T_{r-i}}{\Leftrightarrow} & (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{x}) \models \diamondminus(H_{r-i+1} \wedge t_{r-i+1}(\chi)) \\
& \overset{\text{Def. } T_{r-i+1}}{\Leftrightarrow} & \exists \overline{y} \in T_{r-i+1} \times W_2 : \overline{x} \longrightarrow_1 \overline{y} \text{ and} \\
& & \qquad\qquad (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{y}) \models t_{r-i+1}(\chi) \\
& \overset{\text{Def. } \longrightarrow'_1}{\Leftrightarrow} & \exists \overline{y} \in T_{r-i+1} \times W_2 : \overline{x} \longrightarrow'_1 \overline{y} \text{ and} \\
& & \qquad\qquad (\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{y}) \models t_{r-i+1}(\chi) \\
& \overset{\mathsf{rank}_1(\chi)=i-1,\text{hyp}}{\Leftrightarrow} & \exists \overline{y} \in T_{r-i+1} \times W_2 : \overline{x} \longrightarrow'_1 \overline{y} \text{ and } (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{y}) \models \chi \\
& \Leftrightarrow & (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{x}) \models \diamondminus\chi \\
& \Leftrightarrow & (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{x}) \models \psi.
\end{array}
$$

Since we assume $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{w}) \models t_0(\varphi)$, we must have $\overline{w} \in T_0 \times W_2$. Because $\mathsf{rank}_1(\varphi) = r$, the above equivalence implies $(\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{w}) \models \varphi$. Hence, φ is id-satisfiable. □

Lemma 8 provides a reduction of $\mathbf{K}^2_{\mathsf{id}}$-SAT to id-satisfiability in $\mathbf{K4} \times \mathbf{K}$. Finally, Proposition 1 together with Proposition 5 yields the following result.

Theorem 9. *Satisfiability in* $\mathbf{K4} \times \mathbf{K}$ *is nonelementary.*

Next, we study combinations of $\mathbf{K}$ with $\mathbf{S5}$ and $\mathbf{S5}_2$. It is well-known that the complexity for checking satisfiability jumps from NP for $\mathbf{S5}$ to PSPACE for $\mathbf{S5}_2$. We will show that also the complexity for deciding satisfiability in the product logics $\mathbf{S5}\times\mathbf{K}$ and $\mathbf{S5}_2\times\mathbf{K}$, respectively, differs. In particular, we will again reduce from $\mathbf{K}^2_{\mathsf{id}}$-SAT in order to show a nonelementary lower bound for the latter logic, which is in sharp contrast to the following result by Marx [21].

Theorem 10 ([21])**.** *Satisfiability in* $\mathbf{S5} \times \mathbf{K}$ *is* NEXP*-complete.*

PSPACE-hardness for satisfiability in $\mathbf{S5}_2$ is shown by a straightforward reduction from $\mathbf{K}$ [3]. We adapt this reduction to the two-dimensional case by defining a translation † by

$$
\begin{array}{rcl}
q^\dagger & \overset{\text{def}}{=} & p^* \wedge q \\
(\varphi_1 \wedge \varphi_2)^\dagger & \overset{\text{def}}{=} & \varphi_1^\dagger \wedge \varphi_2^\dagger \\
(\neg\varphi)^\dagger & \overset{\text{def}}{=} & p^* \wedge \neg(\varphi^\dagger) \\
(\diamondvert\varphi)^\dagger & \overset{\text{def}}{=} & \diamondvert(\varphi^\dagger) \\
(\diamondminus\varphi)^\dagger & \overset{\text{def}}{=} & p^* \wedge \diamondminus_1(\neg p^* \wedge \diamondminus_2(p^* \wedge \varphi^\dagger))
\end{array}
$$

where $\diamondminus_1$ and $\diamondminus_2$ refer to the two modalities in $\mathbf{S5}_2$ and p^* is a fresh propositional variable in the left signature. Intuitively, *one* transition in $\mathbf{K}$ is simulated by *two* transitions in $\mathbf{S5}_2$. This is possible since the composition of two equivalence relations is neither symmetric nor transitive in general and using the fresh variable p^* we can enforce a non-trivial transition, i.e., no loops. It can be proven along the lines of the proof in [3] that † preserves id-satisfiability.

Lemma 11. *For every unimodal $\mathbf{K}^2$-formula φ we have: φ is id-satisfiable in $\mathbf{K}^2$ iff $\varphi^\dagger$ is id-satisfiable in $\mathbf{S5}_2 \times \mathbf{K}$.*

Proof. For the remainder of the proof assume that φ is defined over $\mathsf{P} = \mathsf{P}_1 \cup \mathsf{P}_2$ for disjoint P_1 and P_2 with $p^* \notin \mathsf{P}_1 \cup \mathsf{P}_2$. Again, we write $\mathfrak{S}_2 \times \mathfrak{S}_2$ for $\prod^{\mathsf{id}}_{i\in[1,2]} \mathfrak{S}_i$.

Assume first that φ is id-satisfiable. Thus, there are $\mathfrak{S}_1 = (W_1, \xrightarrow{a}, \{W_{1,p} \mid p \in \mathsf{P}_1\})$, $\mathfrak{S}_2 = (W_2, \xrightarrow{b}, \{W_{2,p} \mid p \in \mathsf{P}_2\})$, and $\overline{s} \in W_1 \times W_2$ such that $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{s}) \models \varphi$. Define an $\mathbf{S5}_2$-structure $\mathfrak{S}'_1 = (W'_1, \overset{1}{\equiv}, \overset{2}{\equiv}, \{W'_{1,p} \mid p \in \mathsf{P}_1 \cup \{p^*\}\})$ as follows:

- $W'_1 \overset{\text{def}}{=} W_1 \uplus \xrightarrow{a}$,
- $\overset{1}{\equiv}$ is the reflexive, transitive, and symmetric closure of $\{(w, (w, w')) \mid w \xrightarrow{a} w'\}$,
- $\overset{2}{\equiv}$ is the reflexive, transitive, and symmetric closure of $\{((w, w'), w') \mid w \xrightarrow{a} w'\}$,
- $W'_{1,p} \overset{\text{def}}{=} W_{1,p}$ for $p \in \mathsf{P}_1$,
- $W'_{1,p^*} \overset{\text{def}}{=} W_1$.

Now, one can prove by induction on the structure of a formula ψ that for every world $\overline{w} \in W_1 \times W_2$ we have:

$$(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{w}) \models \psi \quad \Leftrightarrow \quad (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{w}) \models \psi^\dagger.$$

In particular, we obtain $(\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{s}) \models \varphi^\dagger$.

Assume now that $\varphi^\dagger$ is id-satisfiable. Hence, there is an $\mathbf{S5}_2$-structure

$$\mathfrak{S}_1 = (W_1, \overset{1}{\equiv}, \overset{2}{\equiv}, \{W_{1,p} \mid p \in \mathsf{P}_1 \cup \{p^*\}\}),$$

a structure $\mathfrak{S}_2 = (W_2, \xrightarrow{b}, \{W_{2,p} \mid p \in \mathsf{P}_2\})$, and $\overline{s} \in W_1 \times W_2$ such that $(\mathfrak{S}_1 \times \mathfrak{S}_2, s) \models \varphi^\dagger$. Define a structure $\mathfrak{S}'_1 = (W'_1, \xrightarrow{a}, \{W'_{1,p} \mid p \in \mathsf{P}_1\})$ as follows:

- $W'_1 \overset{\text{def}}{=} W_{1,p^*}$
- $\xrightarrow{a} \overset{\text{def}}{=} \{(u, v) \mid \exists w \in W_1 \setminus W_{1,p^*} : u \overset{1}{\equiv} w \overset{2}{\equiv} v\}$
- $W'_{1,p} \overset{\text{def}}{=} W_{1,p} \cap W_{1,p^*}$ for all $p \in \mathsf{P}_1$

We can prove by induction on the structure of a formula ψ that for every world $\overline{w} \in W'_1 \times W_2$ we have:

$$(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{w}) \models \psi^\dagger \quad \Leftrightarrow \quad (\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{w}) \models \psi$$

Again, the details are straightforward. Observe now that $(\mathfrak{S}_1 \times \mathfrak{S}_2, \overline{s}) \models \varphi^\dagger$ implies $\overline{s} \in W'_1 \times W_2$ by the definition of †. Therefore, we get $(\mathfrak{S}'_1 \times \mathfrak{S}_2, \overline{s}) \models \varphi$. $\square$

The following theorem is an immediate consequence of Lemma 11, Proposition 5, and Proposition 1.

Theorem 12. *Satisfiability in $\mathbf{S5}_2 \times \mathbf{K}$ is nonelementary.*

5 Feferman-Vaught decompositions for products

The Feferman-Vaught decomposition theorem for multimodal $\mathbf{K}^d$ can be formulated as follows, and was proven in [13]. Recall the notion of an interpretation σ from Section 2.3.

Theorem 13 ([13])**.** *From an interpretation σ and a multimodal $\mathbf{K}^d$-formula φ over (A,P) with $\mathsf{A} = \biguplus_{i\in[1,d]}$, $\mathsf{P} = \biguplus_{i\in[1,d]} P_i$, one can compute a tuple $(\Psi_1,\ldots,\Psi_d,\beta)$ with $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ a finite set of multimodal formulas over $(\mathsf{A}_i,\mathsf{P}_i)$ and β a positive boolean formula with variables from $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ such that for every $(\mathsf{A}_i,\mathsf{P}_i)$-structure $\mathfrak{S}_i$ and every world w_i of $\mathfrak{S}_i$ ($i \in [1,d]$):*

$$(\prod_{i\in[1,d]}^{\sigma} \mathfrak{S}_i, \langle w_1,\ldots,w_n\rangle) \models \varphi \qquad \Leftrightarrow \qquad \mu \models \beta$$

Here, $\mu : X \to \{0,1\}$ is defined by $\mu(x_i^j) = 1$ iff $(\mathfrak{S}_i, w_i) \models \psi_i^j$.

We call $\mathsf{D} \overset{\text{def}}{=} (\Psi_1,\ldots,\Psi_d,\beta)$ the *decomposition* of φ and define $|\mathsf{D}| \overset{\text{def}}{=} |\beta| + \sum_{i,j} |\psi_i^j|$ to be its *size*. We note that in the same way one can define decompositions for the unimodal variant and for extensions of multimodal logic.

Note that Theorem 13 only holds in the presence of an interpretation σ for the atomic propositions since interpretations establish the connection between the product and component structures. We also mention that Theorem 13 has been proven in [13] for much more elaborated notions of interpretations. However, note that not every logic admits decomposition: For instance the property $\mathsf{EG}p$ meaning "there is a maximal path (a path is maximal if it is either infinite or ends in a dead-end) on which every world satisfies p" is not decomposable, as shown in [13].

An analogous theorem can be stated for first-order sentences, see [11] for a survey. We assume standard definitions concerning first-order logic. We will consider only relational signatures τ. For a finite set A of action labels and a finite set of propositions P we identify the pair (A,P) with the signature, where every $a \in \mathsf{A}$ has arity 2 and every $p \in \mathsf{P}$ has arity 1. This allows to consider Kripke structures as ordinary relational structures. In the following we consider decomposition theorems for finite variable fragments FO^k of first-order logic. A formula φ is in FO^k if at most k different variables occur in φ. Note that a formula, in which every subformula has at most k free variables is equivalent to an FO^k-formula.

Theorem 14 ([10])**.** *From an interpretation σ and an FO^k-sentence φ over the signature (A,P) with $\mathsf{A} = \biguplus_{i\in[1,d]}$, $\mathsf{P} = \biguplus_{i\in[1,d]} P_i$, one can compute a tuple $(\Psi_1,\ldots,\Psi_d,\beta)$ with $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ a finite set of FO^k-sentences over the signature $(\mathsf{A}_i,\mathsf{P}_i)$ and β a positive boolean formula with variables from $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ such that for every $(\mathsf{A}_i,\mathsf{P}_i)$-structure $\mathfrak{S}_i$ ($i \in [1,d]$):*

$$\prod_{i\in[1,d]}^{\sigma} \mathfrak{S}_i \models \varphi \qquad \Leftrightarrow \qquad \mu \models \beta.$$

Here, $\mu : X \to \{0,1\}$ is defined by $\mu(x_i^j) = 1$ iff $\mathfrak{S}_i \models \psi_i^j$.

Note that the proofs of both Theorem 13 and Theorem 14 yield decompositions of *nonelementary* size. In this section we provide matching lower bounds for Feferman-Vaught decompositions for multimodal $\mathbf{K}^d$ and FO^k for $k \geq 2$. Having enforced nonelementarily branching trees with small 2-dimensional unimodal formulas (Theorem 4) allows us to prove a nonelementary lower bound for the sizes of Feferman-Vaught decompositions for 2-dimensional unimodal logic. Without making this explicit in the statement, our lower bound is more general than the nonelementary lower bound for 2-dimensional unimodal logic from [17] in the following sense. We provide a family of small formulas which are "inherently hard to decompose": When assuming, by contradiction, the existence of small decompositions for our formulas, *any* model for them can be used to deduce the desired contradiction, whereas in [17] *appropriately chosen models* had to be defined for this. Our proof strategy is similar to the proof of Theorem 5.1 in [12].

Theorem 15. *Feferman-Vaught decompositions for unimodal logic w.r.t. asynchronous product are inherently nonelementary. More precisely, for every elementary function $f(n)$ there exists $\ell \geq 1$ such that the unimodal $\mathbf{K}^2$-formula $\varphi_{\ell,2}$ from Definition 3 has no decomposition D_ℓ in the sense of Theorem 13 with $|\mathsf{D}_\ell| \leq f(|\varphi_{\ell,2}|)$. The same lower bound holds when relativized to product structures $\mathfrak{T} \times \mathfrak{T}'$, where $\mathfrak{F}(\mathfrak{T})$ and $\mathfrak{F}(\mathfrak{T}')$ are finite trees.*

Proof. Assume by contradiction that there were an elementary function $f : \mathbb{N} \to \mathbb{N}$ such that for each $\ell \geq 1$ there is a decomposition $\mathsf{D}_\ell = (\Psi_1^{(\ell)}, \Psi_2^{(\ell)}, \beta_\ell)$ of $\varphi_{\ell,2}$ in the sense of Theorem 13 with $|\mathsf{D}_\ell| \leq f(|\varphi_{\ell,2}|)$. In particular, $|\beta_\ell| \leq f(|\varphi_{\ell,2}|)$. Since $|\varphi_{\ell,2}| \leq \exp(\ell)$ by Theorem 4(b), there exists an elementary function g such that $|\beta_\ell| \leq g(\ell)$ for all $\ell \geq 0$. Thus, there exists an $h_0 \geq 0$ with $2^{g(h-1)} < \mathsf{Tower}(h, 2)$ for all $h \geq h_0$; let us fix such an h_0.

By Theorem 4(a), $\varphi_{h,2}$ is id-satisfiable. Assume that $(\mathfrak{S} \times \mathfrak{S}', \langle w, w' \rangle) \vDash \varphi_{h_0,2}$ for some left pointed structure $(\mathfrak{S}, w)$ and some right pointed structure $(\mathfrak{S}', w')$. By Theorem 4(a), $(\mathfrak{S}, w)$ and $(\mathfrak{S}', w')$ are $(h_0, 2)$-treelike and $\mathsf{val}(\mathfrak{S}, w) = \mathsf{val}(\mathfrak{S}', w') = k$ for some $k \in [0, \mathsf{Tower}(h_0 + 1, 2) - 1]$. By the definition of $(h_0, 2)$-treelike structures, there exist for each $i \in [0, \mathsf{Tower}(h_0, 2) - 1]$ worlds $v_i \in N_{\mathfrak{S}}(w)$ and $v_i' \in N_{\mathfrak{S}'}(w')$ such that $(\mathfrak{S}, v_i)$ and $(\mathfrak{S}', v_i')$ are $(h_0 - 1, 2)$-treelike and $\mathsf{val}(\mathfrak{S}, v_i) = \mathsf{val}(\mathfrak{S}', v_i') = i$. Also note that

$$(\mathfrak{S} \times \mathfrak{S}', \langle v_i, v_j' \rangle) \vDash \varphi_{h_0-1,2} \iff i = j \tag{2}$$

for all $i, j \in [0, \mathsf{Tower}(h_0, 2) - 1]$. Consider our decomposition $\mathsf{D}_{h_0-1} = (\Psi_1^{(h_0-1)}, \Psi_2^{(h_0-1)}, \beta_{h_0-1})$ of $\varphi_{h_0-1,2}$. Assume that $\Psi_1^{(h_0-1)} = \{\psi_j \mid j \in J\}$ and $\Psi_2^{(h_0-1)} = \{\psi_j' \mid j \in J'\}$. Recall that β_{h_0-1} is a positive boolean formula with variables from $X = \{x_j \mid j \in J\} \cup \{x_j' \mid j \in J'\}$ and that $|\beta_{h_0-1}| \leq g(h_0 - 1)$. Hence, $|X| \leq g(h_0 - 1)$.

For each $r \in [0, \mathsf{Tower}(h_0, 2) - 1]$ we define a truth assignment $\mu_r : X \to \{0, 1\}$ as follows:

$$\begin{aligned} \mu_r(x_j) = 1 \quad &\Leftrightarrow \quad (\mathfrak{S}, v_r) \vDash \psi_j \\ \mu_r(x_j') = 1 \quad &\Leftrightarrow \quad (\mathfrak{S}', v_r') \vDash \psi_j' \end{aligned}$$

Since for β_{h_0-1} there are $2^{|X|} \leq 2^{g(h_0-1)} < \mathsf{Tower}(h_0, 2)$ many truth assignments, there exist $0 \leq r < s < \mathsf{Tower}(h_0, 2)$ with $\mu_r = \mu_s$. In other words, $(\mathfrak{S}, v_r) \vDash \psi_j$ iff $(\mathfrak{S}, v_s) \vDash \psi_j$ and $(\mathfrak{S}', v_r') \vDash \psi_j'$ iff $(\mathfrak{S}', v_s') \vDash \psi_j'$. By the definition of a Feferman-Vaught decomposition and the fact that $(\mathfrak{S} \times \mathfrak{S}', \langle v_r, v_r' \rangle) \vDash \varphi_{h_0-1,2}$, we obtain $(\mathfrak{S} \times \mathfrak{S}', \langle v_r, v_s' \rangle) \vDash \varphi_{h_0-1,2}$. But this contradicts (2).

Our lower bound also holds when only products of finite trees are allowed as models, since for every h, n, there exists an (h, n)-treelike structure $\mathfrak{S}$ such that $\mathfrak{F}(\mathfrak{S})$ is a finite tree (of height h). □

Note that the lower bound from Theorem 15 would even hold if we defined the size of a decomposition $(\Psi_1, \ldots, \Psi_d, \beta)$ as the size of the boolean formula β only (and not accounting for the sizes of the Ψ_i); the same proof works for this variant. In contrast to [17] the proof of Theorem 15 allows to derive nonelementary lower bounds on decompositions for any decomposable logic (in the sense of Theorem 13) that is at least as expressive as unimodal logic and only elementarily less succinct than unimodal logic.

Corollary 16. *Every logic that is at least as expressive as and at most elementary less succinct as unimodal logic does not have elementary sized Feferman-Vaught decompositions with respect to asynchronous product.*

Proof. We exemplarily provide the proof for FO^2 sentences. The proof for any other logic that satisfies the properties from Corollary 16 works analogously. It is well known that a unimodal logic formula φ can be translated (in polynomial time) into an equivalent FO^2-formula $\widehat{\varphi}(x)$ with one free variable. The family of FO^2-sentences that witnesses that there are no elementarily-sized decompositions is simply $\{\exists x : \widehat{\varphi_{\ell,2}}(x) \mid \ell \geq 1\}$. ◻

6 Feferman-Vaught decompositions for sum

So far, we only considered Feferman-Vaught decompositions for asynchronous products. Another important and natural operation on structures is the disjoint sum. Let us fix a relational signature τ and for $i \in [1,d]$ let $\mathfrak{S}_i = (D_i, \{P_{i,a} \mid a \in \tau\})$ be a τ-structure such that $D_i \cap D_j = \varnothing$ for $i \neq j$. Let $A_i \notin \tau$ be a fresh unary predicate symbol for each $i \in [1,d]$. The the *disjoint sum* $\sum_{i=1}^{d} \mathfrak{S}_i$ is the following structure over the signature $\tau \cup \{A_1, \ldots, A_d\}$:

$$\sum_{i=1}^{d} \mathfrak{S}_i \stackrel{\text{def}}{=} \Big(\bigcup_{i\in[1,d]} D_i, \{\bigcup_{i\in[1,d]} P_{i,a} \mid a \in \tau\} \cup \{D_i \mid i \in [1,d]\}\Big).$$

Here, $\bigcup_{i\in[1,d]} P_{i,a}$ is the interpretation for $a \in \tau$ and D_i is the interpretation for the fresh symbol A_i. Note that the fresh symbol A_i allows to recover the component structure $\mathfrak{S}_i$. In other words, we can express in FO over $\sum_{i=1}^{d} \mathfrak{S}_i$ that $\mathfrak{S}_i$ satisfied a certain FO-sentence. The following result is again classical [10, 11].

Theorem 17. *For every FO^k-sentence φ over the signature $\tau \uplus \{A_1, \ldots, A_d\}$ one can compute a tuple $(\Psi_1, \ldots, \Psi_d, \beta)$, where each $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ is a finite set of FO^k-sentences over the signature τ and where β is a positive boolean formula with variables from $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ such that for all τ-structures $\mathfrak{S}_1, \ldots, \mathfrak{S}_d$:*

$$\sum_{i=1}^{d} \mathfrak{S}_i \models \varphi \qquad \text{if and only if} \qquad \mu \models \beta.$$

Here, $\mu : X \to \{0,1\}$ is defined by: $\mu(x_i^j) = 1$ iff $\mathfrak{S}_i \models \psi_i^j$.

The following result is a simple corollary of Corollary 16.

Corollary 18. *For every $k \geq 3$, there is no elementary function f such that every FO^k-formula φ has a Feferman-Vaught decomposition w.r.t. disjoint sum of size $f(|\varphi|)$.*

Proof. Recall that the unimodal $\mathbf{K}^2$-formula $\varphi_{\ell,2}$ was defined over $(\{a,a'\},\mathsf{P}_2\cup\mathsf{Q}_2)$. For $q\in\mathsf{Q}_2$ let $\tilde{q}$ be the corresponding proposition in P_2 (e.g., $\tilde{q}_b=p_b$) and for a right structure $\mathfrak{S}'$ (i.e. over $(\{a'\},Q_2)$) let $\tilde{\mathfrak{S}}'$ be the corresponding left structure (i.e. over $(\{a\},P_2)$) . We translate the unimodal $\mathbf{K}^2$-formula $\varphi_{\ell,2}$ from above (see the proof of Theorem 15) into an FO^3-formula $\overline{\varphi_{\ell,2}}(x,x')$ with *two* free variables over the signature $\tau=\{a\}\cup\mathsf{P}_2\cup\{A_1,A_2\}$ inductively as follows:

- $\overline{p}(x,x')\overset{\text{def}}{=}p(x)$ for each $p\in\mathsf{P}_2$
- $\overline{q}(x,x')\overset{\text{def}}{=}\tilde{q}(x')$ for each $q\in\mathsf{Q}_2$
- $\overline{\neg\psi}(x,x')\overset{\text{def}}{=}\neg\overline{\psi}(x,x')$
- $\overline{\psi_1\wedge\psi_2}(x,x')\overset{\text{def}}{=}\overline{\psi_1}(x,x')\wedge\overline{\psi_2}(x,x')$
- $\overline{\ominus\psi}(x,x')\overset{\text{def}}{=}\exists y:(A_1(y)\wedge a(x,y)\wedge\overline{\psi}(y,x'))$
- $\overline{\oplus\psi}(x,x')\overset{\text{def}}{=}\exists y':(A_2(y')\wedge a(x',y')\wedge\overline{\psi}(x,y'))$

Note that this translation indeed yields an FO^3-formula because every subformula has at most three free variables.

Recall the FO^2 sentence $\widehat{\varphi_{\ell,2}}$ from the proof of Corollary 16. The reader can easily verify that for every left structure $\mathfrak{S}$ and every right structure $\mathfrak{S}'$ we have $\mathfrak{S}\times\mathfrak{S}'\vDash\exists x:\widehat{\varphi_{\ell,2}}(x)$ if and only if $\mathfrak{S}+\mathfrak{S}'\vDash\exists x\exists x':A_1(x)\wedge A_2(x')\wedge\overline{\varphi_{\ell,2}}(x,x')$. □

Corollary 18 raises the question whether even Feferman-Vaught decompositions for FO^2 w.r.t. disjoint sum become nonelementary. We give a negative answer to this question.

Theorem 19. *The following is computable in doubly exponential time:*
INPUT: An FO^2*-sentence* φ *over* $\tau\uplus\{A_1,\dots,A_d\}$.
OUTPUT: A decomposition $(\Psi_1,\dots,\Psi_d,\beta)$*, where* $\Psi_i=\{\psi_i^j\mid j\in J_i\}$ *is a finite set of* FO^2*-sentences over* τ *and* β *is a positive boolean formula with variables from* $X=\{x_i^j\mid i\in[1,d],j\in J_i\}$ *such that for all* τ*-structures* $\mathfrak{S}_1,\dots,\mathfrak{S}_d$*:*

$$\sum_{i=1}^{d}\mathfrak{S}_i\vDash\varphi\qquad\textit{if and only if}\qquad\mu\vDash\beta.$$

Here, $\mu:X\to\{0,1\}$ *is defined by:* $\mu(x_i^j)=1$ *iff* $\mathfrak{S}_i\vDash\psi_i^j$*.*

We will prove Theorem 19 only for the case $d=2$; the general case can be shown in the same way. Hence, let us fix a signature τ of relational symbols and let $A_1,A_2\notin\tau$ be two additional unary symbols. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be relational structures over the signature τ.

We define a partial order $\leq$ on the set of all first-order formulas by setting $\psi_1\leq\psi_2$ if ψ_1 is a subformula of ψ_2. For a formula φ we denote with $\mathcal{Q}_\varphi$ the set of all subformulas of φ that start with a quantifier. With $\mathcal{Q}_\varphi^{\mathsf{cl}}$ we denote the set of those formulas in $\mathcal{Q}_\varphi$ that are closed, i.e., do not have free variables. In a formula $\exists x:A_i(x)\wedge\psi$ (resp. $\forall x:A_i(x)\to\psi$), where $i\in\{1,2\}$, we say that x *is relativized to* A_i, and for better readability we write $\exists x\in A_i:\psi$ (resp. $\forall x\in A_i:\psi$) for that formula.

A formula φ over the signature $\tau\cup\{A_1,A_2\}$ is called *pure* if φ is a boolean combination of formulas $\varphi_1,\dots,\varphi_n$ such that for every $1\leq i\leq n$ there exists $j\in\{1,2\}$ such that for every

$(Qx:\psi) \in \mathcal{Q}_{\varphi_i}$ (where $Q \in \{\exists, \forall\}$), x is relativized to A_j in $Qx:\psi$. Equivalently, φ is pure, if for all $(Q_1x:\psi_1),(Q_2y:\psi_2) \in \mathcal{Q}_\varphi$ with $(Q_1x:\psi_1) \leq (Q_2y:\psi_2)$, x is relativized in $(Q_1x:\psi_1)$ to the same A_i as y in $(Q_2y:\psi_2)$. To prove Theorem 19 (for $d=2$), it suffices to transform an FO^2-sentence over the signature $\tau \cup \{A_1, A_2\}$ in doubly exponential time into an equivalent pure FO^2-sentence over the signature $\tau \cup \{A_1, A_2\}$.

A formula φ over the signature $\tau \cup \{A_1, A_2\}$ is called *almost pure* if it satisfies the following conditions:

- For all $(Qx:\psi) \in \mathcal{Q}_\varphi$, x is relativized in $(Qx:\psi)$ to either A_1 or A_2.
- If $(Q_1x:\psi_1),(Q_2y:\psi_2) \in \mathcal{Q}_\varphi$ with $(Q_1x:\psi_1) \leq (Q_2y:\psi_2)$, then x is relativized in $(Q_1x:\psi_1)$ to the same A_i as y in $(Q_2y:\psi_2)$, or there exists $\theta \in \mathcal{Q}^{\mathsf{cl}}_\varphi$ with $(Q_1x:\psi_1) \leq \theta \leq \psi_2$.

In other words, whenever a chain of subformulas $(Q_1x:\psi_1) \leq (Q_2y:\psi_2) \leq \varphi$ does not satisfy the pureness condition, then $(Q_1x:\psi_1)$ occurs within a proper subsentence of $(Q_2y:\psi_2)$ that moreover starts with a quantifier. Clearly, every pure formula is almost pure. Vice versa, we have:

Lemma 20. *From a given almost pure formula φ over the signature $\tau \cup \{A_1, A_2\}$ one can compute a logically equivalent pure formula φ' of size $2^{|\mathcal{Q}^{\mathsf{cl}}_\varphi|} \cdot O(|\varphi|)$. If φ is an FO^2-formula then φ' is an FO^2-formula as well.*

Proof. The idea is to replace the topmost occurrences of sentences from the set $\mathcal{Q}^{\mathsf{cl}}_\varphi$ by truth values in all possible ways in a big disjunction over all possible truth assignments. Since sentences from $\mathcal{Q}^{\mathsf{cl}}_\varphi$ may also violate the pureness condition, we have to iterate this replacement step.

Let φ be almost pure and let $\mathcal{F}$ be the set of all mappings from $\mathcal{Q}^{\mathsf{cl}}_\varphi \setminus \{\varphi\}$ to $\{\mathsf{true}, \mathsf{false}\}$. For $f \in \mathcal{F}$ and a formula θ let $\theta[f]$ be the formula that results from θ by replacing every $\leq$-maximal formula ψ from the set $\mathcal{Q}^{\mathsf{cl}}_\varphi \setminus \{\theta\}$ by the truth value $f(\psi)$. Then, we define φ' as the disjunction

$$\bigvee_{f \in \mathcal{F}} \Big(\varphi[f] \wedge \bigwedge_{\psi \in \mathcal{Q}^{\mathsf{cl}}_\varphi \setminus \{\varphi\}} (f(\psi) \leftrightarrow \psi[f])\Big).$$

Clearly, φ' is equivalent to φ and φ' is pure. Moreover, the size of the formula

$$\varphi[f] \wedge \bigwedge_{\psi \in \mathcal{Q}^{\mathsf{cl}}_\varphi \setminus \{\varphi\}} (f(\psi) \leftrightarrow \psi[f])$$

is in $O(|\varphi|)$ since the formulas $\varphi[f]$, $\psi[f]$ (for $\psi \in \mathcal{Q}^{\mathsf{cl}}_\varphi \setminus \{\varphi\}$) form a kind of partition of the whole formula ψ. Hence, the size of φ' is bounded by $2^{|\mathcal{Q}^{\mathsf{cl}}_\varphi|} \cdot O(|\varphi|)$. □

Lemma 21. *From a given FO^2-formula $\varphi(x)$ over the signature $\tau \cup \{A_1, A_2\}$ with at most one free variable x, one can compute FO^2-formulas $\varphi'(x)$ and $\varphi''(x)$ of size $2^{O(|\varphi|^2)}$ such that the following holds for all structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ over the signature τ.*

- *$Qx \in A_1 : \varphi'(x)$ and $Qx \in A_2 : \varphi''(x)$ are almost pure (where $Q \in \{\forall, \exists\}$).*
- *For all $a \in \mathfrak{S}_1$, $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi(a)$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi'(a)$.*
- *For all $a \in \mathfrak{S}_2$, $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi(a)$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi''(a)$.*

Moreover, $|\mathcal{Q}^{\mathsf{cl}}_{\varphi'(x)}| \in 2^{O(|\varphi|)}$ *and* $|\mathcal{Q}^{\mathsf{cl}}_{\varphi''(x)}| \in 2^{O(|\varphi|)}$.

Proof. Let us construct the formula $\varphi'(x)$ ($\varphi''(x)$ is constructed analogously) by induction over the structure of the formula $\varphi(x)$. For this, we assume that $\varphi(x)$ is in negation normal form, i.e., negations appear only in front of atomic formulas. The case when $\varphi(x)$ is quantifier-free is easy: simply replace every occurrence of $A_1(x)$ by true and every occurrence of $A_2(x)$ by false.

The case that the top-most operator in $\varphi(x)$ is a boolean operator is clear, e.g., set $(\varphi_1 \wedge \varphi_2)' = \varphi_1' \wedge \varphi_2'$.

Let us now assume that $\varphi(x) = \exists y : \psi(x,y)$. Since $\varphi(x)$ is an FO^2-formula, the formula $\psi(x,y)$ can be obtained from a positive boolean formula $B(p_1,\dots,p_k)$ by replacing every propositonal variable p_i by

(a) some $\alpha(x) \in \mathcal{Q}_\varphi$, where only x may occur freely, or by

(b) some $\beta(y) \in \mathcal{Q}_\varphi$, where only y may occur freely, or by

(c) a possibly negated atomic formula (i.e., a literal) that involves a subset of the variables $\{x,y\}$.

Let $\psi'(x,y)$ be the formula that results from $\psi(x,y)$ by replacing every subformula $\alpha(x)$ (resp. $\beta(y)$) of type (a) (resp. (b)) by $\alpha'(x)$ (resp. $\beta'(y)$). Since by induction, every formula $\exists x \in A_1 : \alpha'(x)$ and every formula $\exists y \in A_1 : \beta'(y)$ is almost pure, also $\exists x \in A_1 \exists y \in A_1 : \psi'(x,y)$ is almost pure.

We can write B as a DNF formula $B = \bigvee_{i=1}^r B_i$ of size $2^{O(|B|)}$, where every B_i is a conjunction of formulas of the types (a)–(c). Hence, we can write B_i as

$$B_i = \alpha_i(x) \wedge \beta_i(y) \wedge \gamma_i(x,y),$$

where α_i is a conjunction of type-(a) formulas, β_i is a conjunction of type-(b) formulas, and $\gamma_i(x,y)$ is a conjunction of type-(c) formulas.

Clearly, over a structure $\mathfrak{S}_1 + \mathfrak{S}_2$, the formula $\exists y : \psi(x,y)$ is equivalent to $\exists y \in A_1 : \psi(x,y) \vee \exists y \in A_2 : \psi(x,y)$, i.e., to

$$\exists y \in A_1 : \psi(x,y) \ \vee \ \bigvee_{i=1}^{r} \exists y \in A_2 : (\alpha_i(x) \wedge \beta_i(y) \wedge \gamma_i(x,y)).$$

By induction, for all $x \in \mathfrak{S}_1$, this formula is equivalent to

$$\exists y \in A_1 : \psi'(x,y) \ \vee \ \bigvee_{i=1}^{r} \exists y \in A_2 : (\alpha_i'(x) \wedge \beta_i''(y) \wedge \gamma_i(x,y)).$$

In this formula, every occurrence of a literal in $\gamma_i(x,y)$, in which both x and y occur, can be replaced either by true (if the literal is negative) or false (if the literal is positive). The reason for this is that no atomic relations of $\mathfrak{S}_1 + \mathfrak{S}_2$ involve both elements of $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Clearly, if a literal in $\gamma_i(x,y)$ is replaced by false then we can remove the whole disjunct $\exists y \in A_2 : (\alpha_i'(x) \wedge \beta_i''(y) \wedge \gamma_i(x,y))$; let us assume that this occurs for $q+1 \le i \le r$. We therefore obtain an equivalent formula of the form

$$\exists y \in A_1 : \psi'(x,y) \ \vee \ \bigvee_{i=1}^{q} (\alpha_i'(x) \wedge \delta_{i,1}(x) \wedge \exists y \in A_2 : (\beta_i''(y) \wedge \delta_{i,2}(y))).$$

Here $\delta_{i,1}(x)$ (resp. $\delta_{i,2}(y)$) is the conjunction of all literals in $\gamma_i(x,y)$ that only involve the variable x (resp. y). Let $\varphi'(x)$ be the above formula. We have to show that the formula

$$\exists x \in A_1 \left(\exists y \in A_1 : \psi'(x,y) \ \vee \ \bigvee_{i=1}^{q} (\alpha_i'(x) \wedge \delta_{i,1}(x) \wedge \exists y \in A_2 : (\beta_i''(y) \wedge \delta_{i,2}(y))) \right)$$

is almost pure. This follows inductively from the fact that $\exists x \in A_1 \exists y \in A_1 : \psi'(x,y)$, $\exists x \in A_1 : \alpha_i'(x)$, and $\exists y \in A_2 : \beta_i''(y)$ are almost pure, and the fact that $\exists y \in A_2 : (\beta_i''(y) \wedge \delta_{i,2}(y))$ is closed. This concludes the case $\varphi(x) = \exists y : \psi(x,y)$. The case $\varphi(x) = \forall y : \psi(x,y)$ can be treated analogously.

If we allow $\wedge$'s and $\vee$'s of arbitrary width, then the depth (i.e., the height of the syntax tree) of $\varphi'(x)$ is bounded by $O(|\varphi|)$. Due to forming CNFs and DNFs, the width of $\wedge$'s and $\vee$'s can be bounded by $2^{|\varphi(x)|}$. Hence, the syntax tree of $\varphi'(x)$ has height $O(|\varphi|)$ and branching degree $2^{|\varphi(x)|}$, and therefore has $2^{O(|\varphi|^2)}$ nodes. Replacing $\wedge$'s and $\vee$'s of arbitrary width $\le 2^{|\varphi(x)|}$ by 2-ary $\wedge$'s and $\vee$'s only multiplies the number of nodes by $2^{|\varphi(x)|}$. Hence, $\varphi'(x)$ is of size $2^{O(|\varphi|^2)}$.

For the bound $|\mathcal{Q}^{\mathsf{cl}}_{\varphi'(x)}| \in 2^{O(|\varphi|)}$ note that in the above construction, the number of closed subformulas that start with a quantifier is increased by at most $q+1 \le r+1$ (due to the formulas $\exists y \in A_2 : (\beta_i''(y) \wedge \delta_{i,2}(y))$ for $i \in [1,q]$ and possibly $\exists y \in A_1 : \psi'(x,y)$). Since r is exponential in the size of the boolean formula B, the bound $|\mathcal{Q}^{\mathsf{cl}}_{\varphi'(x)}| \in 2^{O(|\varphi|)}$ follows. □

Theorem 22. *From a given closed FO^2-formula φ over the signature $\tau \cup \{A_1, A_2\}$ one can compute a pure closed FO^2-formula ψ of size $2^{2^{O(|\varphi|)}}$ such that for all structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ over the signature τ, $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \psi$.*

Proof. We first apply Lemma 21 to φ and obtain a closed almost pure FO^2-formula θ such that $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \theta$. The size of θ is bounded by $2^{O(|\varphi|^2)}$. Finally, we apply Lemma 20 to θ and obtain an equivalent pure FO^2-formula ψ of size $2^{|\mathcal{Q}^{\mathsf{cl}}_\theta|} \cdot O(|\theta|)$. Since $|\theta| \in 2^{O(|\varphi|^2)}$ and $|\mathcal{Q}^{\mathsf{cl}}_\theta| \in 2^{O(|\varphi|)}$ this yields the upper bound $2^{2^{O(|\varphi|)}}$ for the size of ψ. □

Let us conclude this section with a (non-matching) lower bound on Feferman-Vaught decompositions for FO^2.

Proposition 23. *There is no function $f(n) \in o(\sqrt{n})$ and $c > 1$ such that every FO^2-formula φ has a Feferman-Vaught decompositions w.r.t. disjoint sum of size $c^{f(|\varphi|)}$.*

Proof. Let us define the family of unary predicate symbols $\mathsf{P}_n = \{p_0, \dots, p_{n-1}, p_{\mathsf{bit}}\}$ and $\tau_n = \mathsf{P}_n \cup \{A_1, A_2\}$ for each $n \ge 0$.

One can define a family of FO^2-sentences $\{\varphi_n \mid n \ge 0\}$, where each formula φ_n is defined over the signature τ_n that has precisely models of the form $\mathfrak{S}_1 + \mathfrak{S}_2$, where

- $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are both P_n-structures,
- $\mathfrak{S}_1$ has precisely 2^n elements $u_0, \dots, u_{2^n-1}$,
- $\mathfrak{S}_1 \models p_j(u_i)$ if and only if the j^{th} least significant bit of the binary representation of i is 1 (where $j \in [0, n-1]$),
- $\mathfrak{S}_2$ has precisely 2^n elements $v_0, \dots, v_{2^n-1}$,

- $\mathfrak{S}_2 \models p_j(u_i)$ if and only if the j^{th} least significant bit of the binary representation of i is 1, and
- $\mathfrak{S}_1 \models p_{\mathsf{bit}}(u_i)$ if and only if $\mathfrak{S}_2 \models p_{\mathsf{bit}}(v_i)$ for every $i \in [0, 2^n - 1]$.

It is a standard exercise to construct FO^2-formulas φ_n of size $O(n^2)$ that realize the above-mentioned properties. Moreover, we can assign both to $\mathfrak{S}_1$ and to $\mathfrak{S}_2$ a number in $[0, 2^{2^n} - 1]$ by simply interpreting the 2^n worlds as positions of a binary string of length 2^n. Formally, let $b_i \in \{0, 1\}$ for $i \in [0, 2^n - 1]$, where $b_i = 1$ iff $\mathfrak{S}_1 \models p_{\mathsf{bit}}(u_i)$ (resp. $\mathfrak{S}_2 \models p_{\mathsf{bit}}(v_i)$). We define

$$\mathsf{val}(\mathfrak{S}_j) \stackrel{\text{def}}{=} \sum_{i=0}^{2^n-1} b_i 2^i \in [0, 2^{2^n} - 1].$$

for each $j \in \{1, 2\}$. Recall that formula φ_n enforces $\mathsf{val}(\mathfrak{S}_1) = \mathsf{val}(\mathfrak{S}_2)$. Also note that conversely for each $i \in [0, 2^{2^n} - 1]$ there is a unique P_n-structure $\mathfrak{S}_{1,i}^{(n)}$ and a unique P_n-structure $\mathfrak{S}_{2,i}^{(n)}$ such that $\mathfrak{S}_{1,i}^{(n)} + \mathfrak{S}_{2,i}^{(n)} \models \varphi_n$ and $\mathsf{val}(\mathfrak{S}_{1,i}^{(n)}) = \mathsf{val}(\mathfrak{S}_{2,i}^{(n)}) = i$. In fact, we have

$$\mathfrak{S}_{1,i}^{(n)} + \mathfrak{S}_{2,j}^{(n)} \models \varphi_n \iff i = j. \tag{3}$$

Assume by contradiction that there were some $c > 1$, a function $f(n) \in o(\sqrt{n})$, and for every $n \geq 1$ a decomposition

$$\mathsf{D}_n = (\Psi^{(n)}, \Theta^{(n)}, \beta_n)$$

where

- each $\Psi^{(n)} = \{\psi_j^{(n)} \mid j \in J_n\}$ is a finite set of FO^2 sentences over the signature P_n,
- each $\Theta^{(n)} = \{\theta_h^{(n)} \mid h \in H_n\}$ is a finite set of FO^2 sentences over the signature P_n,
- β_n is a positive boolean formula with variables $\{x_j^{(n)} \mid j \in J_n\} \cup \{y_h^{(n)} \mid h \in H_n\}$, and
- $|\mathsf{D}_n| \leq c^{f(|\varphi_n|)} \leq c^{f(O(n^2))}$.

such that for every two P_n-structure $\mathfrak{S}_1$ and $\mathfrak{S}_2$ we have

$$\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi_n \qquad \Leftrightarrow \qquad \mu \models \beta_n.$$

Here, μ assigns variables of β_n as follows:

- $\mu(x_j^{(n)}) = 1$ if and only if $\mathfrak{S}_1 \models \psi_j^{(n)}$ and
- $\mu(y_h^{(n)}) = 1$ if and only if $\mathfrak{S}_2 \models \theta_h^{(n)}$.

Note that the number of variables of β_n is bounded by $c^{f(O(n^2))}$. Since $f(n) \in o(\sqrt{n})$ (and thus $f(dn^2) \in o(n)$ for every constant d) there exists an n such that the number of variables of β_n is strictly smaller than by 2^n. Let us fix this n in the following consideration.

For $i \in [0, 2^{2^n} - 1]$ define the truth assignment $\mu_i^{(n)}$ as follows:

- $\mu_i^{(n)}(x_j^{(n)}) \stackrel{\text{def}}{=} 1$ if and only if $\mathfrak{S}_{1,i}^{(n)} \models \psi_j^{(n)}$ and
- $\mu_i^{(n)}(y_h^{(n)}) \stackrel{\text{def}}{=} 1$ if and only if $\mathfrak{S}_{2,i}^{(n)} \models \theta_h^{(n)}$.

Since there are strictly less than 2^{2^n} truth assignments for β_n (by the choice of n), there exist $i < j$ such that $\mu_i^{(n)} = \mu_j^{(n)}$. Since $\mathfrak{S}_{1,i}^{(n)} + \mathfrak{S}_{2,i}^{(n)} \models \varphi_n$ we must have $\mathfrak{S}_{1,i}^{(n)} + \mathfrak{S}_{2,j}^{(n)} \models \varphi_n$ as well. Hence $i = j$ by (3), which is a contradiction. □

7 Gaifman normal form

Our technique from the proof of Theorem 19 can be used to prove a doubly exponential upper bound on the size (and construction) of Gaifman normal forms [14]. Let us start with a few definitions.

Let $\mathfrak{S} = (D, \{P_a \mid a \in \tau\})$ be a structure over a relational signature τ. The *Gaifman graph* of $\mathfrak{S}$ is the undirected graph $G(\mathfrak{S}) = (D, E)$, where the edge relation E contains a pair $(u,v) \in D \times D$ with $u \neq v$ if and only if there exists a relation P_a ($a \in \tau$) of arity say n and a tuple $(u_1, \ldots, u_n) \in P_a$ such that $u, v \in \{u_1, \ldots, u_n\}$. For $u, v \in D$, the distance $d_{\mathfrak{S}}(u,v)$ is the length (number of edges) of a shortest path from u to v in $G(\mathfrak{S})$. For a tuple $\overline{u} = (u_1, \ldots, u_n) \in D^n$ and $v \in D$, let $d_{\mathfrak{S}}(\overline{u}, v) = \min\{d_{\mathfrak{S}}(u_i, v) \mid 1 \leq i \leq n\}$. For $n \in \mathbb{N}$, the *n-sphere* around $\overline{u}$ is $S_{\mathfrak{S},n}(\overline{u}) = \{v \in D \mid d_{\mathfrak{S}}(\overline{u}, v) \leq n\}$. We write $S_n(\overline{u})$ for $S_{\mathfrak{S},n}(\overline{u})$, if $\mathfrak{S}$ is clear from the context.

Note that for every $n \in \mathbb{N}$, there exists a first-order formula $d_n(\overline{x}, y)$ such that for all structures $\mathfrak{S}$ and all elements $\overline{u}, v$ of $\mathfrak{S}$, $\mathfrak{S} \models d_n(\overline{u}, v)$ if and only if $d_{\mathfrak{S}}(\overline{u}, v) \leq n$. For better readability, we write $d(\overline{x}, y) \leq n$ instead of $d_n(\overline{x}, y)$. The formula $d(\overline{x}, y) > n$ should be understood similarly. In a formula of the form $\exists y : d(\overline{x}, y) \leq r \wedge \psi$ or $\forall y : d(\overline{x}, y) \leq r \rightarrow \psi$, we say that the variable y is *relativized* to $S_r(\overline{x})$. A formula φ is called *r-local around $\overline{x}$* if for every subformula $(Qy : \psi) \in \mathcal{Q}_\varphi$, the variable y is relativized in $(Qy : \psi)$ to a sphere $S_q(\overline{x})$ for some $q \leq r$. A sentence ψ is called an *r-local Gaifman-sentence* if it is of the form

$$\exists x_1, \ldots, x_n : \bigwedge_{1 \leq i < j \leq n} d(x_i, x_j) > 2q \wedge \bigwedge_{1 \leq i \leq n} \varphi(x_i),$$

where $\varphi(x_i)$ is q-local around (the single variable) x_i for some $q \leq r$.

Theorem 24 (Gaifman's theorem [14])**.** *Every first-order formula $\varphi(\overline{x})$ is equivalent to a boolean combination $\psi(\overline{x})$ of r-local formulas around $\overline{x}$ and q-local Gaifman-sentences for suitable r and q (that are exponential in the size of $\varphi(\overline{x})$).*

We call the formula $\psi(\overline{x})$ from Theorem 24 a *Gaifman normal form* for $\varphi(\overline{x})$. In [12] it was shown that (for FO^4-formulas already) the size of equivalent formulas in Gaifman normal form cannot be bounded elementarily. By using our formulas $\varphi_{\ell,n}$ from Section 3 and analogous ideas as in [12], we can strengthen the latter result to FO^3.

Proposition 25. *There is no elementary function f such that every FO^3-formula φ has an equivalent formula in Gaifman normal form of size $f(|\varphi|)$.*

Proof (sketch). We only give a sketch of the proof because the overall proof strategy is very similar to the proof of Theorem 4.2 in [12].

Recall that the unimodal $\mathbf{K}^2$-formula $\varphi_{\ell,2}$ was defined over $(\{a, a'\}, \mathsf{P}_2 \cup \mathsf{Q}_2)$. Recall the translation of the $\mathbf{K}^2$-formula $\varphi_{\ell,2}$ into the FO^3-formula $\overline{\varphi_{\ell,2}}$ over the signature $\tau = \{a\} \cup \mathsf{P}_2 \cup \{A_1, A_2\}$ from the proof of Corollary 18.

We define a τ-structure $\mathfrak{S}$ to be a *left $(\ell, 2)$-tree* if

- $\mathfrak{S}$ is a tree rooted at some element s such that $(\mathfrak{S}, s)$ satisfies the previous definition of $(\ell, 2)$-treelike structures over the signature $\{a\} \cup \mathsf{P}_2$,
- the unary predicate A_1 holds everywhere, and
- the unary predicate A_2 holds nowhere.

We define a τ-structure $\mathfrak{S}$ to be a *right* $(\ell, 2)$*-tree* if

- $\mathfrak{S}$ is a tree rooted at some element s such that $(\mathfrak{S}, s)$ satisfies the previous definition of $(\ell, 2)$-treelike structures over the signature $\{a\} \cup \mathsf{P}_2$,
- the unary predicate A_2 holds everywhere, and
- the unary predicate A_1 holds nowhere.

For each $i \in [0, \mathsf{Tower}(\ell + 1, 2) - 1]$, let us fix an arbitrary left $(\ell, 2)$-tree $(\mathfrak{S}_i, s_i)$ with $\mathsf{val}(\mathfrak{S}_i, s_i) = i$ and an arbitrary right $(\ell, 2)$-tree $(\mathfrak{S}'_i, s'_i)$ with $\mathsf{val}(\mathfrak{S}'_i, s'_i) = i$.

Consider the structure

$$\mathfrak{F}_\ell \quad \stackrel{\text{def}}{=} \quad \biguplus_{i \in [0, \mathsf{Tower}(\ell+1,2)-1]} \mathfrak{S}_i \uplus \mathfrak{S}'_i,$$

where $\uplus$ denotes disjoint union. For each $\ell \geq 1$ let us define the FO^3-formula φ_ℓ as the conjunction of the following two formulas:

- $\exists x, x' : A_1(x) \wedge A_2(x') \wedge \overline{\varphi_{\ell,2}}(x, x') \wedge \overline{\mathsf{first}_{\ell,2}}(x, x')$
- $\forall x, x' : \Big(A_1(x) \wedge A_2(x') \wedge \overline{\varphi_{\ell,2}}(x, x') \rightarrow$
$$\overline{\mathsf{last}_{\ell,2}}(x, x') \vee \exists x' : (\overline{\mathsf{succ}_{\ell,2}}(x, x') \wedge \exists x : \overline{\varphi_{\ell,2}}(x, x'))\Big)$$

Let us interpret the formula φ_ℓ on the structure $\mathfrak{F}_\ell$. The two $(\ell, 2)$-trees $\mathfrak{S}_0$ and $\mathfrak{S}'_0$ (each of value 0) witness the first conjunct of the formula φ_ℓ. The second conjunct of φ_ℓ holds in $\mathfrak{F}_\ell$ since for each left $(\ell, 2)$-tree $\mathfrak{S}_i$ (recall that $\mathfrak{S}_i$ has value $i \in [0, \mathsf{Tower}(\ell + 1, 2) - 1]$), we either have $i = \mathsf{Tower}(\ell + 1, 2) - 1$ or there is some right tree $\mathfrak{S}'$ with value $i + 1$ in $\mathfrak{F}_\ell$, namely $\mathfrak{S}' = \mathfrak{S}'_{i+1}$.

For each $i \in [0, \mathsf{Tower}(\ell + 1, 2) - 1]$ let $\mathfrak{F}_\ell^{-i}$ be the τ-structure that one obtains from $\mathfrak{F}_\ell$ by entirely removing $\mathfrak{S}'_i$ from it. Note that $\mathfrak{F}_\ell \models \varphi_\ell$, but $\mathfrak{F}_\ell^{-i} \not\models \varphi_\ell$ for every $i \in [0, \mathsf{Tower}(\ell+1, 2)-1]$.

Assume now that there is an elementary function f such that every formula φ_ℓ has an equivalent formula ψ_ℓ in Gaifman normal form of size at most $f(|\varphi_\ell|)$. Note that $f(|\varphi_\ell|)$ is elementarily bounded in ℓ. Hence, there exists ℓ such that $f(|\varphi_\ell|) < \mathsf{Tower}(\ell + 1, 2)$. Since $\mathfrak{F}_\ell \models \varphi_\ell$, we also have $\mathfrak{F}_\ell \models \psi_\ell$. We can now prove in exactly the same way as in the proof of Theorem 4.2 in [12] that there must exist $i \in [0, \mathsf{Tower}(\ell+1, 2)-1]$ with $\mathfrak{F}_\ell^{-i} \models \psi_\ell$, i.e., $\mathfrak{F}_\ell^{-i} \models \varphi_\ell$, which is a contradiction. □

Next, we show that for the fragment FO^2 such an elementary (in fact, doubly exponential) bound is possible: The *quantifier rank* of a first-order formula φ is the maximal nesting depth of quantifiers in φ; it is denoted by $\mathsf{qr}(\varphi)$.

Theorem 26. *Every* FO^2*-formula* $\varphi(x)$ *is equivalent to a boolean combination* $\psi(x)$ *of* r*-local formulas around* x *and* q*-local Gaifman-sentences with* $r \leq 3\mathsf{qr}(\varphi)$, $q \leq 6\mathsf{qr}(\varphi)$, *and* $|\psi| \leq 2^{2^{O(|\varphi|)}}$.

In Theorem 26, x is a single variable. This is no restriction, since every FO^2-formula can be written as a boolean combination of formulas that (i) start with a quantifier, and (ii) that have at most one free variable. In the rest of this section, all r-local formulas will be

r-local around a single variable x. For the proof of Theorem 26 it is useful to define *almost r-local formulas* around x and *almost r-local Gaifman-sentences.* We do this by simultaneous induction:

- Every formula that is built up from atomic formulas and almost p-local Gaifman-sentences (for arbitrary p) using boolean operators and quantifiers relativized to $S_q(x)$ for arbitrary $q \le r$ is an almost r-local formula around x (hence, every r-local formula around x is almost r-local around x).
- If for some $q \le r$ every formula $\varphi_i(x_i)$ is almost q-local around x_i ($1 \le i \le n$), then the sentence
$$\exists x_1, \dots, x_n : \bigwedge_{1 \le i < j \le n} d(x_i, x_j) > 2q \wedge \bigwedge_{1 \le i \le n} \varphi(x_i) \tag{4}$$
is an almost r-local Gaifman-sentence.

For a formula φ, let $G(\varphi)$ be the set of all almost p-local Gaifman-sentences ψ (for arbitrary p) with $\psi \preceq \varphi$.

Lemma 27. *From an almost r-local formula $\varphi(x)$ (around x) one can compute a logically equivalent Boolean combination $\varphi'(x)$ of r-local formulas around x and q-local Gaifman sentences. Here, the size of $\varphi'(x)$ is bounded by $2^{|G(\varphi)|} \cdot O(|\varphi|)$ and q is the maximum of all p such that $G(\varphi)$ contains an almost p-local Gaifman sentence.*

Proof. Let $\varphi(x)$ be almost r-local around x and let $\mathcal{F}$ be the set of all mappings from $G(\varphi)$ to $\{\mathsf{true}, \mathsf{false}\}$. For $f \in \mathcal{F}$ and a formula θ let $\theta[f]$ be the formula that results from θ by replacing every $\preceq$-maximal formula ψ from the set $G(\varphi) \setminus \{\theta\}$ by the truth value $f(\psi)$. Then, we define φ' as the disjunction
$$\bigvee_{f \in \mathcal{F}} \Big(\varphi[f] \wedge \bigwedge_{\psi \in G(\varphi)} (f(\psi) \leftrightarrow \psi[f]) \Big).$$
Clearly, φ' is equivalent to φ and φ' is r-local around x. □

Lemma 28. *From an FO^2-formula $\varphi(x)$ with at most one free variable x, one can compute an equivalent almost r-local formula $\varphi^\ell(x)$ of size $2^{O(|\varphi|^2)}$ with $r \le 3\mathsf{qr}(\varphi)$, $|G(\varphi^\ell)| \le 2^{O(|\varphi|)}$, and every $\psi \in G(\varphi^\ell)$ is an almost $2r$-local Gaifman sentence.*

Proof. We prove the lemma by induction over the structure of the formula $\varphi(x)$. The case that the top-most operator in $\varphi(x)$ is a boolean operator is clear, e.g., set $(\varphi_1 \wedge \varphi_2)^\ell = \varphi_1^\ell \wedge \varphi_2^\ell$.

Now, assume that $\varphi(x) = \exists y : \psi(x, y)$. Since $\varphi(x)$ is an FO^2-formula, the formula $\psi(x, y)$ can be obtained from a positive boolean formula $B(p_1, \dots, p_k)$ by replacing every propositonal variable p_i by

(a) a formula $\alpha(x) \in \mathcal{Q}_\varphi$, which may only contain x freely, or by

(b) a formula $\beta(y) \in \mathcal{Q}_\varphi$, which may only contain y freely, or by

(c) a possibly negated atomic formula (i.e., a literal) that involves a subset of the variables $\{x, y\}$.

Inductively, we replace each of the formulas $\alpha(x)$ and $\beta(y)$ in (a) and (b), respectively, by $\alpha^\ell(x)$ and $\beta^\ell(y)$, respectively. These formulas are almost r-local with $r \le 3\mathsf{qr}(\psi)$. Let us denote the resulting formula by $\exists y : \psi^\ell(x,y)$ It is clearly equivalent to

$$\exists y : (d(x,y) \le 1 \wedge \psi^\ell(x,y)) \ \vee \ \exists y : (d(x,y) \ge 2 \wedge \psi^\ell(x,y)).$$

The formula $\exists y : (d(x,y) \le 1 \wedge \psi^\ell(x,y))$ can be transformed into an almost $(r+1)$-local formula around x. To see this, note that $d(x,y) \le 1$ implies that every quantification that is relativized to $S_r(y)$ can be replaced by a quantification that is relativized to $S_{r+1}(x)$, see also [14]. We will use this argument several times below.

So, let us concentrate on the second formula $\exists y : (d(x,y) \ge 2 \wedge \psi^\ell(x,y))$. We transform the boolean formula $B(p_1,\dots,p_k)$ into disjunctive normal form. Hence, for $\exists y : (d(x,y) \ge 2 \wedge \psi^\ell(x,y))$ we obtain an equivalent formula of the form

$$\exists y : (d(x,y) \ge 2 \wedge \bigvee_{i=1}^{p} (\alpha_i(x) \wedge \beta_i(y) \wedge \gamma_i(x,y))),$$

where

- $\alpha_i(x)$ is a conjunction of formulas $\alpha^\ell(x)$, where $\alpha(x)$ is of type (a),
- $\beta_i(y)$ is a conjunction of formulas $\beta^\ell(y)$, where $\beta(y)$ is of type (b), and
- $\gamma_i(x,y)$ is a conjunction of literals in the free variables x and y.

Note that all $\alpha_i(x)$ and $\beta_i(y)$ are almost r-local. Since we assume that $d(x,y) \ge 2$, every occurrence of an atomic formula in $\gamma_i(x,y)$, in which both x and y occurs, can be replaced by false. We thus obtain an equivalent formula of the form

$$\exists y : (d(x,y) \ge 2 \wedge \bigvee_{i=1}^{p} (\alpha_i(x) \wedge \beta_i(y) \wedge \delta_{i,1}(x) \wedge \delta_{i,2}(y))).$$

Here $\delta_{i,1}(x)$ (resp. $\delta_{i,2}(y)$) is the conjunction of all literals in $\gamma_i(x,y)$ that only involve the variable x (resp. y). The above formula is equivalent to

$$\bigvee_{i=1}^{p} (\alpha_i(x) \ \wedge \ \delta_{i,1}(x) \ \wedge \ \exists y : (d(x,y) \ge 2 \wedge \beta_i(y) \wedge \delta_{i,2}(y))).$$

The formulas $\alpha_i(x) \wedge \delta_{i,1}(x)$ are almost r-local around x. So, let us concentrate on the formulas $\exists y : (d(x,y) \ge 2 \wedge \beta_i(y) \wedge \delta_{i,2}(y))$. Let us consider a specific such formula and let us just write

$$\exists y : (d(x,y) \ge 2 \wedge \theta(y)) \tag{5}$$

for it, where $\theta(y)$ is almost r-local around y. Consider the sentence

$$\begin{aligned} \rho \ = \ & \exists x_1, x_2 : (d(x_1,x_2) \ge 2r+1 \wedge \theta(x_1) \wedge \theta(x_2)) \ \wedge \\ & \exists x_1, x_2 : (3 \le d(x_1,x_2) \le 2r \wedge \theta(x_1) \wedge \theta(x_2)). \end{aligned}$$

The part in the first line is an almost r-local Gaifman-senctence. The part in the second line can be rewritten as

$$\exists z : (\exists x_1, x_2 \in S_r(z) : d(x_1,x_2) \ge 3 \wedge \theta(x_1) \wedge \theta(x_2)). \tag{6}$$

This sentence is an almost $2r$-local Gaifman-sentence (with $n = 1$ in (4)): Since $\theta(y)$ is almost r-local around y, the formula $\exists x_1, x_2 \in S_r(z) : d(x_1, x_2) \geq 3 \wedge \theta(x_1) \wedge \theta(x_2)$ is almost $2r$-local around z.

Hence, ρ is the conjunction of an almost r-local Gaifman-senctence and an almost $2r$-local Gaifman-sentence, which states that there exist two elements with distance at least 3 that satisfy θ.

We claim that the formula in (5) is equivalent to the following almost $(r+3)$-local formula around x:

$$(\exists z : \theta(z)) \;\wedge\; \Big(\neg(\exists z \in S_1(x) : \theta(z)) \;\vee\; \rho \;\vee\; (\exists z \in S_3(x) : d(x,z) \geq 2 \wedge \theta(z))\Big).$$

Let us first assume that there exists y with $d(x,y) \geq 2$ and $\theta(y)$. Hence, $\exists z : \theta(z)$ holds. Moreover, assume that $\exists z \in S_1(x) : \theta(z)$ and $\neg\rho$ holds. We have to show that $\exists z \in S_3(x) : d(x,z) \geq 2 \wedge \theta(z)$ holds. Since we have $\neg\rho$, all elements that satisfy the formula θ have pairwise distance at most 2. Since there is an element of distance at most 1 from x that satisfies θ, the element y with $d(x,y) \geq 2$ and $\theta(y)$ has distance at most 3 from x.

For the other direction, assume that $\exists z : \theta(z)$ and one of $\neg(\exists z \in S_1(x) : \theta(z))$, ρ, or $\exists z \in S_3(x) : d(x,z) \geq 2 \wedge \theta(z)$ holds. We have to show that there exists y with $d(x,y) \geq 2$ and $\theta(y)$. The case that $\neg(\exists z \in S_1(x) : \theta(z))$ or $\exists z \in S_3(x) : d(x,z) \geq 2 \wedge \theta(z)$ holds is clear. If ρ holds, then there exist two elements with distance at least 3 that satisfy θ. Since two elements in $S_1(x)$ have distance at most 2, there must exist $y \notin S_1(x)$ satisfying θ.

We have shown that $\varphi(x) = \exists y : \psi(x,y)$ is equivalent to an almost $(r+3)$-local formula around x (with $r \leq \mathsf{qr}(\psi)$), which we can take for $\varphi^\ell(x)$. Hence, $\varphi^\ell(x)$ is indeed almost r'-local for some $r' \leq 3\mathsf{qr}(\varphi)$. Moreover, every sentence in $G(\varphi^\ell(x))$ is an almost $2r'$-local Gaifman sentence (the factor 2 comes from the formula (6)).

In order to bound the size of $\varphi^\ell(x)$, note that the depth of $\varphi^\ell(x)$ is bounded by $O(|\varphi|)$ if we allow $\wedge$'s and $\vee$'s of arbitrary width. Since the width can be bounded by $2^{|\varphi(x)|}$, the size of $\varphi^\ell(x)$ can be bounded by $2^{O(|\varphi|^2)}$, see the proof of Lemma 21 for an analogous argument.

For the bound $|G(\varphi^\ell)| \leq 2^{O(|\varphi|)}$ note that in the above construction, the number of almost local Gaifman sentences that are introduced is bounded by $O(p)$. Since p is exponential in the size of the boolean formula B, the bound $|G(\varphi^\ell)| \leq 2^{O(|\varphi|)}$ follows. Again, see the proof of Lemma 21 for an analogous argument. □

Let us finally prove Theorem 26. We first apply Lemma 28 to $\varphi(x)$ and obtain an equivalent almost r-local formula $\theta(x)$ with $|\theta| \leq 2^{O(|\varphi|^2)}$. Moreover $r \leq 3\mathsf{qr}(\varphi)$ and every sentence in $G(\theta)$ is an almost $2r$-local Gaifman sentence. Finally, we apply Lemma 27 to θ and obtain an equivalent Boolean combination $\psi(x)$ of r-local formulas around x and $2r$-local Gaifman sentences. The size of $\psi(x)$ is bounded by $2^{|G(\theta)|} \cdot O(|\theta|)$. Since $|\theta| \leq 2^{O(|\varphi|^2)}$ and $|G(\theta)| \leq 2^{O(|\varphi|)}$, this yields the upper bound $2^{2^{O(|\varphi|)}}$ for the size of $\psi(x)$. □

Finally, we give a (non-matching) lower bound on the size of equivalent formulas in Gaifman normal form for FO^2; the proof is again based on techniques from [12].

Proposition 29. *There is no function $f(n) \in o(\sqrt{n})$ and $c > 1$ such that every FO^2-formula φ has an equivalent formula in Gaifman normal form of size $c^{f(|\varphi|)}$.*

Proof. This proof uses a very similar strategy as in [12]. Let us define the signature $\tau_n = \{p_0, \ldots, p_{n-1}\}$ of solely unary predicate symbols for each $n \geq 1$. For each $n \geq 1$ it is standard to define an FO^2-sentence φ_n of size $O(n^2)$ such that there is a unique τ_n-structure $\mathfrak{S}_n$ with $\mathfrak{S}_n \vDash \varphi_n$, where $\mathfrak{S}_n$ satisfies the following properties:

- $\mathfrak{S}_n$ has 2^n elements $u_0, \ldots, u_{2^n-1}$ and
- $\mathfrak{S}_n \vDash p_j(u_i)$ if and only if the j^{th} least significant bit of the binary representation of i is 1 (where $j \in [0, n-1]$).

Assume by contradiction that there is a function $f(n) \in o(\sqrt{n})$ and $c > 1$ such that for every $n \geq 1$, φ_n has an equivalent sentence ψ_n in Gaifman normal form with $|\psi_n| \leq c^{f(|\varphi_n|)} = c^{f(O(n^2))}$. Hence, we have $\mathfrak{S}_n \vDash \psi_n$. Since $f(n) \in o(\sqrt{n})$ there is an n such that $|\psi_n| < 2^n$. Let us fix such an n in the following.

The sentence ψ_n is a boolean combination of sentences $\chi_1, \ldots, \chi_\ell$, where each sentence

$$\chi_i = \exists x_1, \ldots, x_{n_i} : \chi_i'$$

is an r_i-local Gaifman-sentence. Without loss of generality, we can assume that there is some $h \in [1, \ell]$ such that

(1) $\mathfrak{S}_n \vDash \chi_i$ for each $i \in [1, h]$ and

(2) $\mathfrak{S}_n \nvDash \chi_i$ for each $i \in [h+1, \ell]$.

Recall that $\mathfrak{S}_n$ consists precisely of the elements $u_0, \ldots, u_{2^n-1}$. By (1) we have that for each $i \in [1, h]$, we can fix elements $w_1^{(i)}, \ldots, w_{n_i}^{(i)} \in \{u_0, \ldots, u_{2^n-1}\}$ that witness $\mathfrak{S}_n \vDash \chi_i$, i.e. $\mathfrak{S}_n \vDash \chi_i'(w_1^{(i)}, \ldots, w_{n_i}^{(i)})$. Let $d \stackrel{\text{def}}{=} \sum_{i=1}^h n_i \leq \sum_{i=1}^\ell n_i \leq |\psi_n| < 2^n$. For each $j \in [0, 2^n - 1]$, let the τ_n-structure $\mathfrak{S}_n^{-j}$ be obtained from $\mathfrak{S}_n$ by removing the element u_j. Note that obviously $\mathfrak{S}_n^{-j} \nvDash \varphi_n$ for *every* $j \in [0, 2^n - 1]$. Since $d < 2^n$, by the pigeonhole principle, there exists some $j \in [0, 2^n - 1]$ such that

- $u_j \notin \{w_k^{(i)} \mid i \in [1, h], k \in [1, n_i]\}$ and thus
- $\mathfrak{S}_n^{-j} \vDash \chi_i$ (since χ_i is an r_i-local Gaifman-sentence) for each $i \in [1, h]$.

Recall that by (2) we have $\mathfrak{S}_n \nvDash \chi_i$, or equivalently

$$\mathfrak{S}_n \vDash \neg\exists x_1, \ldots, x_{n_i} : \chi_i'$$

for each $i \in [h+1, \ell]$. Too, note that since each formula χ_i is an r_i-local Gaifman-sentence it follows $\mathfrak{S}_n^{-j} \vDash \neg\chi_i$ for each $i \in [h+1, \ell]$. In total we have

- $\mathfrak{S}_n^{-j} \vDash \chi_i$ for each $i \in [1, h]$ and
- $\mathfrak{S}_n^{-j} \nvDash \chi_i$ for each $i \in [h+1, \ell]$.

and hence $\mathfrak{S}_n^{-j} \vDash \psi_n$, contradicting $\mathfrak{S}_n^{-j} \nvDash \varphi_n$. □

8 Open problems

The main open problem concerns the size of Feferman-Vaught decompositions (w.r.t. disjoint sum) and equivalent formulas in Gaifman normal form for FO^2. For both formalisms, we proved a doubly exponential upper bound and a lower bound of the form $c^{o(\sqrt{n})}$ (for any constant $c > 1$). We conjecture that the upper bound can be improved to a singly exponential bound.

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