Word Problems and Confluence Problems for Restricted Semi-Thue Systems

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Abstract. We investigate word problems and confluence problems for the following four classes of terminating semi-Thue systems: length-reducing systems, weight-reducing systems, length-lexicographic systems, and weight-lexicographic systems. For each of these four classes we determine the complexity of several variants of the word problem and confluence problem. Finally we show that the variable membership problem for quasi context-sensitive grammars is EXPSPACE-complete.

1 Introduction

The main purpose of semi-Thue systems is to solve word problems for finitely presented monoids. But since there exists a fixed semi-Thue system $\mathcal{R}$ such that the word problem for the monoid presented by the set of equations that corresponds to $\mathcal{R}$ (in the following we will briefly speak of the word problem for $\mathcal{R}$) is undecidable [Mar47,Pos47], also semi-Thue systems cannot help for the effective solution of arbitrary word problems. This motivates the investigation of restricted classes of semi-Thue systems which give rise to decidable word problems. One of the most prominent class of semi-Thue systems with a decidable word problem is the class of all terminating and confluent semi-Thue systems. But if we want to have efficient algorithms for the solution of word problems also this class might be too large: It is known that for every $n \geq 3$ there exists a terminating and confluent semi-Thue system $\mathcal{R}$ such that the (characteristic function of the) word problem for $\mathcal{R}$ is contained in the nth Grzegorczyk class but not in the $(n-1)$th Grzegorczyk class [BO84]. Thus the complexity of the word problem for a terminating and confluent semi-Thue system can be extremely high. One way to reduce the complexity of the word problem is to force bounds on the length of derivation sequences. Such a bound can be forced by restricting to certain subclasses of terminating systems. For instance it is known that for a length-reducing and confluent semi-Thue system the word problem can be solved in linear time [Boo82]. On the other hand in [Loh99] it was shown that a uniform variant of the word problem for length-reducing and confluent semi-Thue systems (where the semi-Thue system is also part of the input) is P-complete.

In this paper we will continue the investigation of the word problem for restricted classes of terminating and confluent semi-Thue systems. We will study the following four classes of semi-Thue systems, see e.g. also [BO93], pp 41–42: length-reducing systems, weight-reducing systems, length-lexicographic systems, and weight-lexicographic systems. Let $\mathcal{C}$ be one of these four classes. We will study the following five decision problems for $\mathcal{C}$: (i) the word problem for a fixed confluent $\mathcal{R} \in \mathcal{C}$, where the input consists of two words, (ii) the uniform word problem for a fixed alphabet Σ, where the input consists of a confluent semi-Thue system $\mathcal{R} \in \mathcal{C}$ over the alphabet Σ and two words, (iii) the uniform word problem, where the input consists of a confluent $\mathcal{R} \in \mathcal{C}$ and two words, (iv) the confluence problem for a fixed alphabet Σ, where the input consists of a semi-Thue system $\mathcal{R} \in \mathcal{C}$ over the alphabet Σ, and finally (v) the confluence problem, where the input consists of an arbitrary $\mathcal{R} \in \mathcal{C}$. For each of the resulting 20 decision problems we will determine its complexity, see Table 1, p 11, and Table 2, p 13. Finally we consider a problem from [BL94], the variable membership problem for quasi context-sensitive grammars. This problem was shown to be in EXPSPACE but NEXPTIME-hard in [BL94]. In this paper we will prove that this problem is EXPSPACE-complete. We assume that the reader is familiar with the basic notions of complexity theory, in particular with the complexity classes P, PSPACE, EXPTIME, and EXPSPACE, see e.g. [Pap94].

2 Preliminaries

In the following let Σ be a finite alphabet. The empty word will be denoted by ϵ. A *weight-function* is a homomorphism $f : \Sigma^* \to \mathbb{N}$ from the free monoid Σ^* with concatenation to the natural numbers with addition such that $f(s) = 0$ if and only if $s = \epsilon$. The weight-function f with $f(a) = 1$ for all $a \in \Sigma$ is called the *length-function*. In this case for a word $s \in \Sigma^*$ we abbreviate $f(s)$ by $|s|$ and call it the *length* of s. Furthermore for every $a \in \Sigma$ we denote by $|s|_a$ the number of different occurrences of the symbol a in s. For a binary relation $\to$ on some set, we denote by $\overset{+}{\to}$ ($\overset{*}{\to}$) the transitive (reflexive and transitive) closure of $\to$.

In this paper, a *deterministic Turing-machine* is a tuple $\mathcal{M} = (Q, \Sigma, \delta, q_0, q_f)$, where Q is the finite set of states, Σ is the tape alphabet with $\Sigma \cap Q = \emptyset$, $\delta : (Q \backslash \{q_f\}) \times \Sigma \to Q \times \Sigma \times \{-1, +1\}$ is the transition function, where -1 ($+1$) means that the read-write head moves to the left (right), $q_0 \in Q$ is the initial state, and $q_f \in Q$ is the unique final state. The tape alphabet Σ always contains a blank symbol $\Box$. We assume that $\mathcal{M}$ has a one-sided infinite tape, whose cells can be identified with the natural numbers. Note that $\mathcal{M}$ cannot perform any transition out of the final state q_f. These assumptions do not restrict the computational power of Turing-machines and will always be assumed in this paper. An input for $\mathcal{M}$ is a word $w \in (\Sigma \backslash \{\Box\})^*$. A word of the form uqv, where $u, v \in \Sigma^*$ and $q \in Q$, codes the configuration, where the machine is in state q, the cells 0 to $|uv| - 1$ contain the word uv, all cells k with $k \geq |uv|$ contain the blank symbol $\Box$, and the read-write head is scanning cell $|u|$. We write $sqt \Rightarrow_{\mathcal{M}} upv$ if $\mathcal{M}$ can move in one step from the configuration sqt to the configuration upv, where $q, p \in Q$ and $st, uv \in \Sigma^*$. The language that is accepted by $\mathcal{M}$ is defined by $L(\mathcal{M}) = \{w \in (\Sigma \backslash \{\Box\})^* \mid \exists u, v \in \Sigma^* : q_0 w \overset{+}{\Rightarrow}_{\mathcal{M}} u q_f v\}$. Note that $w \in L(\mathcal{M})$ if and only if $\mathcal{M}$ terminates on the input w. A *deterministic linear bounded automaton* is a deterministic Turing-machine that operates in space $n+1$ on an input of length n.

A *semi-Thue system* $\mathcal{R}$ over Σ, briefly STS, is a finite set $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$, whose elements are called rules. See [BO93] for a good introduction to the theory of semi-Thue systems. A rule (s,t) will also be written as $s \to t$. The sets $\mathrm{dom}(\mathcal{R})$ of all left-hand sides and $\mathrm{ran}(\mathcal{R})$ of all right-hand sides are defined by $\mathrm{dom}(\mathcal{R}) = \{s \mid \exists t : (s,t) \in \mathcal{R}\}$ and $\mathrm{ran}(\mathcal{R}) = \{t \mid \exists s : (s,t) \in \mathcal{R}\}$. We define the two binary relations $\to_{\mathcal{R}}$ and $\leftrightarrow_{\mathcal{R}}$ as follows, where $x, y \in \Sigma^*$:

- $x \to_{\mathcal{R}} y$ if there exist $u, v \in \Sigma^*$ and $(s,t) \in \mathcal{R}$ with $x = usv$ and $y = utv$.
- $x \leftrightarrow_{\mathcal{R}} y$ if ($x \to_{\mathcal{R}} y$ or $y \to_{\mathcal{R}} x$).

The relation $\overset{*}{\leftrightarrow}_{\mathcal{R}}$ is a congruence relation with respect to the concatenation of words, it is called the *Thue-congruence* associated with $\mathcal{R}$. Hence we can define the quotient monoid $\Sigma^* / \overset{*}{\leftrightarrow}_{\mathcal{R}}$, which is briefly denoted by $\Sigma^*/\mathcal{R}$. We say that $\mathcal{R}$ is *terminating* if there does not exist an infinite sequence of words $s_i \in \Sigma^*$ ($i \in \mathbb{N}$) with $s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \cdots$. The set of *irreducible words* with respect to $\mathcal{R}$ is $\mathrm{IRR}(\mathcal{R}) = \Sigma^* \backslash \{stu \in \Sigma^* \mid s, u \in \Sigma^*, t \in \mathrm{dom}(\mathcal{R})\}$. A word t is a *normal form* of s if $s \overset{*}{\to}_{\mathcal{R}} t \in \mathrm{IRR}(\mathcal{R})$. We say that $\mathcal{R}$ is *confluent* if for all $s, t, u \in \Sigma^*$ with $s \overset{*}{\to}_{\mathcal{R}} t$ and $s \overset{*}{\to}_{\mathcal{R}} u$ there exists $w \in \Sigma^*$ with $t \overset{*}{\to}_{\mathcal{R}} w$ and $u \overset{*}{\to}_{\mathcal{R}} w$. We say that $\mathcal{R}$ is *locally confluent* if for all $s, t, u \in \Sigma^*$ with $s \to_{\mathcal{R}} t$ and $s \to_{\mathcal{R}} u$ there exists $w \in \Sigma^*$ with $t \overset{*}{\to}_{\mathcal{R}} w$ and $u \overset{*}{\to}_{\mathcal{R}} w$. If $\mathcal{R}$ is terminating then by Newman's lemma [New43] $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is locally confluent.

Two decision problems that are of fundamental importance in the theory of semi-Thue systems are the (uniform) word problem and the confluence problem. Let $\mathcal{C}$ be a class of STSs. The *uniform word problem*, briefly UWP, for the class $\mathcal{C}$ is the following decision problem:
INPUT: An STS $\mathcal{R} \in \mathcal{C}$ (over some alphabet Σ) and two words $u, v \in \Sigma^*$.
QUESTION: Does $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ hold?
The *confluence problem*, briefly CP, for the class $\mathcal{C}$ is the following decision problem:
INPUT: An STS $\mathcal{R} \in \mathcal{C}$.
QUESTION: Is $\mathcal{R}$ confluent?
The UWP for a singleton class $\{\mathcal{R}\}$ is called the *word problem*, briefly WP, for the STS $\mathcal{R}$.

For the class of all terminating STSs the CP is known to be decidable [NB72]. This classical result is based on the so called critical pairs of an STS, which result from overlapping left-hand sides. A pair $(s_1, s_2) \in \Sigma^* \times \Sigma^*$ is a critical pair of $\mathcal{R}$ if there exist rules $(t_1, u_1), (t_2, u_2) \in \mathcal{R}$ such that one of the following two cases holds:

- $t_1 = v t_2 w$, $s_1 = u_1$, and $s_2 = v u_2 w$ for some $v, w \in \Sigma^*$ (here the word $t_1 = v t_2 w$ is an overlapping of t_1 and t_2).

– $t_1 = vt$, $t_2 = tw$, $s_1 = u_1w$, and $s_2 = vu_2$ for some $t, v, w \in \Sigma^*$ with $t \neq \epsilon$ (here the word vtw is an overlapping of t_1 and t_2).

Note that there are only finitely many critical pairs of $\mathcal{R}$. In order to check whether a terminating STS $\mathcal{R}$ is confluent it suffices to calculate for every critical pair (s, t) of $\mathcal{R}$ arbitrary normal forms of s and t. If for some critical pair these normal forms are not identical then $\mathcal{R}$ is not confluent, otherwise $\mathcal{R}$ is confluent.

Similarly for the class of all terminating and confluent STSs the UWP is decidable [KB67]: In order to check whether $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ holds for given words $s, t \in \Sigma^*$ we compute arbitrary normal forms of s and t. Then $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ if and only if these normal forms are the same.

In this paper we consider the following classes of terminating systems. An STS $\mathcal{R}$ is *length-reducing* if $|s| > |t|$ for all $(s, t) \in \mathcal{R}$. An STS $\mathcal{R}$ is *weight-reducing* if there exists a weight-function f such that $f(s) > f(t)$ for all $(s, t) \in \mathcal{R}$. An STS $\mathcal{R}$ is *length-lexicographic* if there exists a linear order $\succ$ on the alphabet Σ such that for all $(s, t) \in \mathcal{R}$ it holds $|s| > |t|$ or ($|s| = |t|$ and there exist $u, v, w \in \Sigma^*$ and $a, b \in \Sigma$ with $s = uav$, $t = ubw$, and $a \succ b$). An STS $\mathcal{R}$ is *weight-lexicographic* if there exist a linear order $\succ$ on the alphabet Σ and a weight-function f such that for all $(s, t) \in \mathcal{R}$ it holds $f(s) > f(t)$ or ($f(s) = f(t)$ and there exist $u, v, w \in \Sigma^*$ and $a, b \in \Sigma$ with $s = uav$, $t = ubw$, and $a \succ b$).

For all these classes, restricted to confluent STSs, the UWP is decidable. Since we want to determine the complexity of the UWP for these classes, we have to define the *length of an STS*. For an STS $\mathcal{R}$ it is natural to define its length $\|\mathcal{R}\|$ by $\|\mathcal{R}\| = \sum_{(s,t)\in\mathcal{R}} |st|$.

3 Length-reducing semi-Thue systems

In [Loh99] it was shown that the UWP for the class of all length-reducing and confluent STSs over $\{a, b\}$ is P-complete. In this section we prove that for a fixed STS the complexity decreases to LOGCFL. Recall that LOGCFL is the class of all problems that are log space reducible to the membership problem for a context-free language [Sud78]. It is strongly conjectured that LOGCFL is a proper subset of P.

Theorem 1. *Let $\mathcal{R}$ be a fixed length-reducing and confluent STS. Then the WP for $\mathcal{R}$ is in LOGCFL.*

Proof. Let $\mathcal{R}$ be a fixed length-reducing STS over Σ and let $s \in \Sigma^*$. From the results of [DW86] [1] it follows immediately that the following problem is in LOGCFL:
INPUT: A word $t \in \Sigma^*$.
QUESTION: Does $t \overset{*}{\rightarrow}_{\mathcal{R}} s$ hold?

[1] The main result of [DW86] is that the membership problem for a fixed growing context-sensitive grammar is in LOGCFL. Note that the uniform variant of this problem is NP-complete [KN87,CH90,BL92].

Now let $\mathcal{R}$ be a fixed length-reducing and confluent STS over Σ and let $u, v \in \Sigma^*$. Let $\overline{\Sigma} = \{\overline{a} \mid a \in \Sigma\}$ be a disjoint copy of Σ. For a word $s \in \Sigma^*$ define the word $\overline{s}^{\text{rev}} \in \overline{\Sigma}^*$ inductively by $\overline{\epsilon}^{\text{rev}} = \epsilon$ and $\overline{at}^{\text{rev}} = \overline{t}^{\text{rev}}\overline{a}$ for $a \in \Sigma$ and $t \in \Sigma^*$. Define the length-reducing STS $\mathcal{P}$ by

$$\mathcal{P} = \mathcal{R} \cup \{\overline{s}^{\text{rev}} \to \overline{t}^{\text{rev}} \mid (s,t) \in \mathcal{R}\} \cup \{a\overline{a} \to \epsilon \mid a \in \Sigma\}.$$

Since $\mathcal{R}$ is confluent, it holds $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ if and only if $u\overline{v}^{\text{rev}} \overset{*}{\to}_{\mathcal{P}} \epsilon$. The later property can be checked in LOGCFL. Clearly $u\overline{v}^{\text{rev}}$ can be constructed in log space from u and v. □

4 Weight-reducing semi-Thue systems

Weight-reducing STSs were investigated for instance in [Die87,Jan88,NO88] and [BL92] as a grammatical formalism. The WP for a fixed weight-reducing and confluent STS can be easily reduced to the WP for a fixed length-reducing and confluent STS. Thus the WP for every fixed weight-reducing and confluent STS can also be solved in LOGCFL:

Theorem 2. *Let $\mathcal{R}$ be a fixed weight-reducing and confluent STS. Then the WP for $\mathcal{R}$ is in LOGCFL.*

Proof. Let $\mathcal{R}$ be a weight-reducing and confluent STS over Σ and let $u, v \in \Sigma^*$. Let f be a weight-function such that $f(s) > f(t)$ for all $(s,t) \in \mathcal{R}$. Let $\$ \notin \Sigma$ and define the morphism $\varphi : \Sigma^* \to (\Sigma \cup \{\$\})^*$ by $\varphi(a) = \$^{f(a)}a^{f(a)}$ for all $a \in \Sigma$. Note that non-trivial overlappings between two words $\varphi(a)$ and $\varphi(b)$ are not possible. It follows that the STS $\varphi(\mathcal{R}) = \{\varphi(s) \to \varphi(t) \mid (s,t) \in \mathcal{R}\}$ is length-reducing and confluent, and we see that $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ if and only if $\varphi(u) \overset{*}{\leftrightarrow}_{\varphi(\mathcal{R})} \varphi(v)$. Since $\varphi(u)$ and $\varphi(v)$ can be constructed in log space, the theorem follows from Theorem 1. □

Next we will consider the UWP for weight-reducing and confluent STSs over a fixed alphabet Σ. In order to get an upper bound for this problem we need the following lemma, which we state in a slightly more general form for later applications.

Lemma 1. *Let Σ be a finite alphabet with $|\Sigma| = n$ and let $\mathcal{R}$ be an STS over Σ with $\alpha = \max\{|s|_a \mid s \in dom(\mathcal{R}) \cup ran(\mathcal{R}), a \in \Sigma\}$. Let g be a weight-function with $g(s) \geq g(t)$ for all $(s,t) \in \mathcal{R}$. Then there exists a weight-function f such that for all $(s,t) \in \mathcal{R}$ the following holds:*

- *If $g(s) > g(t)$ then $f(s) > f(t)$, and if $g(s) = g(t)$ then $f(s) = f(t)$.*
- *$f(a) \leq (n+1)(\alpha n)^n$ for all $a \in \Sigma$.*

Proof. We use the following result about solutions of integer (in)equalities from [vZGS78]: Let A, B, C, D be $(m \times n)$-, $(m \times 1)$-, $(p \times n)$-, $(p \times 1)$-matrices, respectively, with integer entries. Let $r = \text{rank}(A)$, $s = \text{rank}\begin{pmatrix} A \\ C \end{pmatrix}$. Let M be an upper bound on the absolute values of all $(s-1)\times(s-1)$- or $(s \times s)$-subdeterminants of the $(m+p)\times(n+1)$-matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, which are formed with at least r rows from the matrix $(A \;\; B)$. Then the system $Ax = B$, $Cx \geq D$ has an integer solution if and only if it has an integer solution x such that the absolute value of every entry of x is bounded by $(n+1)M$.

Now let Σ, n, $\mathcal{R}$, α, and g be as specified in the lemma. Let $\Sigma = \{a_1, \ldots, a_n\}$ and $\mathcal{R} = \{(s_i, t_i) \mid 1 \leq i \leq k\} \cup \{(u_i, v_i) \mid 1 \leq i \leq \ell\}$, where $g(s_i) = g(t_i)$ for $1 \leq i \leq k$ and $g(u_i) > g(v_i)$ for $1 \leq i \leq \ell$. Define the $(k \times n)$-matrix A by $A_{i,j} = |s_i|_{a_j} - |t_i|_{a_j}$ and define the $(\ell \times n)$-matrix C' by $C'_{i,j} = |u_i|_{a_j} - |v_i|_{a_j}$. Let $C = \begin{pmatrix} C' \\ \text{Id}_n \end{pmatrix}$, where Id_n is the $(n \times n)$-identity matrix. Finally let $(j)_i$ be the i-dimensional column vector with all entries equal to j. Then the n-dimensional column vector x with $x_j = g(a_j)$ is a solution of the following system:

$$Ax = (0)_k \qquad Cx \geq (1)_{\ell+n} \tag{1}$$

Note that $r = \text{rank}(A) \leq n$ and $s = \text{rank}\begin{pmatrix} A \\ C \end{pmatrix} \leq n$. Furthermore every entry of the matrix $E = \begin{pmatrix} A & (0)_k \\ C & (1)_{\ell+n} \end{pmatrix}$ is bounded by α. Thus the absolute value of every $(s-1)\times(s-1)$- or $(s \times s)$-subdeterminant of E is bounded by $s! \cdot \alpha^s \leq (\alpha n)^n$. By the result of [vZGS78] the system (1) has a solution y with $y_j \leq (n+1)(\alpha n)^n$ for all $1 \leq j \leq n$. If we define the weight-function f by $f(a_j) = y_j$ then f has the properties stated in the lemma. □

Theorem 3. *Let Σ be a fixed alphabet with $|\Sigma| \geq 2$. Then the UWP for the class of all weight-reducing and confluent STSs over Σ is P-complete.*

Proof. Let $|\Sigma| = n \geq 2$. Let $\mathcal{R}$ be a weight-reducing and confluent STS over Σ and let $u, v \in \Sigma^*$. By Lemma 1 there exists a weight-function f such that $f(s) > f(t)$ for all $(s,t) \in \mathcal{R}$ and $f(a) \leq (n+1)(\alpha n)^n$ for all $a \in \Sigma$. Thus every derivation that starts from the word u has a length bounded by $|u| \cdot (n+1) \cdot (\alpha n)^n$, which is polynomial in the input length $\|\mathcal{R}\| + |uv|$. Thus a normal form of u can be calculated in polynomial time and similarly for v. This proves the upper bound. P-hardness follows from the fact that the UWP for the class of all length-reducing and confluent STSs over $\{a, b\}$ is P-complete [Loh99]. □

Finally for the class of all weight-reducing and confluent STSs the complexity of the UWP increases to EXPTIME:

Theorem 4. *The UWP for the class of all weight-reducing and confluent STSs is EXPTIME-complete.*

Proof. The EXPTIME-upper bound can be shown by using the arguments from the previous proof. Just note that this time the upper bound of $(n+1)(\alpha n)^n$ for a weight-function is exponential in the length of the input. For the lower bound let $\mathcal{M} = (Q, \Sigma, \delta, q_0, q_f)$ be a deterministic Turing-machine such that for some polynomial p it holds: If $w \in L(\mathcal{M})$ then $\mathcal{M}$, started on w, reaches the final state q_f after at most $2^{p(|w|)}$ many steps. Let $w \in (\Sigma \backslash \{\Box\})^*$ be an arbitrary input for $\mathcal{M}$. Let $m = p(|w|)$ and let

$$\Gamma = Q \cup \Sigma \cup \bigcup_{i=0}^{m} (\Sigma_i \cup \{\triangleright_i, 2^{(i)}, A_i, B_i\}) \cup \{\#, \triangleright\}.$$

Here $2^{(i)}$ is a single symbol and $\Sigma_i = \{a_i \mid a \in \Sigma\}$ is a disjoint copy of Σ for $0 \leq i \leq m$. Let $\Sigma_\triangleright = \Sigma \cup \{\triangleright\}$ and let $\mathcal{R}$ be the STS over Γ that consists of the following rules:

(1)	$2^{(i)}ab \to a_m 2^{(i-1)} \cdots 2^{(1)} 2^{(0)} b$	for $0 \leq i \leq m$, $a \in \Sigma_\triangleright$, $b \in \Sigma$
(2)	$2^{(i)}a_k \to a_{k-1} 2^{(i)}$	for $0 \leq i \leq m$, $1 \leq k \leq m$, $a \in \Sigma_\triangleright$
(3)	$\# a_k \to a \#$	for $0 \leq k \leq m$, $a \in \Sigma_\triangleright$
(4)	$\# x \to x$	for $x \in \Sigma \cup Q \backslash \{q_f\}$
(5)	$2^{(i)} q \to q$	for $q \in Q \backslash \{q_f\}$
(6)	$2^{(i)} cqa \to cbp$	for $0 \leq i \leq m$, $c \in \Sigma_\triangleright$, $\delta(q, a) = (p, b, +1)$
(7)	$2^{(i)} cqa \to pcb$	for $0 \leq i \leq m$, $c \in \Sigma_\triangleright$, $\delta(q, a) = (p, b, -1)$
(8)	$A_i \to A_{i+1} A_{i+1}$	for $0 \leq i < m$
(9)	$B_i \to B_{i+1} B_{i+1}$	for $0 \leq i < m$
(10)	$A_m \to \# 2^{(m)}$	
(11)	$B_m \to \Box$	
(12)	$x q_f \to q_f$	for $x \in \Gamma$
(13)	$q_f x \to q_f$	for $x \in \Gamma$

We claim that $\mathcal{R}$ is weight-reducing. For this we define the weight-function f as follows: [2]

$$f(A_i) = 2 \cdot f(A_{i+1}) + 1 \text{ for } 0 \leq i < m \qquad f(A_m) = 2^m + 2$$

$$f(B_i) = 2 \cdot f(B_{i+1}) + 1 \text{ for } 0 \leq i < m \qquad f(B_m) = 2$$

$$f(x) = 1 \text{ for } x \in Q \cup \Sigma_\triangleright \cup \{\#\} \qquad f(2^{(i)}) = 2^i \text{ for } 0 \leq i \leq m$$

$$f(a_i) = 1 + \frac{i+1}{m+2} \text{ for } 0 \leq i \leq m,\ a \in \Sigma_\triangleright$$

Then it is easy to check that $f(s) > f(t)$ for all $(s, t) \in \mathcal{R}$. All non-trivial critical pairs of $\mathcal{R}$ are of the form (sq_f, tq_f) (where $(sx, t) \in \mathcal{R}$, $x \in \Sigma$), $(q_f s, q_f t)$ (where $(xs, t) \in \mathcal{R}$, $x \in \Sigma$), or $(xq_f, q_f y)$ (where $x, y \in \Sigma$). By the rules in (12) and (13) both components of these critical pairs can be reduced to q_f. Thus $\mathcal{R}$ is confluent. Finally we claim that $A_0 \triangleright q_0 w B_0 \overset{*}{\leftrightarrow}_{\mathcal{R}} q_f$ if and only if $w \in L(\mathcal{M})$.

Before we prove this claim let us first explain the effect of the rules from $\mathcal{R}$. For $0 \leq i \leq 2^m$ let $\text{sum}(i) = 2^{(i_1)} \cdots 2^{(i_k)} \in \Gamma^*$ if $i_1 > \cdots > i_k$ and $i = 2^{i_1} + \cdots + 2^{i_k}$ (note that $\text{sum}(0) = \epsilon$). Let us call a word of the form

[2] Here we use rational weights, but of course they can be replaced by integer weights.

$\#\mathrm{sum}(i) \in \Gamma^*$ a counter with value i. The effect of the rules in (1), (2), and (3) is to move counters to the right in words from $\triangleright\Sigma^*$. Here the symbol $\triangleright$ is a left-end marker. If a whole counter moves one step to the right, its value is decreased by one. More generally for all $u \in \Sigma^*$, $b \in \Sigma$, and all $|u| < i \leq 2^m$ we have $\#\mathrm{sum}(i)\triangleright ub \xrightarrow{*}_{\mathcal{R}} \triangleright u\#\mathrm{sum}(i-|u|-1)b$. If a counter has reached the value 0, i.e, it consists only of the symbol # then the counter is deleted with a rule in (4). Also if a counter collides with a state symbol from Q at its right end, then the counter is deleted with the rules in (4) and (5). Note that such a collision may occur after an application of a rule in (7). The rules in (6) and (7) simulate the machine $\mathcal{M}$. In order to be weight-reducing, these rules consume the right-most symbol of the right-most counter. The rules in (8) and (10) produce 2^m many counters of the form $\#2^{(m)}$. Each of these counters can move at most 2^m cells to the right. But since $\mathcal{M}$ terminates after at most 2^m many steps, the distance between the left end of the tape and the read-write head is also at most 2^m. This implies that with each of the 2^m many counters that are produced from A_0, at least one step of $\mathcal{M}$ can be simulated. The rules in (9) and (11) produce 2^m many blank symbols, which is enough in order to simulate 2^m many steps of $\mathcal{M}$. Finally the rules in (12) and (13) make the final state q_f absorbing.

Now if $w \in L(\mathcal{M})$ then $A_0\triangleright q_0 w B_0 \xrightarrow{*}_{\mathcal{R}} (\#2^{(m)})^{2^m}\triangleright q_0 w\Box^{2^m} \xrightarrow{*}_{\mathcal{R}} uq_f v \xrightarrow{*}_{\mathcal{R}} q_f$ for some $u, v \in \Gamma^*$. On the other hand if $w \notin L(\mathcal{M})$ then $\mathcal{M}$ does not terminate on w. By simulating $\mathcal{M}$ long enough, and thereby consuming all 2^m many initial counters, we obtain $A_0 \triangleright q_0 w B_0 \xrightarrow{*}_{\mathcal{R}} (\#2^{(m)})^{2^m} \triangleright q_0 w\Box^{2^m} \xrightarrow{*}_{\mathcal{R}} \triangleright uqv \in \mathrm{IRR}(\mathcal{R})$ for some $q \in Q\backslash\{q_f\}$, $u, v \in \Sigma^*$. Since also $q_f \in \mathrm{IRR}(\mathcal{R})$ and $\mathcal{R}$ is confluent, $A_0 \triangleright q_0 w B_0 \xleftrightarrow{*}_{\mathcal{R}} q_f$ cannot hold. $\square$

Since P is a proper subclass of EXPTIME, it follows from Theorem 3 and Theorem 4 that in general it is not possible to encode the alphabet of a weight-reducing and confluent STS into a fixed alphabet with a polynomial blow-up such that the resulting STS is still weight-reducing and confluent. For length-reducing systems this is always possible, see [Loh99] and the coding function from the proof of Theorem 5.

5 Length-lexicographic semi-Thue systems

In this section we consider length-lexicographic semi-Thue systems, see for instance [KN85]. The complexity bounds that we will achieve in this section are the same that are known for preperfect systems. An STS $\mathcal{R}$ is *preperfect* if for all $s, t \in \Sigma^*$ it holds $s \xleftrightarrow{*}_{\mathcal{R}} t$ if and only if there exists $u \in \Sigma^*$ with $s \overset{*}{\mapsto}_{\mathcal{R}} u$ and $t \overset{*}{\mapsto}_{\mathcal{R}} u$, where the relation $\mapsto_{\mathcal{R}}$ is defined by $v \mapsto_{\mathcal{R}} w$ if $v \leftrightarrow_{\mathcal{R}} w$ and $|v| \geq |w|$. Since every length-preserving STS is preperfect and every linear bounded automaton can easily be simulated by a length-preserving STS, there exists a fixed preperfect STS $\mathcal{R}$ such that the WP for $\mathcal{R}$ is PSPACE-complete [BJM+81]. The following theorem may be seen as a stronger version of this well-known fact in the sense that a deterministic linear bounded automaton can even be simulated by a length-lexicographic, length-preserving, and confluent STS.

Theorem 5. *The WP for a length-lexicographic and confluent STS is contained in PSPACE. Furthermore there exists a fixed length-lexicographic and confluent STS $\mathcal{R}$ over $\{a, b\}$ such that the WP for $\mathcal{R}$ is PSPACE-complete.*

Proof. The first statement of the theorem is obvious. For the second statement let $\mathcal{M} = (Q, \Sigma, \delta, q_0, q_f)$ be a deterministic linear bounded automaton such that the question whether $w \in L(\mathcal{M})$ is PSPACE-complete. Such a linear bounded automaton exists, see e.g. [BO84]. We may assume that $\mathcal{M}$ operates in phases, where a single phase consists of a sequence of $2 \cdot n$ transitions of the form $q_1 w_1 \overset{*}{\Rightarrow}_{\mathcal{M}} w_2 q_2 \overset{*}{\Rightarrow}_{\mathcal{M}} q_3 w_3$, where $w_1, w_2, w_3 \in \Sigma^*$ and $q_1, q_2, q_3 \in Q$. During the sequence $q_1 w_1 \overset{*}{\Rightarrow}_{\mathcal{M}} w_2 q_2$ only right-moves are made, and during the sequence $w_2 q_2 \overset{*}{\Rightarrow}_{\mathcal{M}} q_3 w_3$ only left-moves are made. A similar trick is used for instance also in [CH90]. Let $c > 0$ be constant such that if $w \in L(\mathcal{M})$ then $\mathcal{M}$, started on w, reaches the final state q_f after at most $2^{c \cdot n}$ phases. Let $w \in (\Sigma \backslash \{\Box\})^*$ be an input for $\mathcal{M}$ with $|w| = n$. As usual let $\overline{\Sigma}$ be a disjoint copy of Σ and similarly for $\overline{Q}$. Let $\Gamma = Q \cup \overline{Q} \cup \Sigma \cup \overline{\Sigma} \cup \{\triangleleft, 0, 1, \overline{1}\}$ and let $\mathcal{R}$ be the STS over Γ that consists of the following rules: [3]

$0\overline{q} \to \overline{q}\,\overline{1}$	for all $q \in Q$
$1\overline{q} \to 0q$	for all $q \in Q$
$q\overline{1} \to 1q$	for all $q \in Q$
$qa \to \overline{b}p$	if $\delta(q, a) = (p, b, +1)$
$q\triangleleft \to \overline{q}\triangleleft$	for all $q \in Q \backslash \{q_f\}$
$\overline{a}\,\overline{q} \to \overline{p}b$	if $\delta(q, a) = (p, b, -1)$
$xq_f \to q_f$	for all $x \in \Gamma$
$q_f x \to q_f$	for all $x \in \Gamma$

First we claim that $\mathcal{R}$ is length-lexicographic. For this choose a linear order $\succ$ on the alphabet Γ that satisfies $Q \succ 1 \succ 0 \succ \overline{\Sigma} \succ \overline{Q}$ (here for instance $Q \succ 1$ means that $q \succ 1$ for every $q \in Q$). Furthermore $\mathcal{R}$ is confluent. Finally we claim that $10^{c \cdot n} q_0 w \triangleleft \overset{*}{\leftrightarrow}_{\mathcal{R}} q_f$ if and only if $w \in L(\mathcal{M})$. For $v = b_k \cdots b_0 \in \{0, 1\}^*$ ($b_i \in \{0, 1\}$) let $\mathrm{val}(v) = \sum_{i=0}^{k} b_i \cdot 2^i$. Note that for every $q \in Q$ and $s, t \in \{0, 1\}^+$ with $s \neq 0^{|s|}$ it holds $s\overline{q} \overset{*}{\rightarrow}_{\mathcal{R}} tq$ if and only if $|s| = |t|$ and $\mathrm{val}(t) = \mathrm{val}(s) - 1$. First assume that $w \in L(\mathcal{M})$. Then $10^{c \cdot n} q_0 w \triangleleft \overset{*}{\rightarrow}_{\mathcal{R}} v q_f u \triangleleft \overset{*}{\rightarrow}_{\mathcal{R}} q_f$ for some $u \in \Sigma^*$ and $v \in \{0, 1\}^+$. Now assume that $w \notin L(\mathcal{M})$. Then $\mathcal{M}$ does not terminate on w and we obtain $10^{c \cdot n} q_0 w \triangleleft \overset{*}{\rightarrow}_{\mathcal{R}} 0^{c \cdot n + 1} \overline{q} u \triangleleft \overset{*}{\rightarrow}_{\mathcal{R}} \overline{q}\,\overline{1}^{c \cdot n + 1} u \triangleleft \in \mathrm{IRR}(\mathcal{R})$, where $u \in \Sigma^*$ and $q \in Q \backslash \{q_f\}$. Since also $q_f \in \mathrm{IRR}(\mathcal{R})$ and $\mathcal{R}$ is confluent, $10^{c \cdot n} q_0 w \triangleleft \overset{*}{\leftrightarrow}_{\mathcal{R}} q_f$ cannot hold.

Finally, we have to encode the alphabet Γ into the alphabet $\{a, b\}$. For this let $\Gamma = \{a_1, \ldots, a_k\}$ and let $a_1 \succ a_2 \succ \cdots \succ a_k$ be the chosen linear order on Γ. Define a morphism $\varphi : \Gamma^* \longrightarrow \{a, b\}^*$ by $\varphi(a_i) = ab^i ab^{2k+1-i}$ and let $a \succ b$. Then the STS $\varphi(\mathcal{R})$ is also length-lexicographic and confluent and for all $u, v \in \Gamma^*$ it holds $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ if and only if $\varphi(u) \overset{*}{\leftrightarrow}_{\varphi(\mathcal{R})} \varphi(v)$, see [BO93], p 60. ◻

[3] It will be always clear from the context whether e.g. 1 denotes the symbol $1 \in \Gamma$ or the natural number 1.

6 Weight-lexicographic semi-Thue systems

The widest class of STSs that we study in this paper are weight-lexicographic STSs. Let $\mathcal{R}$ be a weight-lexicographic STS over an alphabet Σ with $|\Sigma| = n$ and let $u \in \Sigma^*$. Thus there exists a weight-function f with $f(s) \geq f(t)$ for all $(s,t) \in \mathcal{R}$. If $u = u_0 \to_{\mathcal{R}} u_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} u_n$ is some derivation then for all $0 \leq i \leq n$ it holds $|u_i| \leq f(u_i) \leq f(u)$. By Lemma 1 we may assume that $f(a) \leq (n+1)(\alpha n)^n$ for all $a \in \Sigma$ and thus $|u_i| \leq |u| \cdot (n+1)(\alpha n)^n$. Together with Theorem 5 it follows that the UWP for weight-lexicographic and confluent STSs over a fixed alphabet is PSPACE-complete and furthermore that there exists a fixed weight-lexicographic and confluent STS whose WP is PSPACE-complete. For arbitrary weight-lexicographic and confluent STSs we have the following result.

Theorem 6. *The UWP for the class of all weight-lexicographic and confluent STSs is EXPSPACE-complete.*

Proof. The EXPSPACE-upper bound can be shown by using the arguments above. For the lower bound let $\mathcal{M} = (Q, \Sigma, \delta, q_0, q_f)$ be a deterministic Turing-machine which uses for every input w at most space $2^{p(|w|)}$, where p is some polynomial. Similarly to the proof of Theorem 5 we may assume that $\mathcal{M}$ operates in phases. There exists a polynomial q such that if $w \in L(\mathcal{M})$ then $\mathcal{M}$, started on w, reaches q_f after at most $2^{2^{q(|w|)}}$ many phases. Let $w \in (\Sigma \backslash \{\Box\})^*$ be an arbitrary input for $\mathcal{M}$. Let $m = p(|w|)$, $n = q(|w|)$, and

$$\Gamma = Q \cup \overline{Q} \cup \Sigma \cup \overline{\Sigma} \cup \{\triangleleft, 0, 1, \overline{1}\} \cup \{A_i \mid 0 \leq i \leq n\} \cup \{B_i \mid 0 \leq i \leq m\}.$$

Let $\mathcal{R}$ be the STS over Γ that consists of the following rules:

$0\overline{q} \to \overline{q}\overline{1}$	for all $q \in Q$
$1\overline{q} \to 0q$	for all $q \in Q$
$q\overline{1} \to 1q$	for all $q \in Q$
$qa \to \overline{b}p$	if $\delta(q,a) = (p, b, +1)$
$q\triangleleft \to \overline{q}\triangleleft$	for all $q \in Q\backslash\{q_f\}$
$\overline{a}\,\overline{q} \to \overline{p}b$	if $\delta(q,a) = (p, b, -1)$
$A_i \to A_{i+1}A_{i+1}$	for $0 \leq i < n$
$B_i \to B_{i+1}B_{i+1}$	for $0 \leq i < m$
$A_n \to 0$	
$B_m \to \Box$	
$xq_f \to q_f$	for all $x \in \Gamma$
$q_f x \to q_f$	for all $x \in \Gamma$

Note that the first six rules are exactly the same rules that we used for the simulation of a linear bounded automaton in the proof of Theorem 5. We claim that $\mathcal{R}$ is weight-lexicographic. For this define the weight-function f by $f(x) = 1$ for all $x \in Q \cup \overline{Q} \cup \Sigma \cup \overline{\Sigma} \cup \{\triangleleft, 0, 1, \overline{1}, A_n, B_m\}$ and $f(A_i) = 2 \cdot f(A_{i+1})$, $f(B_j) = 2 \cdot f(B_{j+1})$ for $0 \leq i < n$, $0 \leq j < m$. Then the last two rules are weight-reducing and all other rules are weight-preserving. Now choose a linear order $\succ$ on Γ that satisfies $Q \succ 1 \succ 0 \succ \overline{\Sigma} \succ \overline{Q}$, $A_0 \succ A_1 \succ \cdots \succ A_n \succ 0$, and $B_0 \succ B_1 \succ \cdots \succ B_m \succ \Box$. It is easy to see that $\mathcal{R}$ is confluent. Finally we have $w \in L(\mathcal{M})$ if and only if $1A_0q_0wB_0\triangleleft \overset{*}{\leftrightarrow}_{\mathcal{R}} q_f$. This can be shown by using the arguments from the proof of Theorem 5. Just note that this time from the word $1A_0$ we can generate the word 10^{2^n} which allows the simulation of 2^{2^n} many phases. Analogously to the proof of Theorem 4 the symbol B_0 generates enough blank symbols in order to satisfy the space requirements of $\mathcal{M}$. □

7 Confluence problems

The CP for the class of all STSs is undecidable [BO84]. On the other hand, the CP for the class of all terminating STSs is decidable [NB72]. For length-reducing STSs the CP is in P [BO81], the best known algorithm is the $O(\|\mathcal{R}\|^3)$-algorithm from [KKMN85]. Furthermore in [Loh99] it was shown that the CP for the class of all length-reducing STSs is P-complete. This was shown by using the following log space reduction from the UWP for length-reducing and confluent STSs to the CP for length-reducing STSs, see also [VRL98], Theorem 24: Let $\mathcal{R}$ be a length-reducing and confluent STS over Σ. Furthermore let A and B be new symbols. Then for all $s, t \in \Sigma^*$ the length-reducing STS $\mathcal{R} \cup \{A^{|st|}B \to s,\ A^{|st|}B \to t\}$ is confluent if and only if $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ holds. Finally the alphabet $\Sigma \cup \{A, B\}$ can be reduced to the alphabet $\{a, b\}$ by using the coding function from the end of the proof of Theorem 5. The same reduction can be also used for weight-reducing, length-lexicographic, and weight-lexicographic STSs. Thus a lower bound for the UWP for one of the classes considered in the preceding sections carrys over to the CP for this class. Furthermore also the given upper bounds hold for the CP for the corresponding class: Our upper bound algorithms for UWPs are all based on the calculation of normal forms. But since every STS has only polynomially many critical pairs, any upper bound for the calculation of normal forms also gives an upper bound for the CP. The resulting complexity results are summarized in Table 1.

Table 1. Complexity results for confluence problems

	length-reducing STSs	weight-reducing STSs	length-lexicographic STSs	weight-lexicographic STSs
CP for a fixed alphabet	P-complete	P-complete	PSPACE-complete	PSPACE-complete
CP	P-complete	EXPTIME-complete	PSPACE-complete	EXPSPACE-complete

8 Quasi context-sensitive grammars

A *quasi context-sensitive grammar*, briefly QCSG, is a (type-0) grammar $G = (N, T, S, P)$ (here N is the set of non-terminals, T is the set of terminals, $S \in N$ is the start non-terminal, and $P \subseteq (N \cup T)^* N (N \cup T)^* \times (N \cup T)^*$ is a finite set of productions) such that for some weight-function $f : (N \cup T)^* \to \mathbb{N}$ we have $f(u) \leq f(v)$ for all $(u, v) \in P$, see [BL94]. The *variable membership problem for QCSGs* it the following problem:
INPUT: A QCSG G with terminal alphabet T and a terminal word $v \in T^*$.
QUESTION: Does $v \in L(G)$ hold?
In [BL94] it was shown that this problem is in EXPSPACE and furthermore that it is NEXPTIME-hard. Using some ideas from Section 4 we can prove that this problem is in fact EXPSPACE-hard.

Theorem 7. *The variable membership problem for QCSGs is EXPSPACE-complete.*

Proof. It remains to show that the problem is EXPSPACE-hard. For this let $\mathcal{M} = (Q, \Sigma, \delta, q_0, q_f)$ be a Turing-machine, which uses for every input w at most space $2^{p(|w|)} - 2$ for some polynomial p. Let $w \in (\Sigma \backslash \{\Box\})^*$ be an input for $\mathcal{M}$ and let $m = p(|w|)$. We will construct a QCSG $G = (N, T, S, P)$ and a word $v \in T^*$ such that $w \in L(\mathcal{M})$ if and only if $v \in L(G)$. The non-terminal and terminal alphabet of G are $N = \{S, B\} \cup \{A_i \mid 0 \leq i < m\} \cup Q \cup \Sigma$ and $T = \{A_m\}$. The set P consists of the following productions:

$S \to q_0 w B$	
$B \to \Box B$	
$qa \to bp$	if $\delta(q, a) = (p, b, +1)$
$cqa \to pcb$	if $\delta(q, a) = (p, b, -1)$, $c \in \Sigma$
$q_f \to A_0$	
$xA_0 \to A_0 A_0$	for $x \in \Sigma$
$A_0 x \to A_0 A_0$	for $x \in \Sigma \cup \{B\}$
$A_i A_i \to A_{i+1}$	for $0 \leq i < m$

In order to show that G is quasi context-sensitive we define the weight-function $f : (N \cup T)^* \to \mathbb{N}$ by $f(x) = 1$ for all $x \in \{S, B\} \cup \Sigma \cup Q$ and $f(A_i) = 2^i$ for all $0 \leq i \leq m$. Then $f(s) \leq f(t)$ for all $(s, t) \in P$. If $w \in L(\mathcal{M})$ then $A_m \in L(G)$ by the following derivation:

$$S \to q_0 w B \xrightarrow{*}_P q_0 w \Box^{2^m - |w| - 2} B \xrightarrow{*}_P s q_f t B \to_P s A_0 t B \xrightarrow{*}_P A_0^{2^m} \xrightarrow{*}_P A_m,$$

where $s, t \in \Sigma^*$ and $|st| = 2^m - 2$. On the other hand, if $A_m \in L(G)$ then a sentential form $u q_f v$ with $uv \in \Gamma^*$ and $|uv| = 2^m - 1$ must be reachable from S, i.e, reachable from $q_0 w B$. This is only possible if $w \in L(\mathcal{M})$. Furthermore G and v can be calculated from $\mathcal{M}$ and w in log space. This concludes the proof. $\Box$

Note that from the previous proof it follows immediately that the following problem is also EXPSPACE-complete:

INPUT: A context-sensitive grammar G with terminal alphabet $\{a\}$ and a number $n \in \mathbb{N}$ coded in binary.
QUESTION: Does $a^n \in L(G)$ hold?
The same problem for context-free grammars is NP-complete [Huy84].

9 Summary and open problems

The complexity results for WPs are summarized in Table 2. Here the statement in the first row that e.g. the WP for length-lexicographic and confluent STSs is PSPACE-complete means that for every length-lexicographic and confluent STS the WP is in PSPACE and furthermore there exists a fixed length-lexicographic and confluent STS whose WP is PSPACE-complete. Furthermore the completeness results in the second row already hold for the alphabet $\{a, b\}$.

Table 2. Complexity results for word problems

	length-reducing & confluent STSs	weight-reducing & confluent STSs	length-lexicographic & confluent STSs	weight-lexicographic & confluent STSs
WP	LOGCFL	LOGCFL	PSPACE-complete	PSPACE-complete
UWP for a fixed alphabet	P-complete	P-complete	PSPACE-complete	PSPACE-complete
UWP	P-complete	EXPTIME-complete	PSPACE-complete	EXPSPACE-complete

One open question that remains concerns the WP for a fixed length-reducing (weight-reducing) and confluent STS. Does there exist such a system whose WP is LOGCFL-complete or are these WPs always contained for instance in the subclass LOGDCFL, the class of all languages that are log space reducible to a deterministic context-free language? Since there exits a fixed deterministic context-free language whose membership problem is LOGDCFL-complete [Sud78], Theorem 2.2 of [MNO88] implies that there exists a fixed length-reducing and confluent STS whose WP is LOGDCFL-hard.

Another interesting open problem is the descriptive power of the STSs considered in this paper. Let $\mathcal{M}_{\ell r}$ ($\mathcal{M}_{wr}$, $\mathcal{M}_{\ell\ell}$, $\mathcal{M}_{w\ell}$) be the class of all monoids (modulo isomorphism) of the form $\Sigma^*/\mathcal{R}$, where $\mathcal{R}$ is a length-reducing (weight-reducing, length-lexicographic, weight-lexicographic) and confluent STS over Σ. In [Die87] it was shown that the monoid $\{a, b, c\}^*/\{ab \to c^2\}$ is not contained in $\mathcal{M}_{\ell r}$. Since the STS $\{ab \to c^2\}$ is of course confluent, weight-reducing, and length-lexicographic, it follows that $\mathcal{M}_{\ell r}$ is strictly contained in $\mathcal{M}_{wr}$ and $\mathcal{M}_{\ell\ell}$. Furthermore the monoid $\{a, b\}^*/\{ab \to ba\}$ is contained in $\mathcal{M}_{\ell\ell} \backslash \mathcal{M}_{wr}$ [Die90], p

90. If there exists a monoid in $\mathcal{M}_{wr} \backslash \mathcal{M}_{\ell\ell}$ then $\mathcal{M}_{wr}$ and $\mathcal{M}_{\ell\ell}$ are incomparable and both are proper subclasses of $\mathcal{M}_{w\ell}$. But we do not know whether this holds.

Finally, another interesting class of rewriting systems, for which (uniform) word problems and confluence problems were studied, is the class of commutative semi-Thue systems, see for instance [Car75,Huy85,Huy86,Loh99,MM82,VRL98] for several decidability and complexity results. But there are still many interesting open questions, see for instance the remarks in [Huy85,Huy86,Loh99].

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