Word equations over graph products

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Abstract. For a restricted class of monoids, we prove that the decidability of the existential theory of word equations is preserved under graph products. Furthermore, we show that the positive theory of a graph product of groups can be reduced to the positive theories of some of the factor monoids and the existential theories of the remaining factors. Both results also include suitable constraints for the variables.

1 Introduction

Since the seminal work of Makanin [13] on equations in free monoids, the decidability of various theories of equations in different monoids and groups has been studied, and several new decidability and complexity results have been shown. Let us mention here the results of [16, 19] for free monoids, [3, 14] for free groups, [7] for free partially commutative monoids (trace monoids), [8] for free partially commutative groups (graph groups), [4] for plain groups (free products of finite and free groups), and [18] for torsion-free hyperbolic groups.

In this paper, we will continue this stream of research by considering graph products (Section 2.2). The graph product construction is a well-known construction in mathematics, see e.g. [11, 12], that generalizes both free products and direct products: An independence relation on the factors of the graph product specifies, which monoids are allowed to commute elementwise. Section 3 deals with existential theories of graph products. Using a general closure result for existential theories (Thm. 2), we will show in Section 3.2 that under some algebraic restriction on the factors of a graph product, the decidability of the existential theory of word equations is preserved under graph products (Thm. 4). This closure result remains also valid if we allow constraints for variables, which means that the value of a variable may be restricted to some specified set. More precisely, we will define an operation, which, starting from a class of constraints for each factor monoid of the graph product, constructs a class of constraints for the graph product. We will also present an upper bound for the space complexity of the existential theory of the graph product in terms of the space complexities for the existential theories of the factor monoids. Using known results from [20] it follows that the existential theory of word equations of a graph product of finite monoids, free monoids, and torsion-free hyperbolic groups is decidable. This result generalizes the decidability result for graph products of finite monoids, free monoids, and free groups groups from [5].

In Section 4 we consider positive theories of equations. It turns out that the positive theory of word equations of a graph product of *groups* with recognizable constraints can be reduced to (i) the positive theories with recognizable constraints of those factors of the graph product that are allowed to commute with all the other factors and (ii) the existential theories of the remaining factors.

Proofs that are omitted in this paper can be found in the full version [6]

2 Preliminaries

Let A be a possibly infinite alphabet. The empty word over A is ε. An *involution* ι on A is a function $\iota : A \to A$ with $\iota(\iota(a)) = a$ for all $a \in A$. Let $\mathcal{M} = (M, \circ, 1)$ be a monoid. A *monoid involution* on $\mathcal{M}$ is an involution $\iota : M \to M$ with $\iota(a \circ b) = \iota(b) \circ \iota(a)$ for all $a, b \in M$ and $\iota(1) = 1$. A *partial monoid involution* on $\mathcal{M}$ is a monoid involution $\iota : \mathcal{I} \to \mathcal{I}$, where $\mathcal{I}$ is a submonoid of $\mathcal{M}$. A subset $L \subseteq M$ is *recognizable* if there exists a homomorphism $h : \mathcal{M} \to Q$ to a finite monoid Q such that $L = h^{-1}(F)$ for some $F \subseteq Q$. With $\mathrm{REC}(\mathcal{M})$ we denote the class of all recognizable subsets of $\mathcal{M}$, it is a Boolean algebra. The set $\mathrm{RAT}(\mathcal{M})$ of *rational* subsets of $\mathcal{M}$ is defined via rational expressions over $\mathcal{M}$, see e.g. [2]. If $\mathcal{M}$ is finitely generated, then $\mathrm{REC}(\mathcal{M}) \subseteq \mathrm{RAT}(\mathcal{M})$.

2.1 Mazurkiewicz traces

For a detailed introduction to trace theory see [9]. An *independence alphabet* is a pair (A, I), where A is a possibly infinite set and $I \subseteq A \times A$ is a symmetric and irreflexive *independence relation*. Its complement $D = (A \times A) \setminus I$ is the *dependence relation*. The pair (A, D) is called a *dependence alphabet*. For $a \in A$ let $I(a) = \{b \in A \mid (a, b) \in I\}$ and $D(a) = A \setminus I(a)$. Let $\equiv_I$ be the smallest congruence on A^* that contains all pairs (ab, ba) with $(a, b) \in I$. The *trace monoid (free partially commutative monoid)* $\mathbb{M}(A, I)$ is the quotient monoid $A^*/{\equiv_I}$, its elements are called *traces*. Since A is not necessarily finite, we do not restrict to finitely generated trace monoids. Extreme cases are *free monoids* (if $D = A \times A$) and *free commutative monoids* (if $D = \{(a, a) \mid a \in A\}$). The trace represented by the word $s \in A^*$ is denoted by $[s]_I$. For $R \subseteq \mathbb{M}(A, I) \times \mathbb{M}(A, I)$ we denote with $\mathbb{M}(A, I)/R$ the quotient monoid of $\mathbb{M}(A, I)$ wrt. to the smallest congruence on $\mathbb{M}(A, I)$ containing R. If A is finite, then it is easy to see that $L \in \mathrm{REC}(\mathbb{M}(A, I))$ if and only if $\{s \in A^* \mid [s]_I \in L\}$ is regular.

We define on A an equivalence relation $\sim$ by $a \sim b$ if and only if $D(a) = D(b)$ (or equivalently $I(a) = I(b)$). Since D is reflexive, $\sim \subseteq D$. An equivalence class of $\sim$ is called a *complete clan* of (A, I). In the sequel we will briefly speak of clans. A clan C is *thin* [8] if $D(a) \neq \emptyset$ for some (and hence all) $a \in C$. The cardinality of the set of thin clans is denoted by $c(A, I)$ – of course it may be infinite. Note that $c(A, I) \neq 1$, and $c(A, I) = 0 \Leftrightarrow I = \emptyset$.

A partial function f on A is *compatible* with I if $(a, b) \in I$ and $a, b \in \mathrm{dom}(f)$ imply $(f(a), f(b)) \in I$. This allows to lift f to a partial function on $\mathbb{M}(A, I)$ by setting $f([a_1 \cdots a_n]_I) = [f(a_n) \cdots f(a_1)]_I$. The domain of this lifting is $\mathbb{M}(\mathrm{dom}(f), I)$. Note that we reverse the order of the symbols in the f-image of a trace. In our applications, f will be always a partial injection on A like for instance an involution $\iota : B \to B$ with $B \subseteq A$. In this case, the lifting of ι to $\mathbb{M}(A, I)$ is a partial monoid involution on $\mathbb{M}(A, I)$ with domain $\mathbb{M}(B, I)$.

2.2 Graph products

Graph products generalize both free products and direct products. They were introduced in [11]. Let (Σ, I_Σ) be an independence alphabet with Σ *finite*, and let $\mathcal{M}_\sigma = (M_\sigma, \circ_\sigma, 1_\sigma)$ be a monoid for every $\sigma \in \Sigma$. Define an independence alphabet (A, I) by $A = \bigcup_{\sigma\in\Sigma} M_\sigma$ and $I = \bigcup_{(\sigma,\tau)\in I_\Sigma} M_\sigma \times M_\tau$ where w.l.o.g. $M_\sigma \cap M_\tau = \emptyset$ for $\sigma \neq \tau$. Define $R \subseteq \mathbb{M}(A, I) \times \mathbb{M}(A, I)$ by

$$R = \bigcup_{\sigma\in\Sigma} \{(ab, c) \mid a, b \in M_\sigma, a \circ_\sigma b = c\} \cup \{1_\sigma \to \varepsilon\}.$$

The *graph product* $\mathbb{P} = \mathbb{P}(\Sigma, I_\Sigma, (\mathcal{M}_\sigma)_{\sigma\in\Sigma})$ is the quotient monoid $\mathbb{M}(A, I)/R$. In case $I_\Sigma = \emptyset$ (resp. $I_\Sigma = (\Sigma \times \Sigma) \setminus \mathrm{Id}_\Sigma$) we obtain the free (resp. direct) product of the $\mathcal{M}_\sigma$.

For $s, t \in \mathbb{M}(A, I)$ we write $s \to_R t$ if there exist $u, v \in \mathbb{M}(A, I)$ and $(\ell, r) \in R$ with $s = u\ell v$ and $t = urv$. Let $\mathrm{IRR} = \{s \in \mathbb{M}(A, I) \mid \neg\exists t : s \to_R t\}$. The relation $\to_R$ is clearly Noetherian and also confluent (see [6, Lemma 2.2]). It follows that there is a natural bijection $\omega : \mathbb{P} \to \mathrm{IRR}$ such that $x \in \mathbb{P}$ is represented by the trace $\omega(x)$. Moreover, for $x, y, z \in \mathbb{P}$ we have $xy = z$ in $\mathbb{P}$ if and only if $\omega(x)\omega(y) \xrightarrow{*}_R \omega(z)$. Note that $\mathbb{M}(A, I)$ is in general not finitely generated.

2.3 Relational structures and logic

Let us fix a relational structure $\mathcal{A} = (A, (R_i)_{i\in J})$, where $R_i \subseteq A^{n_i}$, $i \in J$. Given further relations R_j, $j \in K$, $J \cap K = \emptyset$, we also write $(\mathcal{A}, (R_i)_{i\in K})$ for the structure $(A, (R_i)_{i\in J\cup K})$. *First-order formulas* over $\mathcal{A}$ are built from the atomic formulas $R_i(x_1, \ldots, x_{n_i})$ and $x = y$ (where $i \in J$ and $x_1, \ldots, x_{n_i}, x, y$ are variables ranging over A) using Boolean connectives and quantifications over variables. The notion of a free variable is defined as usual. A formula without free variables is a *sentence*. If $\varphi(x_1, \ldots, x_n)$ is a first-order formula with free variables among $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in A$, then $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ means that φ evaluates to true in $\mathcal{A}$ if x_i evaluates to a_i. The *first-order theory* of $\mathcal{A}$, denoted by $\mathrm{FOTh}(\mathcal{A})$, is the set of all first-order sentences φ with $\mathcal{A} \models \varphi$. The *existential first-order theory* $\exists\mathrm{FOTh}(\mathcal{A})$ of $\mathcal{A}$ is the set of all sentences in $\mathrm{FOTh}(\mathcal{A})$ of the form $\exists x_1 \cdots \exists x_n : \varphi(x_1, \ldots, x_n)$, where $\varphi(x_1, \ldots, x_n)$ is a Boolean combination of atomic formulas. The *positive theory* $\mathrm{posTh}(\mathcal{A})$ is the set of all sentences in $\mathrm{FOTh}(\mathcal{A})$ that do not use negations, i.e., that are built from atomic formulas using conjunctions, disjunctions, and existential and universal quantifications.

We view a monoid $\mathcal{M} = (M, \circ, 1)$ as a relational structure by considering the multiplication $\circ$ as a ternary relation and the constant 1 as a unary relation.

Instead of $\circ(x, y, z))$ we write $x \circ y = z$ or briefly $xy = z$. We also consider extensions $(\mathcal{M}, (R_i)_{i \in J})$ of the structure $\mathcal{M}$, where R_i is a relation of arbitrary arity over M. In many cases, a partial monoid involution ι will belong to the R_i. It is viewed as a binary relation on M. In case $\mathcal{C} \subseteq 2^M$, we also write $(\mathcal{M}, \mathcal{C}, (R_i)_{i \in J})$ instead of $(\mathcal{M}, (L)_{L \in \mathcal{C}}, (R_i)_{i \in J})$ and call formulas of the form $x \in L$ for $L \in \mathcal{C}$ *constraints*. Constants from M can be included as singleton subsets into the R_i. Note that if $\mathcal{M}$ is finitely generated by Γ, then constants from Γ suffice in order to define all monoid elements of $\mathcal{M}$. On the other hand, the further investigations are not restricted to finitely generated monoids.

It is known that already the $\forall\exists^3$-fragment of $\mathrm{FOTh}(\{a, b\}^*, a, b)$ is undecidable [10]. Together with Presburger's result on the decidability of $\mathrm{FOTh}(\mathbb{N})$ it follows that the decidability of the full first-order theory is not preserved under free products. For a restricted class of monoids and existential sentences, we will show such a closure result in Section 3.2 even for general graph products.

3 Existential theories of graph products

Based on results from [8] for finitely generated trace monoids with partial involution, we prove in Section 3.1 a general preservation theorem for existential theories. In Section 3.2 we use this result in order to show that under some restrictions graph products preserve the decidability of the existential theory.

All our decidability results in this section are based on the main result from [8], see also [6, Thm. 3.1]:

Theorem 1. *For every $k \geq 0$, the following problem is in PSPACE:*

INPUT: A finite independence alphabet (A, I) with $c(A, I) \leq k$, a partial involution ι on A that is compatible with I, and an existential sentence ϕ over $(\mathbb{M}(A, I), \iota, \mathrm{REC}(\mathbb{M}(A, I)))$ (with ι lifted to $\mathbb{M}(\mathrm{dom}(\iota), I)$).

QUESTION: Does $(\mathbb{M}(A, I), \iota, \mathrm{REC}(\mathbb{M}(A, I))) \models \phi$ hold?

In Thm. 1, a recognizable set $L \in \mathrm{REC}(\mathbb{M}(A, I))$ has to be represented by a finite automaton for the regular language $\{u \in A^* \mid [u]_I \in L\}$, which is crucial for the PSPACE upper-bound, see e.g. the remarks in [6]. Note that since every singleton subset belongs to $\mathrm{REC}(\mathbb{M}(A, I))$, constants are implicitly allowed in Thm. 1.

Thm. 1 cannot be extended to the case of rational constraints: By [15, Prop. 2.9.2 and 2.9.3], $\exists\mathrm{FOTh}(\mathbb{M}(A, I), \mathrm{RAT}(\mathbb{M}(A, I)))$ is decidable if and only if $I \cup \mathrm{Id}_A$ is an equivalence relation.

3.1 A general preservation theorem

For the further discussion let us fix an independence alphabet (A, I), a partial involution ι on A, a subset $\mathcal{C} \subseteq 2^A$ of constraints, and additional predicates R_j $(1 \leq j \leq m)$ of arbitrary arity on A. Let $\mathbb{M} = \mathbb{M}(A, I)$ and $\mathbb{A} = (A, \iota, (L)_{L \in \mathcal{C}})$. We assume that:

(1) ι is compatible with I,
(2) there are only finitely many clans of (A, I), i.e., there are only finitely many sets $D(a)$,
(3) $\mathrm{dom}(\iota)$ as well as every clan belong to $\mathcal{C}$, and
(4) $\exists\mathrm{FOTh}(\mathbb{A}, (R_j)_{1\leq j\leq m})$ is decidable.

Due to (1), we can lift ι to a partial monoid involution on $\mathbb{M}$. Since I is a union of Cartesian products of clans, (2) and (3) imply that I is definable by a Boolean formula over $(A, (L)_{L\in\mathcal{C}})$.

From the unary predicates in $\mathcal{C}$ we construct a set $\mathcal{L}(\mathcal{C}, I) \subseteq 2^{\mathbb{M}}$ as follows: A $\mathcal{C}$-automaton $\mathcal{A}$ is a finite automaton in the usual sense, except that every edge of $\mathcal{A}$ is labeled with some language $L \in \mathcal{C}$. The language $L(\mathcal{A}) \subseteq A^*$ is the union of all concatenations $L_1L_2\cdots L_n$ for which there exists a path $q_0 \xrightarrow{L_1} q_1 \xrightarrow{L_2} \cdots q_{n-1} \xrightarrow{L_n} q_n$ in $\mathcal{A}$ from the initial state q_0 to a final state q_n. We say that $\mathcal{A}$ is I-closed if $[u]_I = [v]_I$ and $u \in L(\mathcal{A})$ imply $v \in L(\mathcal{A})$. In the following, we will identify $L(\mathcal{A})$ with $\{[u]_I \mid u \in L(\mathcal{A})\} \subseteq \mathbb{M}$. Then $\mathcal{L}(\mathcal{C}, I)$ consists of all languages $L(\mathcal{A}) \subseteq \mathbb{M}$ such that $\mathcal{A}$ is an I-closed $\mathcal{C}$-automaton. For effectiveness statements, it is necessary that languages in $\mathcal{C}$ have some finite representation. Then, also languages from $\mathcal{L}(\mathcal{C}, I)$ have a canonical finite representation.

Since $A \subseteq \mathbb{M}$, we can view every relation R_j also as a relation on $\mathbb{M}$. This is done in the following theorem,[1] which is the main result of this section:

Theorem 2. *If* $\exists\mathrm{FOTh}(A,\ \iota,\ (L)_{L\in\mathcal{C}},\ (R_j)_{1\leq j\leq m})$ *belongs to NSPACE*$(s(n))$, *then* $\exists\mathrm{FOTh}(\mathbb{M},\ \iota,\ \mathcal{L}(\mathcal{C}, I),\ (R_j)_{1\leq j\leq m})$ *belongs to NSPACE*$(2^{O(n)} + s(n^{O(1)}))$.

Reducing the number of generators The main difficulty in the proof of Thm. 2 is to reduce the infinite set A of generators of $\mathbb{M}$ to a finite set of generators B. For this, we will prove a technical lemma (Lemma 2) in this paragraph. In the sequel, we will restrict to some reduct $(A, \iota, (L)_{L\in\mathcal{D}})$ of the structure $\mathbb{A}$ from the previous section, where $\mathcal{D} \subseteq \mathcal{C}$ is finite and contains $\mathrm{dom}(\iota)$ as well as every clan of (A, I). We will denote this reduct by $\mathbb{A}$ as well. Assume that $\mathcal{D} = \{L_0, L_1, \ldots, L_k\}$, where $\mathrm{dom}(\iota) = L_0$ and $L_1, \ldots, L_\ell$ is an enumeration of the clans of (A, I). Thus, $\{L_1, \ldots, L_\ell\}$ is a partition of A.

Given another structure $\mathbb{B} = (B, \zeta, (K_i)_{0\leq i\leq k})$ (with ζ a partial involution on B, $K_i \subseteq B$, and $K_0 = \mathrm{dom}(\zeta)$), a mapping $f : A \to B$ is a *strong homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if for all $a \in A$ and $0 \leq i \leq k$:

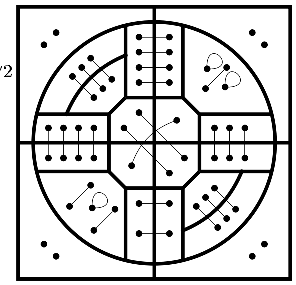
$$a \in L_i \Leftrightarrow f(a) \in K_i \quad \text{and} \quad \forall a \in \mathrm{dom}(\iota) : f(\iota(a)) = \zeta(f(a))$$

Lemma 1. *We can effectively construct a* finite *structure* $\mathbb{B} = (B, \zeta, (K_i)_{0\leq i\leq k})$ *(with* ζ *a partial involution on* B, $K_i \subseteq B$, *and* $\mathrm{dom}(\zeta) = K_0$*) such that* $|B| \leq 2^{k+1}(2^{k+1} + 2)$ *and there exist strong homomorphisms* $f : \mathbb{A} \to \mathbb{B}$ *and* $g : \mathbb{B} \to \mathbb{A}$ *with* f *surjective.*[2]

[1] Recall that in contrast to this, the partial involution ι was lifted from A to the whole trace monoid $\mathbb{M}$.

[2] Effectiveness in this context means that given a finite set $\mathcal{D} \subseteq \mathcal{C}$, we can construct the finite structure $\mathbb{B}$ effectively.

Proof (sketch). The picture on the right visualizes the construction for $k = 2$. The set L_1 (resp. L_2) is represented by the left (resp. lower) half of the whole square, which represents A. The inner circle represents $\mathrm{dom}(\iota) = L_0$, and the thin lines represent the partial involution ι. The 22 regions that are bounded by thick lines represent the (in general infinite) preimages $f^{-1}(b)$ $(b \in B)$. Basically, $\mathbb{B}$ results by contracting every nonempty region to a single point. Nonemptyness of a region can be expressed as an existential sentence over $\mathbb{A}$ and is therefore decidable. □



Since the strong homomorphism f is surjective in the previous lemma and $\{L_1, \dots, L_\ell\}$ is a partition of A, also $\{K_1, \dots, K_\ell\}$ is a partition of B.

Now assume that we have given a third structure $\mathbb{C} = (C, \xi, (\Lambda_i)_{0 \le i \le k})$, where C is finite, ξ is a partial involution on C, $\Lambda_i \subseteq C$ for $0 \le i \le k$, $\mathrm{dom}(\xi) = \Lambda_0$, and $\{\Lambda_1, \dots, \Lambda_\ell\}$ is a partition of C (with $\Lambda_i = \emptyset$ allowed). In the sequel, an *embedding of* $\mathbb{C}$ *in* $\mathbb{A}$ is an *injective* strong homomorphism $h : \mathbb{C} \to \mathbb{A}$.

Lemma 1 allows to deduce the following lemma, where $f(I) = \{(f(a), f(b)) \mid (a, b) \in I\}$ (and similarly for $g(J)$).

Lemma 2. *Given* $\mathbb{C}$ *as above, we can effectively construct a* finite *structure* $\mathbb{B} = (B, \zeta, (K_i)_{0 \le i \le k})$ *(with* ζ *a partial involution on* B*,* $K_i \subseteq B$*, and* $\mathrm{dom}(\zeta) = K_0$*) together with an independence relation* $J \subseteq B \times B$ *such that:*

- $C \subseteq B$*,* $|B| \le 2^{k+1}(2^{k+1} + 2) + |C|$*,* ζ *is compatible with* J*, and*
- *for every embedding* $h : \mathbb{C} \to \mathbb{A}$ *there are strong homomorphisms* $f : \mathbb{A} \to \mathbb{B}$ *and* $g : \mathbb{B} \to \mathbb{A}$ *with* $f(I) \subseteq J$*,* $g(J) \subseteq I$*, and* $f(h(c)) = c$*,* $g(c) = h(c)$ *for all* $c \in C$*.*

Proof of Thm. 2. Fix a formula θ over $(\mathbb{M}, \iota, \mathcal{L}(\mathcal{C}, I), (R_j)_{1 \le j \le m})$. We have to decide whether θ is satisfiable in $(\mathbb{M}, \iota, \mathcal{L}(\mathcal{C}, I), (R_j)_{1 \le j \le m})$. For this, we will present a nondeterministic algorithm that constructs a *finitely generated* trace monoid $\mathbb{M}'$ with a partial monoid involution ζ and a Boolean formula ϕ' over $(\mathbb{M}', \zeta, \mathrm{REC}(\mathbb{M}'))$ such that θ is satisfiable in $(\mathbb{M}, \iota, \mathcal{L}(\mathcal{C}, I), (R_j)_{1 \le j \le m})$ if and only if for at least one outcome of our nondeterministic algorithm, ϕ' is satisfiable in $(\mathbb{M}', \zeta, \mathrm{REC}(\mathbb{M}'))$. This allows to apply Thm. 1.

Assume that every $\mathcal{C}$-automaton in θ only uses sets among the finite set $\mathcal{D} \subseteq \mathcal{C}$. Assume that also $\mathrm{dom}(\iota)$ as well as every clan of (A, I) belong to $\mathcal{D}$. Let $\mathcal{D} = \{L_0, \dots, L_k\}$, where $L_0 = \mathrm{dom}(\iota)$ and $L_1, \dots, L_\ell$ is an enumeration of the clans of (A, I). Let $\mathcal{L} = \mathcal{L}(\mathcal{D}, I)$.

First we may push negations to the level of atomic subformulas in θ. Moreover, disjunctions may be eliminated by nondeterministically guessing one of the two corresponding disjuncts. Thus, we may assume that θ is a conjunction of atomic predicates and negated atomic predicates. We replace every negated equation $xy \ne z$ (resp. $\iota(x) \ne z$) by $xy = z' \wedge z \ne z'$, (resp. $\iota(x) = z' \wedge z \ne z'$), where z' is a new variable. Thus, we may assume that all negated predicates in θ are of the form $x \neq y$, $x \notin L$ ($L \in \mathcal{L}$), and $\neg R_j(x_1, \ldots, x_n)$.

We can write θ as a conjunction $\phi \wedge \psi$, where ψ contains all formulas of the form $(\neg)R_j(x_1, \ldots, x_n)$. Let $x \neq y$ be a negated equation in ϕ, where x and y are variables. Since $x \neq y$ is interpreted in the trace monoid $\mathbb{M}$, we can replace $x \neq y$ by either $(x = zau \ \wedge \ y = zbv \ \wedge \ a, b \in L \ \wedge \ a \neq b)$ or $(x = zu \ \wedge \ y = zv \ \wedge \ u \in L \cdot \mathbb{M} \ \wedge \ v \notin L \cdot \mathbb{M})$, where $L \in \mathcal{D}$ is a clan of (A, I) that is guessed nondeterministically. In the first case, we add $a, b \in L \wedge a \neq b$ to the "$\mathbb{A}$-local" part ψ. In the second case, we have to construct an I-closed $\mathcal{D}$-automaton for $L \cdot \mathbb{M}$, which is easy, since all clans belong to $\mathcal{D}$. Thus, in the sequel we may assume that ϕ does not contain negated equations.

So far, we have obtained a conjunction $\phi \ \wedge \ \psi$, where ϕ is interpreted in $(\mathbb{M}, \iota, \mathcal{L})$ and ψ is interpreted in the base structure $(\mathbb{A}, (R_j)_{1 \leq j \leq m})$. The formula ϕ does not contain negated equations. Let Ξ be the set of all variables that occur in $\phi \ \wedge \ \psi$, and let $\Omega \subseteq \Xi$ contain all variables that occur in the $\mathbb{A}$-local part ψ. Thus, all variables from Ω are implicitly restricted to $A \subseteq \mathbb{M}$. Note that variables from Ω may of course also occur in ϕ. In case ϕ contains a constraint $x \in L$ with $L \in \mathcal{L}$ and $x \in \Omega$, then we can guess $L' \in \mathcal{D}$ with $L \cap L' \neq \emptyset$ and replace $x \in L$ by the constraint $x \in L'$, which will be shifted to ψ. Hence, we may assume that for every constraint $x \in L$ that occurs in ϕ, we have $x \in \Xi \setminus \Omega$.

Next, for every variable $x \in \Omega$ we guess whether $x \in L_0 = \mathrm{dom}(\iota)$ or $x \notin \mathrm{dom}(\iota)$ holds and add the corresponding (negated) constraint to ψ. In case $x \in \mathrm{dom}(\iota)$ was guessed, we add a new variable $\overline{x}$ to Ω and add the equation $\iota(x) = \overline{x}$ to ψ. Next, we guess for all different variables $x, y \in \Omega$ (here Ω refers to the new set of variables including the added copys $\overline{x}$), whether $x = y$ or $x \neq y$. In case $x = y$ is guessed, we can eliminate for instance y. Thus, we may assume that for all different variables $x, y \in \Omega$ the negated equation $x \neq y$ belongs to ψ. Finally, for every set L_i with $1 \leq i \leq k$ and every $x \in \Omega$ we guess whether $x \in L$ or $x \notin L$ holds and add the corresponding constraint to ψ. We denote the resulting formula by ψ as well.

Most of the guessed formulas ψ won't be satisfiable in $(\mathbb{A}, (R_j)_{1 \leq j \leq m})$ (e.g., if $L_i \cap L_j = \emptyset$ and the constraints $x \in L_i$ and $x \in L_j$ were guessed). But since $\exists\mathrm{FOTh}(\mathbb{A}, (R_j)_{1 \leq j \leq m})$ is decidable, we can effectively check whether the guessed formula ψ is satisfiable. If it is not satisfiable, then we reject on the corresponding computation path. Let us fix a specific guess, which results in a satisfiable formula ψ, for the further consideration.

Now we define a finite structure $\mathbb{C} = (\widetilde{\Omega}, \xi, (\Lambda_i)_{0 \leq i \leq k})$ as follows: Let $\widetilde{\Omega} = \{\widetilde{x} \mid x \in \Omega\}$ be a disjoint copy of the set of variables Ω. For $0 \leq i \leq k$ let Λ_i be the set of all $\widetilde{x} \in \widetilde{\Omega}$ such that $x \in L_i$ belongs to ψ. Finally, we define the partial involution ξ on $\widetilde{\Omega}$ as follows: The domain of ξ is Λ_0 and $\xi(\widetilde{x}) = \widetilde{y}$ in case $\iota(x) = y$ or $\iota(y) = x$ belongs to the conjunction ψ. Since ψ is satisfiable and $\{L_1, \ldots, L_\ell\}$ is a partition of A, it follows that $\{\Lambda_1 \ldots, \Lambda_\ell\}$ is a partition of $\widetilde{\Omega}$ (with $\Lambda_i = \emptyset$ allowed). Thus, $\mathbb{C}$ satisfies all the requirements from Lemma 2, which can be applied to the structures $\mathbb{A}$ and $\mathbb{C}$. Hence, from $\mathbb{C}$ we can effectively determine a *finite* structure $\mathbb{B} = (B, \zeta, (K_i)_{0 \leq i \leq k})$ together with an independence relation

$J \subseteq B \times B$ such that $\widetilde{\Omega} \subseteq B$, the partial involution ζ is compatible with J, and for every embedding $h : \mathbb{C} \to \mathbb{A}$ there exist strong homomorphisms $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$ with $f(I) \subseteq J$, $g(J) \subseteq I$, and $f(h(\widetilde{x})) = \widetilde{x}$, $g(\widetilde{x}) = h(\widetilde{x})$ for every $x \in \Omega$. We also obtain a size bound of $|\widetilde{\Omega}| + 2^{O(k)} \subseteq 2^{O(|\theta|)}$ for $|B|$. We denote the lifting of ζ to $\mathbb{M}(B, J)$ by ζ as well. Let $\mathbb{M}' = \mathbb{M}(B, J)$.

Recall that we have to check whether there exist assignments $\kappa : \Omega \to A$ and $\lambda : \Xi \setminus \Omega \to \mathbb{M}$ such that κ satisfies ψ in $(\mathbb{A}, (R_j)_{1 \le j \le m})$ and $\kappa \cup \lambda$ satisfies ϕ in $(\mathbb{M}, \iota, \mathcal{L})$. We have already verified that the conjunction ψ is satisfiable in $(\mathbb{A}, (R_j)_{1 \le j \le m})$. For the following consideration let us fix an arbitrary assignment $\kappa : \Omega \to A$ that satisfies ψ in $(\mathbb{A}, (R_j)_{1 \le j \le m})$.[3] Then κ defines an embedding $h : \mathbb{C} \to \mathbb{A}$ by $h(\widetilde{x}) = \kappa(x)$ for $x \in \Omega$. Therefore there exist strong homomorphisms $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$ with $f(\kappa(x)) = \widetilde{x}$, $g(\widetilde{x}) = \kappa(x)$ $(x \in \Omega)$, and all the other properties from Lemma 2. Since f and g preserve the involution on A and B, respectively, and $f(I) \subseteq J$, $J \subseteq g(I)$, we obtain the homomorphisms $f : (\mathbb{M}, \iota) \to (\mathbb{M}', \zeta)$ and $g : (\mathbb{M}', \zeta) \to (\mathbb{M}, \iota)$ between trace monoids with partial involution.

Given a $\mathcal{D}$-automaton $\mathcal{A}$, we define a new automaton $\mathcal{A}'$ by replacing every edge $p \xrightarrow{L_i} q$ in $\mathcal{A}$ by $p \xrightarrow{K_i} q$ (and changing nothing else). Recall that $K_i \subseteq B$. Since $\mathcal{A}$ is I-closed, $\mathcal{A}'$ is easily seen to be J-closed. Moreover, since B is finite, $L(\mathcal{A}') \subseteq \mathbb{M}'$ is a recognizable trace language. Recall that for every $0 \le i \le k$, we have $a \in L_i$ if and only if $f(a) \in K_i$ and $b \in K_i$ if and only if $g(b) \in L_i$. Thus, the following statement is obvious:

Lemma 3. *Let $t \in \mathbb{M}$ and $u \in \mathbb{M}'$. Then $t \in L(\mathcal{A})$ if and only if $f(t) \in L(\mathcal{A}')$ and $u \in L(\mathcal{A}')$ if and only if $g(u) \in L(\mathcal{A})$.*

Next, we transform the conjunction ϕ into a conjunction ϕ', which will be interpreted over $(\mathbb{M}', \zeta, \mathrm{REC}(\mathbb{M}'))$, by replacing in ϕ every occurrence of a variable $x \in \Omega$ by the constant $\widetilde{x} \in \widetilde{\Omega} \subseteq B$. Thus, ϕ' contains constants from $\widetilde{\Omega}$ and variables from $\Xi \setminus \Omega$, which range over the trace monoid $\mathbb{M}'$. Moreover, every constraint $x \in L(\mathcal{A})$ (resp. $x \notin L(\mathcal{A})$) in ϕ is replaced by $x \in L(\mathcal{A}')$ (resp. $x \notin L(\mathcal{A}')$) (note that $x \in \Xi \setminus \Omega$). Thus, all constraint languages in ϕ' belong to $\mathrm{REC}(\mathbb{M}')$.

Lemma 4. *The following two statements are equivalent:*

(1) There exists an assignment $\lambda : \Xi \setminus \Omega \to \mathbb{M}$ such that $\kappa \cup \lambda$ satisfies the Boolean formula ϕ in $(\mathbb{M}, \iota, \mathcal{L})$.
(2) There exists an assignment $\lambda' : \Xi \setminus \Omega \to \mathbb{M}'$ that satisfies the Boolean formula ϕ' in $(\mathbb{M}', \zeta, \mathrm{REC}(\mathbb{M}'))$.

Proof. First, assume that (1) holds. We claim that (2) holds with $\lambda' = f \circ \lambda$. Let $u' = v'$ be an equation of ϕ', which results from the equation $u = v$ of ϕ. The only difference between $u = v$ and $u' = v'$ is that every occurrence of every variable $x \in \Omega$ in $u = v$ is replaced by the constant $\widetilde{x}$ in $u' = v'$. The assignment $\kappa \cup \lambda$ is a solution of $u = v$ in $(\mathbb{M}, \iota)$. Since $f : (\mathbb{M}, \iota) \to (\mathbb{M}', \zeta)$ is a homomorphism between

[3] We do not have to determine κ explicitly, only its existence is important.

trace monoids with partial involution, $f \circ (\kappa \cup \lambda) = f \circ \kappa \,\cup\, f \circ \lambda = f \circ \kappa \,\cup\, \lambda'$ is a solution of $u = v$ in $(\mathbb{M}', \zeta)$. Since $f(\kappa(x)) = \widetilde{x}$ for every $x \in \Omega$, the mapping λ' is a solution of $u' = v'$ in $(\mathbb{M}', \zeta)$. The preservation of (negated) constraints follows from Lemma 3.

Now assume that (2) holds. We claim that (1) holds with $\lambda = g \circ \lambda'$. Consider an equation $u = v$ of ϕ and let $u' = v'$ be the corresponding equation of ϕ'. Thus, λ' is a solution of $u' = v'$ in $(\mathbb{M}', \zeta)$. Let the function π map every variable $x \in \Omega$ to the constant $\widetilde{x} \in B$. By construction of $u' = v'$, $\lambda' \cup \pi$ is a solution of $u = v$ in $(\mathbb{M}', \zeta)$. Since $g : (\mathbb{M}', \zeta) \to (\mathbb{M}, \iota)$ is a homomorphism between trace monoids with partial involution and $g(\pi(x)) = \kappa(x)$ for every $x \in \Omega$, the mapping $g \circ (\lambda' \cup \pi) = \lambda \cup \kappa$ is a solution of $u = v$ in $(\mathbb{M}, \iota)$. For the preservation of (negated) constraints we use again Lemma 3. □

For the previous lemma it is crucial that the conjunction ϕ does not contain negated equations, because the homomorphisms f and g are not injective in general, and therefore do not preserve inequalities.

Since Lemma 4 holds for every $\kappa : \Omega \to A$ that satisfies ψ in the structure $(\mathbb{A}, (R_j)_{1 \le j \le m})$, and we already know that such an assignment exists, it only remains to check whether $(\mathbb{M}', \zeta, \mathrm{REC}(\mathbb{M}')) \models \phi'$. By Thm. 1 this can be done effectively. This proves the decidability statement in Thm. 2. A closer investigation of the outlined decision procedure gives the space bounds in Thm. 2 (note that $c(B, J) = c(A, I)$ is a constant).

3.2 Closure under graph products

Fix a graph product $\mathbb{P} = \mathbb{P}(\Sigma, I_\Sigma, (\mathcal{M}_\sigma)_{\sigma \in \Sigma})$, where $\mathcal{M}_\sigma = (M_\sigma, \circ_\sigma, 1_\sigma)$. Define A, I, R, IRR, and $\omega : \mathbb{P} \to \mathrm{IRR}$ as in Section 2.2. Recall that ω is bijective. Let $\mathrm{inv}_\sigma = \{(a, b) \in M_\sigma \times M_\sigma \mid a \circ_\sigma b = 1_\sigma\}$ and $\mathrm{inv} = \bigcup_{\sigma \in \Sigma} \mathrm{inv}_\sigma$. Let $U_\sigma = \mathrm{dom}(\mathrm{inv}_\sigma)$, $V_\sigma = \mathrm{ran}(\mathrm{inv}_\sigma)$, $U = \bigcup_{\sigma \in \Sigma} U_\sigma$, and $V = \bigcup_{\sigma \in \Sigma} V_\sigma$.

We also include constraints into our considerations. Hence, for every $\sigma \in \Sigma$ let $\mathcal{C}_\sigma \subseteq 2^{M_\sigma}$ be a class of constraints. We assume that $U_\sigma, V_\sigma \in \mathcal{C}_\sigma$. Let $\mathcal{C} = \bigcup_{\sigma \in \Sigma} \mathcal{C}_\sigma$. Recall the definition of the class $\mathcal{L} = \mathcal{L}(\mathcal{C}, I) \subseteq 2^{\mathbb{M}(A,I)}$ from Section 3.1. We define the class $\mathcal{IL} = \mathcal{IL}(\mathcal{C}, I, R) \subseteq 2^{\mathbb{M}(A,I)}$ by $\mathcal{IL} = \{L \cap \mathrm{IRR} \mid L \in \mathcal{L}\}$. Using the one-to-one correspondence between $\mathbb{P}$ and IRR, we may view $L \cap \mathrm{IRR}$ also as a subset of $\mathbb{P}$, hence $\mathcal{IL} \subseteq 2^{\mathbb{P}}$.

Example 1. If $\mathrm{REC}(\mathcal{M}_\sigma) \subseteq \mathcal{C}_\sigma$, then also $\mathrm{REC}(\mathbb{P}) \subseteq \mathcal{IL}$ [6, Lemma 4.8]. A subset $L \subseteq \mathcal{M}$ of a monoid $\mathcal{M}$, which is finitely generated by Γ, is called *normalized rational* if the set of length-lexicographical normalforms from Γ^* (wrt. an arbitrary linear order on Γ) that represent elements from L is rational [6]. It is not hard to see that $\mathcal{IL}$ is the set of normalized rational subsets of $\mathbb{P}$ in case $\mathcal{C}_\sigma$ is the set of normalized rational subsets of $\mathcal{M}_\sigma$.

Throughout this section we make:

Assumption 3 *For all $\sigma \in \Sigma$ and $a, b, c \in M_\sigma$, if $a \circ_\sigma b = a \circ_\sigma c = 1_\sigma$ or $b \circ_\sigma a = c \circ_\sigma a = 1_\sigma$, then $b = c$. Thus inv_σ is a partial injection.*

For example groups, free monoids, the bicyclic monoid $\{a,b\}^*/_{ab=\varepsilon}$, and finite monoids satisfy Assumption 3,[4] whereas $\{a,b,c\}^*/_{ab=ac=\varepsilon}$ does not.

By Assumption 3, inv is a partial injection on A with $\mathrm{dom}(\mathrm{inv}) = U$ and $\mathrm{ran}(\mathrm{inv}) = V$. Since inv is compatible with I, we can lift it to $\mathbb{M}(A,I)$ (see Section 2.1). The resulting partial injection has domain $\mathbb{M}(U,I)$ and range $\mathbb{M}(V,I)$. The following theorem is the main result of this section.

Theorem 4. *If Assumption 3 holds and for all $\sigma \in \Sigma$, $\exists\mathrm{FOTh}(\mathcal{M}_\sigma, \mathcal{C}_\sigma)$ belongs to NSPACE$(s(n))$, then $\exists\mathrm{FOTh}(\mathbb{P}, \mathcal{IL})$ belongs to NSPACE$(2^{O(n)} + s(n^{O(1)}))$.*

Before we go into the details of the proof of Thm. 4 let us first present an application. The existential theory of a finite monoid is decidable for trivial reasons. By Makanin's result, the existential theory with constants of a free monoid is also decidable. Finally, by [18, 20], also the existential theory with constants of a torsion-free hyperbolic group is decidable. Note that every free group is torsion-free hyperbolic. Since finite monoids, free monoids, and groups in general all satisfy Assumption 3 (and either $U_\sigma = V_\sigma = \emptyset$ or $U_\sigma = V_\sigma = M_\sigma$ for these monoids), we obtain the following corollary:

Corollary 1. *Let $\mathbb{P}$ be a graph product of finite monoids, free monoids, and torsion-free hyperbolic groups, and let Γ be a finite generating set for $\mathbb{P}$. Then $\exists\mathrm{FOTh}(\mathbb{P}, (a)_{a\in\Gamma})$ is decidable.*

For the proof of Thm. 4 assume that $\exists\mathrm{FOTh}(\mathcal{M}_\sigma, \mathcal{C}_\sigma)$ belongs to NSPACE$(s(n))$ for every $\sigma \in \Sigma$. Thus, the same holds for $\exists\mathrm{FOTh}(\mathcal{M}_\sigma, \mathrm{inv}_\sigma, \mathcal{C}_\sigma)$. This remains true if we put M_σ and $M_\sigma \setminus \{1_\sigma\}$ into $\mathcal{C}_\sigma$. Then IRR $\in \mathcal{L}$: the language $L = \Sigma^* \setminus \bigcup_{\sigma\in\Sigma} \Sigma^*\sigma I_\Sigma(\sigma)^*\sigma\Sigma^*$ is regular. In order to define a $\mathcal{C}$-automaton for IRR, we just have to replace in a finite automaton for L every label σ by $M_\sigma \setminus \{1_\sigma\}$.

We may also replace $\mathcal{C}$ by its closure under union; this does not change the class $\mathcal{L} = \mathcal{L}(\mathcal{C}, I)$. Thus, the sets U, V, $U \cup V$, and every clan of (A, I) (which is a union of some of the M_σ) belong to $\mathcal{C}$. Then $\mathbb{M}(U,I)$ (the domain of the lifting of inv to $\mathbb{M}(A,I)$) belongs to $\mathcal{L}$.

Since by Assumption 3, $\mathrm{inv} : U \to V$ is a partial injection, we can define a partial involution ι on A with domain $U \cup V \in \mathcal{C}$ by $\iota(a) = b$ if and only if either $\mathrm{inv}(a,b)$ or $\mathrm{inv}(b,a)$ (note that $\mathrm{inv}(a,b)$ and $\mathrm{inv}(b,c)$ implies $a = c$). This involution on A is compatible with I, hence it can be lifted to a partial monoid involution ι on $\mathbb{M}(A,I)$ with domain $\mathbb{M}(U \cup V, I)$.

Since $\exists\mathrm{FOTh}(\mathcal{M}_\sigma, \mathrm{inv}_\sigma, \mathcal{C}_\sigma)$ belongs to NSPACE$(s(n))$ for every $\sigma \in \Sigma$, the same is true for $\exists\mathrm{FOTh}(A, \iota, (L)_{L\in\mathcal{C}}, (\circ_\sigma)_{\sigma\in\Sigma})$. Thm. 2 (with $(\circ_\sigma)_{\sigma\in\Sigma}$ for $(R_j)_{1\le j\le m}$) shows that the theory $\exists\mathrm{FOTh}(\mathbb{M}(A,I), \iota, \mathcal{L}, (\circ_\sigma)_{\sigma\in\Sigma})$ belongs to NSPACE$(2^{O(n)} + s(n^{O(1)}))$ (note that the conditions (1)-(4) from Section 3.1 are all satisfied in the present situation).

Let θ be a Boolean formula with atomic predicates of the form $xy = z$ and $x \in L$, where $L \in \mathcal{IL}$ (atomic predicates of the form $x = 1$ are not necessary since

[4] For a finite monoid note that $a \circ b = 1$ implies that the mapping $x \mapsto b \circ x$ is injective, hence it is surjective. Thus, there exists c with $b \circ c = 1$. Clearly $a = c$, i.e., $b \circ a = 1$, and inv_σ is a partial involution.

$\{1\} \in \mathcal{IL}$). We have to check, whether there exists an assignment for the variables in θ to elements in $\mathbb{P}$ that satisfies θ. For this, we transform θ in polynomial time into an equivalent existential statement over $(\mathbb{M}(A,I), \iota, \mathcal{L}, (\circ_\sigma)_{\sigma\in\Sigma})$. Thus, in some sense we isolate the structure of the factor monoids $\mathcal{M}_\sigma$ into the "$\mathcal{M}_\sigma$-local" $\circ_\sigma$-predicates.

First, we may push negations to the level of atomic subformulas in θ. We replace every negated equation $xy \neq z$ by $xy = z' \wedge z \neq z'$, where z' is a new variable. Thus, we may assume that all negated predicates in θ are of the form $x \neq y$ and $x \notin L$ for variables x and y.

Recall from Section 2.2 that every $x \in \mathbb{P}$ has a unique representative $\omega(x) \in \mathrm{IRR} \subseteq \mathbb{M}(A,I)$ and that $xy = z$ in $\mathbb{P}$ if and only if $\omega(x)\omega(y) \xrightarrow{*}_R \omega(z)$. Moreover, for $L = L' \cap \mathrm{IRR} \in \mathcal{IL}$ with $L' \in \mathcal{L}$ we have $x \in L$ if and only if $\omega(x) \in L'$. Hence, if we add for every variable x in θ the constraint $x \in \mathrm{IRR}$ (recall that $\mathrm{IRR} \in \mathcal{L}$) and replace every equation $xy = z$ in θ by the predicate $xy \xrightarrow{*}_R z$, then we obtain a formula, which is satisfiable in the trace monoid $\mathbb{M}(A,I)$ if and only if the original formula θ is satisfiable in $\mathbb{P}$. Using the following lemma, we can replace the predicates $xy \xrightarrow{*}_R z$ by ordinary equations plus $\circ_\sigma$-predicates. For the proof of this lemma, Assumption 3 is essential.

Lemma 5. *There exists a fixed Boolean formula $\psi(x, y, z, x_1, \ldots, x_m)$ over the structure $(\mathbb{M}(A,I), \iota, (\circ_\sigma)_{\sigma\in\Sigma}, \mathcal{L})$ such that for all $x, y, z \in \mathrm{IRR}$, $xy \xrightarrow{*}_R z$ if and only $\exists x_1 \cdots \exists x_m : \psi(x, y, z, x_1, \ldots, x_m)$ in $(\mathbb{M}(A,I), \iota, \mathcal{L}, (\circ_\sigma)_{\sigma\in\Sigma})$.*

We obtain an equivalent formula over $(\mathbb{M}(A,I), \iota, \mathcal{L}, (\circ_\sigma)_{\sigma\in\Sigma})$ whose size increased by a constant factor. This concludes the proof of Thm. 4.

4 Positive theories of graph products

In this section we consider positive theories. Let $\mathbb{P} = \mathbb{P}(\Sigma, I_\Sigma, (\mathcal{G}_\sigma)_{\sigma\in\Sigma})$ be a graph product, where every $\mathcal{G}_\sigma$ is a *finitely generated group*. Let Γ_σ be a finite generating set for $\mathcal{G}_\sigma$. Then $\mathbb{P}$ is generated by $\Gamma = \bigcup_{\sigma\in\Sigma} \Gamma_\sigma$. A node $\sigma \in \Sigma$ is a *cone* if $I_\Sigma(\sigma) = \Sigma \setminus \{\sigma\}$. Since we restrict to finitely generated groups, we obtain finite representations for recognizable constraints: If $L = h^{-1}(F) \in \mathrm{REC}(\mathbb{P})$, where $h : \mathbb{P} \to Q$ is a homomorphism to a finite monoid S and $F \subseteq Q$, then L can be represented by h and $F \subseteq S$. To represent h, it suffices to specify $h(a)$ for every generator $a \in \Gamma$. The next theorem is our main result for positive theories, its proof is similar to the proof of Corollary 18 in [5], see [6].

Theorem 5. *Assume that: (i) if σ is a cone, then $\mathrm{posTh}(\mathcal{G}_\sigma, (a)_{a\in\Gamma_\sigma}, \mathrm{REC}(\mathcal{G}_\sigma))$ is decidable and (ii) if σ is not a cone, then $\exists\mathrm{FOTh}(\mathcal{G}_\sigma, (a)_{a\in\Gamma_\sigma}, \mathrm{REC}(\mathcal{G}_\sigma))$ is decidable. Then $\mathrm{posTh}(\mathbb{P}, (a)_{a\in\Gamma}, \mathrm{REC}(\mathbb{P}))$ is decidable.*

The theory of $\mathbb{Z}$ with semilinear constraints (which include recognizable constraints over $\mathbb{Z}$) is decidable [17]. Since the same holds for finite groups, Thm. 5 implies that for a graph product $\mathbb{P}$ of finite groups and free groups the theory $\mathrm{posTh}(\mathbb{P}, (a)_{a\in\Gamma}, \mathrm{REC}(\mathbb{P}))$ is decidable. This result was already shown in [5].

Thm. 5 cannot be extended by allowing monoids for the groups $\mathcal{G}_\sigma$. Already the positive $\forall\exists^3$-theory of the free monoid $\{a, b\}^*$ is undecidable [10]. Similarly, Thm. 5 cannot be extended by replacing $\mathrm{REC}(\mathbb{P})$ by $\mathrm{RAT}(\mathbb{P})$, since the latter class contains a free monoid $\{a, b\}^*$ in case $\mathbb{P}$ is the free group of rank 2.

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